Presentations of rings with a chain of semidualizing modules

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**Semidualizing modules**

Throughout $R$ is a commutative Noetherian local ring.

**Definition**

An $R$–module $C$ is called *semidualizing*, if

- $C$ is finite (i.e. finitely generated)
- The natural homothety map $\chi^R_C : R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism
- For all $i > 0$, $\text{Ext}^i_R(C, C) = 0$

**Example**

Examples of semidualizing modules include

- $R$
- The dualizing module of $R$ if it exists (dualizing module is a semidualizing module with finite injective dimension).
Throughout $C$ assumed to be a semidualizing $R$–module.

**Basic properties**

- $\text{Ann}_R(C) = 0$ and $\text{Supp}_R(C) = \text{Spec}(R)$.
- $\text{dim}_R(C) = \text{dim}(R)$ and $\text{Ass}_R(C) = \text{Ass}_R(R)$.
- If $R$ is local, then $\text{depth}_R(C) = \text{depth}(R)$.

If $R$ is Gorenstein and local, then $R$ is the only semidualizing $R$–module. Conversely, if the dualizing $R$–module is just the only semidualizing $R$–module, then $R$ is Gorenstein.
Totally $C$–reflexive modules

**Definition**

A finite $R$–module $M$ is *totally $C$–reflexive* when it satisfies the following conditions.

- The natural homomorphism
  \[ \delta^C_M : M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, C), C) \]
  is an isomorphism.
- For all $i > 0$, $\text{Ext}^i_R(M, C) = 0 = \text{Ext}^i_R(\text{Hom}_R(M, C), C)$.

- Every finite projective $R$–module is totally $C$–reflexive.
- The $G_C$-dimension of a finite $R$–module $M$, denoted $G_C\dim_R(M)$, is defined as

  \[
  G_C - \dim_R(M) = \inf \left\{ n \geq 0 \middle| \begin{array}{l}
  \text{there is an exact sequence of } R - \text{modules} \\
  0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \\
  \text{such that each } G_i \text{ is totally } C - \text{reflexive}
  \end{array} \right\}
  \]
The set $\mathfrak{S}_0(R)$

The set of all isomorphism classes of semidualizing $R$–modules is denoted by $\mathfrak{S}_0(R)$, and the isomorphism class of a semidualizing $R$–module $C$ is denoted $[C]$.

- Write $[C] \preceq [B]$ when $B$ is totally $C$–reflexive.
- Write $[C] \triangleleft [B]$ when $[C] \preceq [B]$ and $[C] \neq [B]$.
- For each $[C] \in \mathfrak{S}_0(R)$ set $\mathfrak{S}_C(R) = \{ [B] \in \mathfrak{S}_0(R) \mid [C] \preceq [B] \}$.
- If $[C] \preceq [B]$, then
  1. $\text{Hom}_R(B, C)$ is a semidualizing, and
  2. $[C] \preceq [\text{Hom}_R(B, C)]$. 

A chain in $G_0(R)$ is a sequence $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$, and such a chain has length $n$ if $[C_i] \neq [C_j]$ whenever $i \neq j$.

**Theorem (Gerko)**

If $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a chain in $G_0(R)$, then one gets

$$C_n \cong C_0 \otimes_R \text{Hom}_R(C_0, C_1) \otimes_R \cdots \otimes_R \text{Hom}_R(C_{n-1}, C_n).$$
Assume that \([C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]\) is a chain in \(\mathfrak{S}_0(R)\).

- For each \(i \in [n]\) set \(B_i = \text{Hom}_R(C_{i-1}, C_i)\).

- For each sequence of integers \(i = \{i_1, \cdots, i_j\}\) with \(j \geq 1\) and \(1 \leq i_1 < \cdots < i_j \leq n\), set \(B_i = B_{i_1} \otimes_R \cdots \otimes_R B_{i_j}\).  
  \((B_{\{i_1\}} = B_{i_1}\) and set \(B_{\emptyset} = C_0\).\)
Proposition (Sather-Wagstaff)

Assume that $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a chain in $\mathcal{G}_0(R)$ such that $\mathcal{G}_{C_1}(R) \subseteq \mathcal{G}_{C_2}(R) \subseteq \cdots \subseteq \mathcal{G}_{C_n}(R)$.

(1) For each sequence $i = \{i_1, \cdots, i_j\} \subseteq [n]$, the $R$–module $B_i$ is a semidualizing.

(2) If $i = \{i_1, \cdots, i_j\} \subseteq [n]$ and $s = \{s_1, \cdots, s_t\} \subseteq [n]$ are two sequences with $s \subseteq i$, then $[B_i] \triangleleft [B_s]$ and $\text{Hom}_R(B_s, B_i) \cong B_i \setminus s$.

(3) If $i = \{i_1, \cdots, i_j\} \subseteq [n]$ and $s = \{s_1, \cdots, s_t\} \subseteq [n]$ are two sequences, then the following conditions are equivalent.

(i) The $R$–module $B_i \otimes_R B_s$ is semidualizing.

(ii) $i \cap s = \emptyset$. 
Chain in $\mathcal{G}_0(R)$

For a semidualizing $R$–module $C$, set $(-)^{t_c} = \text{Hom}_R(-, C)$.

**Definition**

Let $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ be a chain in $\mathcal{G}_0(R)$ of length $n$. For each sequence of integers $i = \{i_1, \cdots, i_j\}$ such that $j \geq 0$ and $1 \leq i_1 < \cdots < i_j \leq n$, set $C_i = C_{i_1}^{i_2} \cdots^{i_j}$. (When $j = 0$, set $C_i = C_{\emptyset} = C_0$).

We say that the above chain is *suitable* if $C_0 = R$ and $C_i$ is totally $C_t$–reflexive, for all $i$ and $t$ with $i_j \leq t \leq n$.

- If $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [R]$ is a suitable chain, then $C_i$ is a semidualizing $R$–module for each $i \subseteq [n]$.
- For each sequence of integers $\{x_1, \cdots, x_m\}$ with $1 \leq x_1 < \cdots < x_m \leq n$, the sequence $[C_{x_m}] \triangleleft \cdots \triangleleft [C_{x_1}] \triangleleft [R]$ is a suitable chain in $\mathcal{G}_0(R)$. 
Chain in \( \mathcal{G}_0(R) \)

**Theorem (Sather-Wagstaff)**

Let \( \mathcal{G}_0(R) \) admit a chain \([C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]\) such that \( \mathcal{G}_{C_1}(R) \subseteq \mathcal{G}_{C_2}(R) \subseteq \cdots \subseteq \mathcal{G}_{C_n}(R) \).

- \(|\mathcal{G}_0(R)| \geq |\{[C_i] \mid i \subseteq [n]\}| = 2^n\).
- If \( C_0 = R \), then \( \{[B_u] \mid u \subseteq [n]\} = \{[C_i] \mid i \subseteq [n]\} \).
Suitable chains

**Lemma (Dibaei and me)**

Assume that $R$ admits a suitable chain $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0] = [R]$ in $\mathcal{G}_0(R)$. Then for any $k \in [n]$, there exists a suitable chain

$[C_n] \triangleleft \cdots \triangleleft [C_{k+1}] \triangleleft [C_k] \triangleleft [C_1^{\dagger c_k}] \triangleleft \cdots \triangleleft [C_{k-2}^{\dagger c_k}] \triangleleft [C_{k-1}^{\dagger c_k}] \triangleleft [R]$ in $\mathcal{G}_0(R)$ of length $n$. 
Proposition (Suitable chains in \( \mathcal{G}_0(R_k) \)) (Dibaei and me)

Let \( R \) be Cohen-Macaulay and \([C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]\) be a suitable chain in \( \mathcal{G}_0(R) \). For any \( k \in [n] \), set \( R_k = R \times C_{k-1}^{\dagger} \) the trivial extension of \( R \) by \( C_{k-1}^{\dagger} \). Set

\[
C_l^{(k)} = \begin{cases} 
\text{Hom}_R(R_k, C_{k-1-l}^{\dagger}) & \text{if } 0 \leq l < k - 1 \\
\text{Hom}_R(R_k, C_{l+1}) & \text{if } k - 1 \leq l \leq n - 1.
\end{cases}
\]

- For all \( l, 0 \leq l \leq n - 1 \), \( C_l^{(k)} \) is a semidualizing \( R_k \)-module.
- For any \( k \in [n] \),

\[
[C_{n-1}^{(k)}] \triangleleft \cdots \triangleleft [C_1^{(k)}] \triangleleft [R_k]
\]

is a suitable chain in \( \mathcal{G}_0(R_k) \) of length \( n - 1 \).
Main result

**Theorem (Dibaei and me)**

Let $R$ be a Cohen–Macaulay ring with a dualizing module $D$. Assume that $R$ admits a suitable chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [R]$ in $\mathcal{G}_0(R)$ and that $C_n \cong D$. Then there exist a Gorenstein local ring $Q$ and ideals $I_1, \cdots, I_n$ of $Q$, which satisfy the following conditions. In this situation, for each $\Lambda \subseteq [n]$, set $R_\Lambda = Q/(\sum_{I \in \Lambda} I)$, in particular $R_\emptyset = Q$.

1. There is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$.
2. For each $\Lambda \subseteq [n]$ with $\Lambda \neq \emptyset$, the ring $R_\Lambda$ is non-Gorenstein Cohen–Macaulay with a dualizing module.
3. For each $\Lambda \subseteq [n]$ with $\Lambda \neq \emptyset$, we have $\bigcap_{I \in \Lambda} I_I = \prod_{I \in \Lambda} I_I$.
4. For subsets $\Lambda, \Gamma$ of $[n]$ with $\Gamma \subsetneq \Lambda$, we have $G - \dim_{R_\Gamma} R_\Lambda = 0$, and $\text{Hom}_{R_\Gamma}(R_\Lambda, R_\Gamma)$ is a non-free semidualizing $R_\Lambda$–module.
Main result

**Theorem (Dibaei and me)**

(5) For subsets $\Lambda$, $\Gamma$ of $[n]$ with $\Lambda \neq \Gamma$, the module $\text{Hom}_{R_{\Lambda \cap \Gamma}}(R_{\Lambda}, R_{\Gamma})$ is not cyclic and

$$\text{Ext}_1^{R_{\Lambda \cap \Gamma}}(R_{\Lambda}, R_{\Gamma}) = 0 = \text{Tor}_1^{R_{\Lambda \cap \Gamma}}(R_{\Lambda}, R_{\Gamma}).$$

(6) For subsets $\Lambda$, $\Gamma$ of $[n]$ with $|\Lambda \setminus \Gamma| = 1$, we have

$$\widehat{\text{Ext}}_i^{R_{\Lambda \cap \Gamma}}(R_{\Lambda}, R_{\Gamma}) = 0 = \widehat{\text{Tor}}_i^{R_{\Lambda \cap \Gamma}}(R_{\Lambda}, R_{\Gamma})$$

for all $i \in \mathbb{Z}$. 
Constructions

We construct the ring $Q$ by induction on $n$. We claim that the ring $Q$, as an $R$–module, has the form $Q = \bigoplus_{i \subseteq [n]} B_i$ and the ring structure on it is as follows.

For two elements $(\alpha_i)_{i \subseteq [n]}$ and $(\theta_i)_{i \subseteq [n]}$ of $Q$

$$(\alpha_i)_{i \subseteq [n]}(\theta_i)_{i \subseteq [n]} = (\sigma_i)_{i \subseteq [n]}, \text{ where } \sigma_i = \sum \alpha_v \cdot \theta_w.$$

- $n = 1$: set $Q = R \rtimes C_1$ and $I_1 = 0 \oplus C_1$. (Proved by Foxby and Reiten)
- $n = 2$: The extension ring $Q$ has the form $Q = R \oplus C_1 \oplus \hat{C}_2 \oplus C_2$ as an $R$–module. The ring structure on $Q$ is given by $(r, c, f, d)(r', c', f', d') = (rr', rc' + r'c, rf' + r'f, f'(c) + f(c') + rd' + r'd)$. (Proved by Jorgensen, Leuschke and Sather-Wagstaff)
Construction

- $n > 2$: Take an element $k \in [n]$. The ring $R_k = R \ltimes C_{k-1}^{\uparrow C_k}$ has the suitable chain $[C_{n-1}^{(k)}] \triangleleft \cdots \triangleleft [C_1^{(k)}] \triangleleft [R_k]$ in $\mathcal{G}_0(R_k)$ of length $n - 1$.

We set $B_i^{(k)} = \text{Hom}_{R_k}(C_{i-1}^{(k)}, C_i^{(k)}), i = 1, \ldots, n - 1$. For two sequences $p = \{p_1, \ldots, p_r\}, q = \{q_1, \ldots, q_s\}$ such that $r, s \geq 1$ and $1 \leq p_1 < \cdots < p_r < k - 1 \leq q_1 < \cdots < q_s \leq n - 1$, we set

$$B_{p,q}^{(k)} = B_{p_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{p_r}^{(k)} \otimes_{R_k} B_{q_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{q_s}^{(k)}.$$ 

By applying the induction hypothesis on $R_k$ there is an extension ring, say $Q_k$, which is Gorenstein local and, as an $R_k$–module, has the form

$$Q_k = \bigoplus_{p \subseteq \{1, \ldots, k-2\}, q \subseteq \{k-1, \ldots, n-1\}} B_{p,q}^{(k)}.$$
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Construction

For each $p, q$ there is an $R$–module isomorphism

$$B_{(k)}^{(p, q)} \cong \begin{cases} B\{k-p_r, \ldots, k-p_1, q_1+1, \ldots, q_s+1\} \oplus B\{k-p_r, \ldots, k-p_1, k, q_1+1, \ldots, q_s+1\}, \\ or \\ B\{1, k-p_r, \ldots, k-p_1, q_2+1, \ldots, q_s+1\} \oplus B\{1, k-p_r, \ldots, k-p_1, k, q_2+1, \ldots, q_s+1\}. \end{cases}$$

Therefore one gets an $R$–module isomorphism $Q_k \cong \bigoplus_{i \subseteq [n]} B_i$.

Set $Q = Q_k$.

We set $l_l = (0 \oplus \cdots \oplus 0) \oplus (\bigoplus_{i \subseteq [n], l \in i} B_i)$, $1 \leq l \leq n$, which is an ideal of $Q$ and $Q/(l_1 + \cdots + l_n) \cong R$. 
Converse of the main result

**Proposition (Dibaei and me)**

Let $R$ be a Cohen–Macaulay ring. Assume that there exist a Gorenstein local ring $Q$ and ideals $I_1, \cdots, I_n$ of $Q$ satisfying the following conditions.

1. There is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$.
2. The ring $R_k = Q/(I_1 + \cdots + I_k)$ is Cohen–Macaulay for all $k \in [n]$.
3. $\text{fd}_{R_j}(R_k) < \infty$ for all $k \in [n]$ and all $1 \leq j \leq k$.
4. For each $k \in [n]$, $I_{R_k}(t) \neq t^e I_{R_{k-1}}(t)$ for any integer $e$.

($R_0 = Q$)

Then there exist integers $g_0, g_1, \cdots, g_{n-1}$ such that

$$[\text{Ext}^{g_0}_{Q}(R, Q)] \triangleleft [\text{Ext}^{g_1}_{R_1}(R, R_1)] \triangleleft \cdots \triangleleft [\text{Ext}^{g_{n-1}}_{R_{n-1}}(R, R_{n-1})] \triangleleft [R]$$

is a chain in $\mathcal{G}_0(R)$ of length $n$. 
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Converse of the main result

Proposition (Dibaei and me)

Let $R$ be a Cohen–Macaulay ring. Assume that there exist a Gorenstein local ring $Q$ and ideals $I_1, \cdots, I_n$ of $Q$ satisfying the following conditions.

1. There is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$.
2. For each $\Lambda \subseteq [n]$, the ring $R_\Lambda = Q/(\sum_{i \in \Lambda} I_i)$ is C-M.
3. For subsets $\Lambda, \Gamma$ of $[n]$ with $\Lambda \cap \Gamma = \emptyset$
   
   (i) $\text{Tor}_1^Q(R_\Lambda, R_\Gamma) = 0$;
   
   (ii) For all $i \in \mathbb{Z}$, $\text{Ext}_i^Q(R_\Lambda, R_\Gamma) = 0 = \text{Tor}_i^Q(R_\Lambda, R_\Gamma)$.
4. For two subsets $\Lambda, \Gamma$ of $[n]$ with $\Lambda \neq \Gamma$ and for any integer $e$,
   
   $I_{R_\Lambda}^R(t) \neq t^e I_{R_\Gamma}^R(t)$.

Then, for each $\Lambda \subseteq [n]$, there is an integer $g_{\Lambda}$ such that $\text{Ext}_{R_\Lambda}^g(R, R_\Lambda)$ is a semidualizing $R$–module.

As conclusion, $R$ admits $2^n$ non-isomorphic semidualizing modules.
Thank You
Theorem (Christensen)

Let $S$ be a Cohen–Macaulay local ring equipped with a module-finite local ring homomorphism $\tau : R \to S$ such that $R$ is Cohen–Macaulay. Assume that $C$ is a semidualizing $R$–module. Then $G_C$-$\text{dim}_R(S) < \infty$ if and only if there exists an integer $g \geq 0$ such that $\text{Ext}^i_R(S, C) = 0$ for all $i, i \neq g$, and $\text{Ext}^g_R(S, C)$ is a semidualizing $S$–module; when these conditions hold, one has $g = G_C$-$\text{dim}_R(S)$.
Tate resolution

**Definition**

Let $M$ be a finite $R$–module. A Tate resolution of $M$ is a diagram $T \xrightarrow{\vartheta} P \xrightarrow{\pi} M$, where $\pi$ is an $R$–projective resolution of $M$, $T$ is an exact complex of projectives such that $\text{Hom}_R(T, R)$ is exact, $\vartheta$ is a morphism, and $\vartheta_i$ is isomorphism for all $i \gg 0$.

**Definition**

Let $M$ be a finite $R$–module of finite $G$-dimension, and let $T \xrightarrow{\vartheta} P \xrightarrow{\pi} M$ be a Tate resolution of $M$. For each integer $i$ and each $R$–module $N$, the $i$th Tate homology and Tate cohomology modules are

$$\widehat{\text{Tor}}_i^R(M, N) = H_i(T \otimes_R N) \quad \widehat{\text{Ext}}_R^i(M, N) = H_{-i}(\text{Hom}_R(T, N)).$$