A note on the rings with flat injective hulls

Fahimeh Khosh-Ahang Ghasr

November 12, 2015
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Let $R$ be a ring and $\mathcal{F}$ be the class of all flat $R$-modules. Then for an $R$-module $M$, an $R$-homomorphism $\varphi : F \rightarrow M$, where $F \in \mathcal{F}$ is called a flat cover of $M$ if
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We briefly say that $F$ is the flat cover of $M$ and denote it by $\mathcal{F}(M)$. 
Definition [Eckmann, B.; Schopf, A. (1953)].

Let $R$ be a ring and $\mathcal{E}$ be the class of all injective $R$-modules. Then for an $R$-module $M$, an $R$-homomorphism $\varphi : M \to E$, where $E \in \mathcal{E}$ is called an injective envelope of $M$ if

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We briefly say that $E$ is the injective envelope of $M$ and denote it by $\mathcal{E}(M)$. 
A module $M$ is Gorenstein flat if there is an exact sequence

$$
\cdots \to F^{-2} \to F^{-1} \to F^0 \to F^1 \to F^2 \to \cdots
$$

of flat modules such that $M = \ker(F^0 \to F^1)$ and such that $E \otimes_R -$ leaves the sequence exact when $E$ is injective.
Definition.

• A module $M$ is **Gorenstein flat** if there is an exact sequence

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of flat modules such that $M = \ker(F^0 \to F^1)$ and such that $E \otimes_R -$ leaves the sequence exact when $E$ is injective.

• The **Gorenstein flat dimension** of $M$ is denoted by $\text{Gfd}(M)$ and defined as

$$\text{Gfd}(M) = \inf\{n | \text{there exists an exact sequence } 0 \to G_n \to \cdots \to G_0 \to M \to 0 \text{ s.th. } G_i\text{'s are Gorenstein flat}\}. $$
Definition.

Let $R$ be a ring and let $\mathcal{GF}$ be the class of all Gorenstein flat $R$-modules. Then for an $R$-module $M$, a morphism $\varphi : G \to M$, where $G \in \mathcal{GF}$ is called a Gorenstein flat cover of $M$ if
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We briefly say that $G$ is the Gorenstein flat cover of $M$ and denote it by $\mathcal{GF}(M)$. 
A module $M$ is **Gorenstein injective** if there is an exact sequence

$$\cdots \rightarrow E^{-2} \rightarrow E^{-1} \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots$$

of injective modules such that $M = \ker(E^0 \rightarrow E^1)$ and such that $\text{Hom}_R(E, -)$ leaves the sequence exact when $E$ is injective.
Remark.

Note that the existence of a flat cover, an injective envelope and a Gorenstein flat cover for any module over any associative ring has been proved.
A module $M$ is called **cotorsion** if $\text{Ext}_R^1(F, M) = 0$ for any flat module $F$. 
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A module $M$ is called **strongly cotorsion** if $\text{Ext}_R^1(X, M) = 0$ for any module $X$ of finite flat dimension.
Definitions.

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Let $R$ be a ring and $n$ a positive integer.

- A module $M$ is called $n$-cotorsion if $\text{Ext}^1_R(X, M) = 0$ for any $R$-module $X$ with flat dimension at most $n$. 

- A module $N$ is called $n$-torsionfree if $\text{Tor}_1^R(N, X) = 0$ for any $R$-module $X$ with flat dimension at most $n$. 

- A module $M$ is called $n$-Gorenstein cotorsion if $\text{Ext}^1_R(X, M) = 0$ for any $R$-module $X$ with Gorenstein flat dimension at most $n$. 

- A module $M$ is called strongly Gorenstein cotorsion if it is $n$-Gorenstein cotorsion for all $n$. 

- A module $N$ is called $n$-Gorenstein torsionfree if $\text{Tor}_1^R(N, X) = 0$ for any $R$-module $X$ with Gorenstein flat dimension at most $n$. 

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Remark.

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\{1\text{-cotorsion modules}\} \supseteq \{2\text{-cotorsion modules}\} \supseteq \cdots \supseteq \{\text{strongly cotorsion modules}\}
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Every flat module $F$ can be uniquely written in the form

$$F = \prod T_p,$$

where $T_p$ is a completion of a free $R_p$-module with respect to $p$-adic topology.
A prime ideal $p$ of $R$ is said to be a **coassociated prime** of $M$ if there exists an Artinian homomorphic image $L$ of $M$ with $p = 0 :_R L$. The set of all coassociated prime ideals of $M$ is denoted by $\text{Coass}(M)$. 
Proposition.

Let $F$ be a flat $R$-module and $E$ be an injective $R$-module, where $R$ is a commutative ring with non-zero identity.

The following conditions are equivalent.

1. $\mathcal{E}(F)$ is flat.
2. $\mathcal{F}(\mathcal{E}(F)) = \mathcal{E}(F)$.
3. $\mathcal{F}(\mathcal{E}(F))$ is injective.
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A flat module over a Noetherian ring $R$ is injective if and only if it is Gorenstein injective.
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- A flat module over a Noetherian ring \( R \) is injective if and only if it is Gorenstein injective.
- An injective module over a commutative ring \( R \) is flat if and only if it is Gorenstein flat.
Proposition.

Assume that for all injective $R$-modules $E$ and $E'$ such that $\text{Ass}(E) \subseteq \text{Ass}(R)$, the $R$-module $\text{Hom}_R(E, E')$ is injective. Then
- $\mathcal{F}(\mathcal{E}(R))$ is injective; and
Proposition.

Assume that for all injective $R$-modules $E$ and $E'$ such that $\text{Ass}(E) \subseteq \text{Ass}(R)$, the $R$-module $\text{Hom}_R(E, E')$ is injective. Then

- $\mathcal{F}(\mathcal{E}(R))$ is injective; and
- $R_p$ is a Gorenstein ring of Krull dimension 0 for all $p \in \text{Ass}(R)$. 
Proposition.

For each injective $R$-module $E$ we have

$$\text{id}_R(\mathcal{F}(E)) \leq \text{fd}_R(\mathcal{E}(R)).$$
Proposition.

- For each injective $R$-module $E$ we have
  \[ \text{id}_R(\mathcal{F}(E)) \leq \text{fd}_R(\mathcal{E}(R)). \]

- For each $R$-module $N$, we have
  \[ \text{id}_R(\mathcal{F}(D_R(N))) \leq \text{fd}_R(\mathcal{E}(N)). \]
Proposition.

- For each injective $R$-module $E$ we have
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- For each $R$-module $N$, we have
  \[ \text{id}_R(\mathcal{F}(D_R(N))) \leq \text{fd}_R(\mathcal{E}(N)). \]

- For each prime ideal $\mathfrak{p}$ of $R$,
  \[ \text{id}_R(\mathcal{E}(\widehat{R}_\mathfrak{p})) = \text{fd}_R(\mathcal{E}(R/\mathfrak{p})). \]
It is well-known that every Gorenstein ring has flat injective hull. [Bass, H., 1963].
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There exists an example of a ring with flat injective hull which is not Gorenstein. [Enochs, E. E.; Huang, Z., 2012]
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There exists an example of a ring with flat injective hull which is not Gorenstein. [Enochs, E. E.; Huang, Z., 2012]

So, the rings with flat injective hulls are generalizations of Gorenstein rings.
The following are equivalent for a commutative Noetherian ring $R$.

1. $\mathcal{E}(R)$ is flat.
2. $R_p$ is a Gorenstein ring of Krull dimension 0 for all $p \in \text{Ass}(R)$.
3. $\mathcal{E}(F)$ is flat for all flat $R$-modules $F$.
4. $\mathcal{F}(E)$ is injective for all injective $R$-modules $E$.
5. $E \otimes E'$ is an injective module for all injective $R$-modules $E$ and $E'$.
6. $S^{-1}R$ is an injective $R$-module where $S$ is the set of non-zero divisors of $R$. 
The following are equivalent for a commutative Noetherian ring $R$.

1. $\mathcal{E}(R)$ is flat.
2. $\mathcal{E}(R)$ has finite flat dimension.
3. $\mathcal{F}(M)$ is injective for any strongly cotorsion module $M$.
4. $\mathcal{E}(M)$ is flat for any strongly torsion free module $M$.
5. $\mathcal{E}(M)$ is flat for any Gorenstein flat module $M$.
6. If $p \in \text{Coass}(E)$ for an injective $R$-module $E$, then $\widehat{R_p}$ is injective.
Theorem [Khashyarmanesh, K.; Salarian, Sh., 2003].

The following are equivalent for a commutative Noetherian ring $R$.

1. $\mathcal{E}(R)$ is flat.
2. $\mathcal{E}(R)$ has finite flat dimension.
3. $\mathcal{F}(M)$ is injective for any strongly cotorsion module $M$.
4. $\mathcal{E}(M)$ is flat for any strongly torsion free module $M$.
5. $\mathcal{E}(M)$ is flat for any Gorenstein flat module $M$.
6. If $p \in \text{Coass}(E)$ for an injective $R$-module $E$, then $\widehat{R_p}$ is injective.

If moreover the Krull dimension of $R$ is finite, then the above conditions are equivalent to

7. $\mathcal{F}(M)$ is injective for any Gorenstein injective module $M$.

For a Commutative Noetherian ring $R$ the following conditions are equivalent.

1. $\mathcal{E}(R)$ is flat.
2. $\mathcal{E}(R)$ is Gorenstein flat.
3. $\mathcal{E}(F)$ is Gorenstein flat for any flat $R$-module $F$.
4. $\mathcal{E}(G)$ is Gorenstein flat for any Gorenstein flat $R$-module $G$.
5. $\mathcal{GF}(M)$ is injective for any 1-Gorenstein cotorsion $R$-module $M$.
6. $\mathcal{GF}(M)$ is injective for any strongly Gorenstein cotorsion $R$-module $M$.
7. $\mathcal{GF}(E)$ is injective for any injective left $R$-module $E$. 
\(E(N)\) is flat for any 1-Gorenstein torsionfree \(R\)-module \(N\).

\(F(M)\) is injective for any 1-cotorsion \(R\)-module \(M\).

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\(E(N)\) is Gorenstein flat for any strongly torsionfree \(R\)-module \(N\).
Theorem.

For a commutative Noetherian ring $R$, the following conditions are equivalent.

1. $\mathcal{E}(R)$ is flat.
2. $\mathcal{F}(\mathcal{E}(R))$ is injective.
3. $\mathcal{F}(\mathcal{E}(R))$ is Gorenstein injective.
4. $\mathcal{E}(R/p)$ is flat for any associated prime ideal $p$ of $R$.
5. $T_p$ is injective for any coassociated prime ideal $p$ of $\mathcal{F}(\mathcal{E}(R))$, where $T_p$ is the completion of a free $\hat{R}_p$-module.
6. $\mathcal{E}(R/p)$ is Gorenstein flat for any associated prime ideal $p$ of $R$.
7. $T_p$ is Gorenstein injective for any coassociated prime ideal $p$ of $\mathcal{F}(\mathcal{E}(R))$. 
(8) $\mathcal{E}(R/p)$ has finite flat dimension for any associated prime ideal $p$ of $R$.

(9) $\mathcal{F}(\mathcal{E}(F))$ is injective for all flat $R$-modules $F$.

(10) $\mathcal{E}(\mathcal{F}(E))$ is flat for all injective $R$-modules $E$.

(11) $\mathcal{F}(\mathcal{E}(F))$ is Gorenstein injective for all flat $R$-modules $F$.

(12) $\mathcal{E}(\mathcal{F}(E))$ is Gorenstein flat for all injective $R$-modules $E$.

(13) $\mathcal{F}(E)$ is Gorenstein injective for all injective $R$-modules $E$.

(14) $\mathcal{F}(M)$ is Gorenstein injective for all strongly cotorsion $R$-modules $M$. 
(15) $\mathcal{E}(F)$ has finite flat dimension for all flat $R$-modules $F$.
(16) $\mathcal{E}(M)$ is flat for all $R$-modules $M$ with $\text{Ass}(M) \subseteq \text{Ass}(R)$.
(17) $\mathcal{E}(M)$ is Gorenstein flat for all $R$-modules $M$ with $\text{Ass}(M) \subseteq \text{Ass}(R)$.
(18) $\mathcal{E}(M)$ has finite flat dimension for all $R$-modules $M$ with $\text{Ass}(M) \subseteq \text{Ass}(R)$.
(19) $R_p$ is injective for all coassociated prime ideals $p$ of $\mathcal{F}(\mathcal{E}(R))$.
(20) There is an injective $R$-module $E$ such that for all $p \in \text{Coass}(E)$, $\widehat{R}_p$ is injective.
(21) For all injective $R$-modules $E$ and $E'$ the $R$-module $E \otimes_R E'$ is injective and flat.
(22) For all injective $R$-modules $E$ and $E'$ such that $\text{Ass}(E) \subseteq \text{Ass}(R)$, the $R$-module $\text{Hom}_R(E, E')$ is injective and flat.
If moreover the Krull dimension of \( R \) is finite, the above conditions are equivalent to:

(23) \( \mathcal{F}(M) \) is Gorenstein injective for all Gorenstein injective \( R \)-modules \( M \).
Also, if every prime ideal in $\text{Ass}(R)$ is a minimal prime ideal of $R$, then the condition "$\mathcal{E}(R)$ is flat" is equivalent to the following conditions.

(24) $\mathcal{F}(\mathcal{E}(R))$ has finite injective dimension.
(25) $T_p$ has finite injective dimension for any coassociated prime ideal $p$ of $\mathcal{F}(\mathcal{E}(R))$.
(26) $\mathcal{F}(E)$ has finite injective dimension for all injective $R$-modules $E$.
(27) Every flat and cotorsion $R$-module $F$ such that $\text{Coass}(F) \subseteq \text{Ass}(R)$ is injective.
(28) Every flat and cotorsion $R$-module $F$ such that $\text{Coass}(F) \subseteq \text{Ass}(R)$ is Gorenstein injective.
(29) $R_p$ is Gorenstein for all coassociated prime ideals $p$ of $\mathcal{F}(\mathcal{E}(R))$. 
Thanks for your patience.