Presentations of rings with a chain of semidualizing modules

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Semidualizing modules

Throughout R is a commutative Noetherian local ring.

Definition

An R–module C is called *semidualizing*, if

- C is finite (i.e. finitely generated)
- The natural homothety map $\chi_C^R : R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism
- For all $i > 0$, $\mathrm{Ext}_R^i(C, C) = 0$

Example

Examples of semidualizing modules include

- R
- The dualizing module of R if it exists (dualizing module is a semidualizing module with finite injective dimension).

Semidualizing modules

Throughout C assumed to be a semidualizing R–module.

Basic properties

- $\mathrm{Ann}_R(C) = 0$ and $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$.
- $\dim_R(C) = \dim(R)$ and $\mathrm{Ass}_R(C) = \mathrm{Ass}_R(R)$.
- If R is local, then $\mathrm{depth}_R(C) = \mathrm{depth}(R)$.

If R is Gorenstein and local, then R is the only semidualizing R–module. Conversely, if the dualizing R–module is just the only semidualizing R–module, then R is Gorenstein.

Totally C–reflexive modules

Definition

A finite R–module M is *totally C–reflexive* when it satisfies the following conditions.

- The natural homomorphism $\delta_M^C : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism.
- For all $i > 0$, $\mathrm{Ext}^i_R(M, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C)$.

- Every finite projective R–module is totally C–reflexive.
- The G_C-dimension of a finite R–module M, denoted $\mathrm{G}_C\text{-}\dim_R(M)$, is defined as

$$\mathrm{G}_C - \dim_R(M) = \inf \left\{ n \geqslant 0 \;\middle|\; \begin{array}{c} \text{there is an exact sequence of } R-\text{modules} \\ 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \\ \text{such that each } G_i \text{ is totally } C-\text{reflexive} \end{array} \right\}$$

The set $\mathfrak{G}_0(R)$

The set of all isomorphism classes of semidualizing R–modules is denoted by $\mathfrak{G}_0(R)$, and the isomorphism class of a semidualizing R–module C is denoted $[C]$.

- Write $[C] \trianglelefteq [B]$ when B is totally C–reflexive.
- Write $[C] \triangleleft [B]$ when $[C] \trianglelefteq [B]$ and $[C] \neq [B]$.
- For each $[C] \in \mathfrak{G}_0(R)$ set $\mathfrak{G}_C(R) = \{[B] \in \mathfrak{G}_0(R) \mid [C] \trianglelefteq [B]\}$.
- If $[C] \trianglelefteq [B]$, then
  (1) $\mathrm{Hom}_R(B, C)$ is a semidualizing, and
  (2) $[C] \trianglelefteq [\mathrm{Hom}_R(B, C)]$.

Chain in $\mathfrak{G}_0(R)$

A chain in $\mathfrak{G}_0(R)$ is a sequence $[C_n] \trianglelefteq \cdots \trianglelefteq [C_1] \trianglelefteq [C_0]$, and such a chain has length n if $[C_i] \neq [C_j]$ whenever $i \neq j$.

Theorem (Gerko)

If $[C_n] \trianglelefteq \cdots \trianglelefteq [C_1] \trianglelefteq [C_0]$ is a chain in $\mathfrak{G}_0(R)$, then one gets

$$C_n \cong C_0 \otimes_R \operatorname{Hom}_R(C_0, C_1) \otimes_R \cdots \otimes_R \operatorname{Hom}_R(C_{n-1}, C_n).$$

Chain in $\mathfrak{G}_0(R)$

Assume that $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a chain in $\mathfrak{G}_0(R)$.

- For each $i \in [n]$ set $B_i = \mathrm{Hom}_R(C_{i-1}, C_i)$.
- For each sequence of integers $\mathbf{i} = \{i_1, \cdots, i_j\}$ with $j \geqslant 1$ and $1 \leqslant i_1 < \cdots < i_j \leqslant n$, set $B_{\mathbf{i}} = B_{i_1} \otimes_R \cdots \otimes_R B_{i_j}$.
  ($B_{\{i_1\}} = B_{i_1}$ and set $B_{\emptyset} = C_0$.)

Chain in $\mathfrak{G}_0(R)$

Proposition (Sather-Wagstaff)

Assume that $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a chain in $\mathfrak{G}_0(R)$ such that $\mathfrak{G}_{C_1}(R) \subseteq \mathfrak{G}_{C_2}(R) \subseteq \cdots \subseteq \mathfrak{G}_{C_n}(R)$.

(1) For each sequence $\mathbf{i} = \{i_1, \cdots, i_j\} \subseteq [n]$, the R–module $B_{\mathbf{i}}$ is a semidualizing.

(2) If $\mathbf{i} = \{i_1, \cdots, i_j\} \subseteq [n]$ and $\mathbf{s} = \{s_1, \cdots, s_t\} \subseteq [n]$ are two sequences with $\mathbf{s} \subseteq \mathbf{i}$, then $[B_{\mathbf{i}}] \trianglelefteq [B_{\mathbf{s}}]$ and $\mathrm{Hom}_R(B_{\mathbf{s}}, B_{\mathbf{i}}) \cong B_{\mathbf{i} \setminus \mathbf{s}}$.

(3) If $\mathbf{i} = \{i_1, \cdots, i_j\} \subseteq [n]$ and $\mathbf{s} = \{s_1, \cdots, s_t\} \subseteq [n]$ are two sequences, then the following conditions are equivalent.

  (i) The R–module $B_{\mathbf{i}} \otimes_R B_{\mathbf{s}}$ is semidualizing.

  (ii) $\mathbf{i} \cap \mathbf{s} = \emptyset$.

Chain in $\mathfrak{G}_0(R)$

For a semidualizing R–module C, set $(-)^{\dagger_C} = \mathrm{Hom}_R(-, C)$.

Definition

Let $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ be a chain in $\mathfrak{G}_0(R)$ of length n. For each sequence of integers $\mathbf{i} = \{i_1, \cdots, i_j\}$ such that $j \geqslant 0$ and $1 \leqslant i_1 < \cdots < i_j \leqslant n$, set $C_{\mathbf{i}} = C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \cdots \dagger_{C_{i_j}}}$.
(When $j = 0$, set $C_{\mathbf{i}} = C_{\emptyset} = C_0$).
We say that the above chain is *suitable* if $C_0 = R$ and $C_{\mathbf{i}}$ is totally C_t–reflexive, for all $\mathbf{i}$ and t with $i_j \leqslant t \leqslant n$.

- If $[C_n] \lhd \cdots \lhd [C_1] \lhd [R]$ is a suitable chain, then $C_{\mathbf{i}}$ is a semidualizing R–module for each $\mathbf{i} \subseteq [n]$.
- For each sequence of integers $\{x_1, \cdots, x_m\}$ with $1 \leqslant x_1 < \cdots < x_m \leqslant n$, the sequence $[C_{x_m}] \lhd \cdots \lhd [C_{x_1}] \lhd [R]$ is a suitable chain in $\mathfrak{G}_0(R)$.

Chain in $\mathfrak{G}_0(R)$

Theorem (Sather-Wagstaff)

Let $\mathfrak{G}_0(R)$ admit a chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ such that $\mathfrak{G}_{C_1}(R) \subseteq \mathfrak{G}_{C_2}(R) \subseteq \cdots \subseteq \mathfrak{G}_{C_n}(R)$.

- $|\mathfrak{G}_0(R)| \geqslant |\{[C_{\mathbf{i}}] \mid \mathbf{i} \subseteq [n]\}| = 2^n$.
- If $C_0 = R$, then $\{[B_{\mathbf{u}}] \mid \mathbf{u} \subseteq [n]\} = \{[C_{\mathbf{i}}] \mid \mathbf{i} \subseteq [n]\}$.

Suitable chains

Lemma (Dibaei and me)

Assume that R admits a suitable chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0] = [R]$ in $\mathfrak{G}_0(R)$. Then for any $k \in [n]$, there exists a suitable chain

$$[C_n] \lhd \cdots \lhd [C_{k+1}] \lhd [C_k] \lhd [C_1^{\dagger_{C_k}}] \lhd \cdots \lhd [C_{k-2}^{\dagger_{C_k}}] \lhd [C_{k-1}^{\dagger_{C_k}}] \lhd [R]$$

in $\mathfrak{G}_0(R)$ of length n.

Proposition (Suitable chains in $\mathfrak{G}_0(R_k)$)(Dibaei and me)

Let R be Cohen-Macaulay and $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ be a suitable chain in $\mathfrak{G}_0(R)$. For any $k \in [n]$, set $R_k = R \ltimes C_{k-1}^{\dagger_{C_k}}$ the trivial extension of R by $C_{k-1}^{\dagger_{C_k}}$. Set

$$C_l^{(k)} = \begin{cases} \mathrm{Hom}_R(R_k, C_{k-1-l}^{\dagger_{C_k}}) & \text{if } \ 0 \leqslant l < k-1 \\ \mathrm{Hom}_R(R_k, C_{l+1}) & \text{if } \ k-1 \leqslant l \leqslant n-1\,. \end{cases}$$

- For all l, $0 \leqslant l \leqslant n-1$, $C_l^{(k)}$ is a semidualizing R_k–module.
- For any $k \in [n]$,

$$[C_{n-1}^{(k)}] \lhd \cdots \lhd [C_1^{(k)}] \lhd [R_k]$$

is a suitable chain in $\mathfrak{G}_0(R_k)$ of length $n-1$.

Main result

Theorem (Dibaei and me)

Let R be a Cohen–Macaulay ring with a dualizing module D. Assume that R admits a suitable chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [R]$ in $\mathfrak{G}_0(R)$ and that $C_n \cong D$. Then there exist a Gorenstein local ring Q and ideals $I_1, \cdots, I_n$ of Q, which satisfy the following conditions. In this situation, for each $\Lambda \subseteq [n]$, set $R_{\Lambda} = Q/(\Sigma_{l \in \Lambda} I_l)$, in particular $R_{\emptyset} = Q$.

1. There is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$.
2. For each $\Lambda \subseteq [n]$ with $\Lambda \neq \emptyset$, the ring R_{Λ} is non-Gorenstein Cohen–Macaulay with a dualizing module.
3. For each $\Lambda \subseteq [n]$ with $\Lambda \neq \emptyset$, we have $\bigcap_{l \in \Lambda} I_l = \prod_{l \in \Lambda} I_l$.
4. For subsets Λ, Γ of $[n]$ with $\Gamma \subsetneq \Lambda$, we have $\mathrm{G}-\dim_{R_{\Gamma}} R_{\Lambda} = 0$, and $\mathrm{Hom}_{R_{\Gamma}}(R_{\Lambda}, R_{\Gamma})$ is a non-free semidualizing R_{Λ}–module.

Main result

Theorem (Dibaei and me)

(5) For subsets Λ, Γ of $[n]$ with $\Lambda \neq \Gamma$, the module $\mathrm{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma)$ is not cyclic and

$$\mathrm{Ext}^{\geqslant 1}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0 = \mathrm{Tor}^{R_{\Lambda\cap\Gamma}}_{\geqslant 1}(R_\Lambda, R_\Gamma).$$

(6) For subsets Λ, Γ of $[n]$ with $|\Lambda \setminus \Gamma| = 1$, we have

$$\widehat{\mathrm{Ext}}^{i}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0 = \widehat{\mathrm{Tor}}^{R_{\Lambda\cap\Gamma}}_{i}(R_\Lambda, R_\Gamma)$$

for all $i \in \mathbb{Z}$.

Construction

We construct the ring Q by induction on n. We claim that the ring Q, as an R–module, has the form $Q = \oplus_{\mathbf{i}\subseteq[n]} B_{\mathbf{i}}$ and the ring structure on it is as follows.
For two elements $(\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq[n]}$ and $(\theta_{\mathbf{i}})_{\mathbf{i}\subseteq[n]}$ of Q

$$(\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq[n]}(\theta_{\mathbf{i}})_{\mathbf{i}\subseteq[n]} = (\sigma_{\mathbf{i}})_{\mathbf{i}\subseteq[n]}\,, \quad \text{where} \quad \sigma_{\mathbf{i}} = \sum_{\substack{\mathbf{v}\subseteq\mathbf{i}\\ \mathbf{w}=\mathbf{i}\setminus\mathbf{v}}} \alpha_{\mathbf{v}}\cdot\theta_{\mathbf{w}}\,.$$

- $n=1$: set $Q = R\ltimes C_1$ and $I_1 = 0\oplus C_1$.
  (Proved by Foxby and Reiten)
- $n=2$: The extension ring Q has the form
  $Q = R\oplus C_1\oplus C_1^{\dagger_{C_2}}\oplus C_2$ as an R–module. The ring structure on Q is given by $(r,c,f,d)(r',c',f',d') =$
  $(rr', rc'+r'c, rf'+r'f, f'(c)+f(c')+rd'+r'd)$.
  (Proved by Jorgensen, Leuschke and Sather-Wagstaff)

Construction

- $n > 2$: Take an element $k \in [n]$. The ring $R_k = R \ltimes C_{k-1}^{\dagger_{C_k}}$ has the suitable chain $[C_{n-1}^{(k)}] \lhd \cdots \lhd [C_1^{(k)}] \lhd [R_k]$ in $\mathfrak{G}_0(R_k)$ of length $n-1$.
  We set $B_i^{(k)} = \mathrm{Hom}_{R_k}(C_{i-1}^{(k)}, C_i^{(k)})$, $i = 1, \cdots, n-1$. For two sequences $\mathbf{p} = \{p_1, \cdots, p_r\}$, $\mathbf{q} = \{q_1, \cdots, q_s\}$ such that $r, s \geqslant 1$ and $1 \leqslant p_1 < \cdots < p_r < k-1 \leqslant q_1 < \cdots < q_s \leqslant n-1$, we set

$$B_{\mathbf{p},\mathbf{q}}^{(k)} = B_{p_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{p_r}^{(k)} \otimes_{R_k} B_{q_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{q_s}^{(k)},$$

  By applying the induction hypothesis on R_k there is an extension ring, say Q_k, which is Gorenstein local and, as an R_k–module, has the form

$$Q_k = \bigoplus_{\substack{\mathbf{p} \subseteq \{1, \cdots, k-2\} \\ \mathbf{q} \subseteq \{k-1, \cdots, n-1\}}} B_{\mathbf{p},\mathbf{q}}^{(k)} .$$

Construction

For each $\mathbf{p}, \mathbf{q}$ there is an R–module isomorphism

$$B^{(k)}_{\mathbf{p},\mathbf{q}} \cong \begin{cases} B_{\{k-p_r,\cdots,k-p_1,q_1+1,\cdots,q_s+1\}} \oplus B_{\{k-p_r,\cdots,k-p_1,k,q_1+1,\cdots,q_s+1\}}, \\ \text{or} \\ B_{\{1,k-p_r,\cdots,k-p_1,q_2+1,\cdots,q_s+1\}} \oplus B_{\{1,k-p_r,\cdots,k-p_1,k,q_2+1,\cdots,q_s+1\}}. \end{cases}$$

Therefore one gets an R–module isomorphism $Q_k \cong \oplus_{\mathbf{i}\subseteq[n]} B_{\mathbf{i}}$.
Set $Q = Q_k$.
We set $I_l = \underbrace{(0 \oplus \cdots \oplus 0)}_{2^{n-1}} \oplus (\oplus_{\mathbf{i}\subseteq[n],\, l\in\mathbf{i}} B_{\mathbf{i}})$, $1 \leqslant l \leqslant n$, which is an ideal of Q and $Q/(I_1 + \cdots + I_n) \cong R$.

Converse of the main result

Proposition (Dibaei and me)

Let R be a Cohen–Macaulay ring. Assume that there exist a Gorenstein local ring Q and ideals $I_1, \cdots, I_n$ of Q satisfying the following conditions.

(1) There is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$.

(2) The ring $R_k = Q/(I_1 + \cdots + I_k)$ is Cohen–Macaulay for all $k \in [n]$.

(3) $\mathrm{fd}_{R_j}(R_k) < \infty$ for all $k \in [n]$ and all $1 \leqslant j \leqslant k$.

(4) For each $k \in [n]$, $\mathrm{I}_{R_k}^{R_k}(t) \neq t^e \mathrm{I}_{R_{k-1}}^{R_{k-1}}(t)$ for any integer e. $(R_0 = Q)$

Then there exist integers $g_0, g_1, \cdots, g_{n-1}$ such that

$$[\mathrm{Ext}_Q^{g_0}(R, Q)] \lhd [\mathrm{Ext}_{R_1}^{g_1}(R, R_1)] \lhd \cdots \lhd [\mathrm{Ext}_{R_{n-1}}^{g_{n-1}}(R, R_{n-1})] \lhd [R]$$

is a chain in $\mathfrak{G}_0(R)$ of length n.

Converse of the main result

Proposition (Dibaei and me)

Let R be a Cohen–Macaulay ring. Assume that there exist a Gorenstein local ring Q and ideals $I_1, \cdots, I_n$ of Q satisfying the following conditions.

(1) There is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$.

(2) For each $\Lambda \subseteq [n]$, the ring $R_{\Lambda} = Q/(\Sigma_{l\in\Lambda} I_l)$ is C-M.

(3) For subsets Λ, Γ of $[n]$ with $\Lambda \cap \Gamma = \emptyset$

  (i) $\mathrm{Tor}^{Q}_{\geqslant 1}(R_{\Lambda}, R_{\Gamma}) = 0$;

  (ii) For all $i \in \mathbb{Z}$, $\widehat{\mathrm{Ext}}^{i}_{Q}(R_{\Lambda}, R_{\Gamma}) = 0 = \widehat{\mathrm{Tor}}^{Q}_{i}(R_{\Lambda}, R_{\Gamma})$.

(4) For two subsets Λ, Γ of $[n]$ with $\Lambda \neq \Gamma$ and for any integer e, $\mathrm{I}^{R_{\Lambda}}_{R_{\Lambda}}(t) \neq t^e \mathrm{I}^{R_{\Gamma}}_{R_{\Gamma}}(t)$.

Then, for each $\Lambda \subseteq [n]$, there is an integer g_{Λ} such that $\mathrm{Ext}^{g_{\Lambda}}_{R_{\Lambda}}(R, R_{\Lambda})$ is a semidualizing R–module.
As conclusion, R admits 2^n non-isomorphic semidualizing modules.

Thank You

Theorem (Christensen)

Let S be a Cohen–Macaulay local ring equipped with a module-finite local ring homomorphism $\tau : R \to S$ such that R is Cohen–Macaulay. Assume that C is a semidualizing R–module. Then $\mathrm{G}_C\text{-}\dim_R(S) < \infty$ if and only if there exists an integer $g \geqslant 0$ such that $\mathrm{Ext}_R^i(S, C) = 0$ for all i, $i \neq g$, and $\mathrm{Ext}_R^g(S, C)$ is a semidualizing S–module; when these conditions hold, one has $g = \mathrm{G}_C\text{-}\dim_R(S)$.

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Tate resolution

Definition

Let M be a finite R–module. A *Tate resolution* of M is a diagram $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$, where π is an R–projective resolution of M, $\mathbf{T}$ is an exact complex of projectives such that $\mathrm{Hom}_R(T, R)$ is exact, ϑ is a morphism, and ϑ_i is isomorphism for all $i \gg 0$.

Definition

Let M be a finite R–module of finite G-dimension, and let $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$ be a Tate resolution of M. For each integer i and each R–module N, the ith *Tate homology* and *Tate cohomology* modules are

$$\widehat{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_i(\mathbf{T} \otimes_R N) \qquad \widehat{\mathrm{Ext}}_R^i(M, N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathbf{T}, N)).$$