Homological invariants of the Stanley-Reisner ring of a k-decomposable simplicial complex

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k-decomposable simplicial complex

Definition[Woodroofe (2009)]. Let Δ be a simplicial complex on the vertex set V. A face σ is called a shedding face if every face τ containing σ satisfies the following exchange property: for every $v \in \sigma$ there is $w \in V \setminus \tau$ such that $(\tau \cup \{w\}) \setminus \{v\}$ is a face of Δ.

Definition[Woodroofe]. A simplicial complex Δ is recursively defined to be k-decomposable if either Δ is a simplex or else has a shedding face σ with $\dim(\sigma) \leq k$ such that both $\Delta \setminus \sigma$ and $\mathrm{lk}(\sigma)$ are k-decomposable.

The complexes $\{\}$ and $\{\emptyset\}$ are considered to be k-decomposable for all $k \geq -1$.

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Let K be a filed, $R = K[V(\Delta)]$ and I_Δ be the Stanley-Reisner ideal of Δ.

Theorem [Khosh-Ahang, Moradi], [Ha, Woodroofe] Let Δ be a vertex decomposable simplicial complex, x a shedding vertex of Δ and $\Delta_1 = \mathrm{del}_\Delta(x)$ and $\Delta_2 = \mathrm{lk}_\Delta(x)$. Then

$$\mathrm{pd}(R/I_\Delta) = \max\{\mathrm{pd}(R/I_{\Delta_1}) + 1, \mathrm{pd}(R/I_{\Delta_2})\},$$

$$\mathrm{reg}(R/I_\Delta) = \max\{\mathrm{reg}(R/I_{\Delta_1}), \mathrm{reg}(R/I_{\Delta_2}) + 1\}.$$

Theorem [Ha]. Let Δ be a simplicial complex and let σ be a face of dimension $d-1$ in Δ. Then

$$\mathrm{reg}(R/I_{\Delta}) \leq \max\{\mathrm{reg}(R/I_{\Delta\setminus\sigma}), \mathrm{reg}(R/I_{\mathrm{lk}(\sigma)}) + d\}.$$

For the monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ in R, the support of u denoted by $\mathrm{supp}\,(u)$ is the set $\{x_i :\ a_i \neq 0\}$.
For the monomial u and the monomial ideal I, set

$$I_u = (M \in \mathcal{G}(I) :\ x_i^{a_i} \nmid M \ \ \forall x_i \in \mathrm{supp}\,(u))$$

and

$$I^u = (M \in \mathcal{G}(I) :\ M \notin \mathcal{G}(I_u))$$

k-decomposable ideal

For a monomial ideal I with $\mathcal{G}(I) = \{M_1, \ldots, M_r\}$, the monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is called a shedding monomial for I if $I_u \neq 0$ and for each $M_i \in \mathcal{G}(I_u)$ and each $x_\ell \in \mathrm{supp}\,(u)$ there exists $M_j \in \mathcal{G}(I^u)$ such that $M_j : M_i = x_\ell$.

Definition[Rahmati-Asghar, Yassemi]. A monomial ideal I with $\mathcal{G}(I) = \{M_1, \ldots, M_r\}$ is called k-decomposable if $r = 1$ or else has a shedding monomial u with $|\mathrm{supp}\,(u)| \leq k + 1$ such that the ideals I_u and I^u are k-decomposable.

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Definition[Rahmati-Asghar, Yassemi]. A monomial ideal I with $\mathcal{G}(I)=\{M_1,\dots,M_r\}$ is called k-decomposable if $r=1$ or else has a shedding monomial u with $|\mathrm{supp}\,(u)|\leq k+1$ such that the ideals I_u and I^u are k-decomposable.

Theorem[Rahmati-Asghar,Yassemi]. A d-dimensional simplicial complex Δ is k-decomposable if and only if $I_{\Delta^\vee}$ is a squarefree k-decomposable ideal for any $k \leq d$.

Theorem. Let I be a k-decomposable ideal with the shedding monomial u. Then

$$\beta_{i,j}(I) = \beta_{i,j}(I^u) + \sum_{\ell=0}^{m} \binom{m}{\ell} \beta_{i-\ell,j-\ell}(I_u)$$

where $m = |\mathrm{supp}\,(u)|$.

Outline of proof. If $f_1 < \cdots < f_t$ is an order of linear quotients on the minimal generators of I^u and $g_{t+1} < \cdots < g_r$ is an order of linear quotients on the minimal generators of I_u, then

$$f_1 < \cdots < f_t < g_{t+1} < \cdots < g_r$$

is an order of linear quotients on the minimal generators of I.

$$\mathrm{set}_I(f_i) = \mathrm{set}_{I^u}(f_i), \quad \forall\ 1 \leq i \leq t$$

and

$$\mathrm{set}_I(g_i) = \mathrm{supp}\,(u) \bigcup \mathrm{set}_{I_u}(g_i) \ \ \forall\ t+1 \leq i \leq r$$

Also for any $t+1 \leq i \leq r$, $\mathrm{supp}\,(u) \bigcap \mathrm{set}_{I_u}(g_i) = \emptyset$. Thus

$$|\mathrm{set}_I(g_i)| = |\mathrm{set}_{I_u}(g_i)| + m$$

$$\beta_{i,j}(I) = \sum_{\deg(f_k)=j-i} \binom{|\mathrm{set}_I(f_k)|}{i} + \sum_{\deg(g_k)=j-i} \binom{|\mathrm{set}_I(g_k)|}{i}$$

Applying the equality

$$\binom{|\mathrm{set}_{I_u}(g_k)| + m}{i} = \sum_{\ell=0}^{m} \binom{m}{\ell} \binom{|\mathrm{set}_{I_u}(g_k)|}{i-\ell}$$

we have

$$\beta_{i,j}(I) = \beta_{i,j}(I^u) + \sum_{\ell=0}^{m} \binom{m}{\ell} \beta_{i-\ell,j-\ell}(I_u)$$

Corollary. Let I be a k-decomposable ideal with the shedding monomial u and $m = |\mathrm{supp}\,(u)|$. Then

- $\mathrm{pd}(I) = \max\{\mathrm{pd}(I^u), \mathrm{pd}(I_u) + m\}$, and
- $\mathrm{reg}(I) = \max\{\mathrm{reg}(I^u), \mathrm{reg}(I_u)\}$.

Theorem. Let Δ be a k-decomposable simplicial complex on the vertex set X with the shedding face σ. Then

- $\mathrm{reg}(R/I_\Delta) = \max\{\mathrm{reg}(R/I_{\Delta\setminus\sigma}), \mathrm{reg}(R/I_{\mathrm{lk}(\sigma)}) + |\sigma|\}$,
- $\mathrm{pd}(R/I_\Delta) = \max\{\mathrm{pd}(R/I_{\Delta\setminus\sigma}), \mathrm{pd}(R/I_{\mathrm{lk}(\sigma)})\}$,

where $I_{\Delta\setminus\sigma}$ and $I_{\mathrm{lk}(\sigma)}$ are Stanley-Reisner ideals of $\Delta\setminus\sigma$ and $\mathrm{lk}(\sigma)$ on the vertex sets X and $X\setminus\sigma$, respectively.

Corollary. Let Δ be a shellable simplicial complex with the shelling order $F_1 < \cdots < F_k$ and $\dim(\Delta) = d$. For any $1 \leq i \leq k$, let $\Delta_i = \langle F_1, \ldots, F_i \rangle$ and $\mathcal{R}(F_i) = \{x \in F_i : \; F_i \setminus \{x\} \in \Delta_{i-1}\}$. Then

$$\mathrm{reg}(R/I_\Delta) = \max\{|\mathcal{R}(F_1)|, \ldots, |\mathcal{R}(F_k)|\}.$$

Definition. Let $\mathcal{H}$ be a clutter. A vertex v of $\mathcal{H}$ is simplicial if for every two edges e_1 and e_2 of $\mathcal{H}$ that contain v, there is a third edge e_3 such that $e_3 \subseteq (e_1 \cup e_2) \setminus \{v\}$.

Definition. A clutter $\mathcal{H}$ is chordal if every minor of $\mathcal{H}$ has a simplicial vertex.

Corollary. Let $\mathcal{H}$ be a chordal clutter, $x \in V(\mathcal{H})$ be a simplicial vertex for $\mathcal{H}$ and $e = \{x, x_1, \ldots, x_d\}$ be an edge of $\mathcal{H}$ containing x. Then

- $\text{reg}(R/I(\mathcal{H})) =$

$$\max\{\text{reg}(R/I(\mathcal{H}')), \text{reg}(R/I(\mathcal{H}/\{x_1, \ldots, x_d\})) + d\}$$

  where
  $E(\mathcal{H}') = \{e \in E(\mathcal{H}) : \{x_1, \ldots, x_d\} \nsubseteq e\} \cup \{\{x_1, \ldots, x_d\}\}$.
- $\text{reg}(R/I(\mathcal{H})) \leq$

$$\max\{\sum_{i=1}^{d} \text{reg}(R/I(\mathcal{H} \setminus x_i)) + (d-1), \text{reg}(R/I(\mathcal{H}/\{x_1, \ldots, x_d\})) + d\}$$

For a graph G, let $J_m(G)$ be the ideal generated by all squarefree monomials u of degree m, such that $\operatorname{supp}(u)$ is an independent set of G.

Theorem. Let G be a chordal graph and x be a simplicial vertex of G. Set $I = J_m(G)$, $J = J_m(G \setminus x)$ and $K = J_{m-1}(G \setminus N_G[x])$. Then $I = J + xK$ is a 0-decomposable ideal. Moreover, if $I \neq 0$, then

(i) $\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i-1,j-1}(J) + \beta_{i,j-1}(K)$

(ii) If $J \neq 0$, then $\mathrm{pd}(I) = \max\{\mathrm{pd}(J) + 1, \mathrm{pd}(K)\}$

(iii) I has a m-linear resolution.

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k-decomposability and expansion

Definition. [Khosh-Ahang, Moradi] Let $\Delta = \langle F_1, \ldots, F_m \rangle$ be a simplicial complex with the vertex set $V(\Delta) = \{x_1, \ldots, x_n\}$ and $s_1, \ldots, s_n \in \mathbb{N}$ be arbitrary integers. We define the $(s_1, \ldots, s_n)$-expansion of Δ to be a simplicial complex with the vertex set $\{x_{11}, \ldots, x_{1s_1}, x_{21}, \ldots, x_{2s_2}, \ldots, x_{n1}, \ldots, x_{ns_n}\}$ and the facets

$$\{\{x_{i_1 r_1}, \ldots, x_{i_{k_i} r_{k_i}}\} \ : \ \{x_{i_1}, \ldots, x_{i_{k_i}}\} \in \mathcal{F}(\Delta), \ (r_1, \ldots, r_{k_i}) \in [s_{i_1}] \times \cdots \times [s_{i_{k_i}}]$$

We denote this simplicial complex by $\Delta^{(s_1, \ldots, s_n)}$

Theorem. [Moradi,Rahmati-Asghar] Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$ and $\alpha \in \mathbb{N}^n$.

$$\Delta \text{ is k-decomposable} \Longleftrightarrow \Delta^\alpha \text{ is k-decomposable}$$

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$$\Delta \text{ is k-decomposable} \iff \Delta^{\alpha} \text{ is k-decomposable}$$

Outline of proof: $\Rightarrow$ Let Δ be k-decomposable, X be the vertex set of Δ, $x_i \in X$ and

$$\Delta' = \Delta \cup \langle (F \setminus \{x_i\}) \cup \{x_i'\} : \ F \in \mathcal{F}(\Delta), x_i \in F \rangle$$

$$I_{\Delta'^\vee} = x_i' I_{\Delta^\vee} + x_i I_{(\mathrm{lk}_\Delta(x_i))^\vee}$$

Also $(I_{\Delta'^\vee})_{x_i'} = x_i I_{(\mathrm{lk}_\Delta(x_i))^\vee}$ and $(I_{\Delta'^\vee})^{x_i'} = x_i' I_{\Delta^\vee}$.

$\mathrm{lk}_\Delta(x_i)$ is k-decomposable

$$\Downarrow$$

$x_i' I_{\Delta^\vee}$ and $x_i I_{(\mathrm{lk}_\Delta(x_i))^\vee}$ are k-decomposable ideals

Also for any minimal generator $x_i x^{X \setminus F} \in (I_{\Delta'^\vee})_{x_i'}$,

$$(x_i' x^{X \setminus F} : x_i x^{X \setminus F}) = (x_i')$$

$\Leftarrow$ Let $\Delta' = \Delta \cup \langle (F \setminus \{x_i\}) \cup \{x_i'\} : \ F \in \mathcal{F}(\Delta), x_i \in F \rangle$ be k-decomposable.

If $\mathcal{F}(\Delta) = \{F\}$, then clearly it is k-decomposable.

Suppose that Δ has more than one facet and σ be a shedding face of Δ' and let $\mathrm{lk}_{\Delta'}\sigma$ and $\Delta' \backslash \sigma$ are k-decomposable. We have two cases:

Case 1. $x_i' \in \sigma$ or $x_i \in \sigma$. Then

$$\Delta = \Delta' \backslash \sigma \text{ and so } \Delta \text{ is k-decomposble}$$

Case 2. $x_i \notin \sigma$ and $x_i' \notin \sigma$.

$\mathrm{lk}_{\Delta'}\sigma$ and $\Delta' \backslash \sigma$ are, respectively, some expansions of $\mathrm{lk}_{\Delta}\sigma$ and $\Delta \backslash \sigma$

So by induction $\mathrm{lk}_{\Delta}\sigma$ and $\Delta \backslash \sigma$ are k-decomposable.

Theorem. [Moradi,Rahmati-Asghar] Let Δ be a k-decomposable simplicial complex on $[n]$ and $\alpha = (s_1, \ldots, s_n)$. Then

- $\mathrm{pd}(S^\alpha/I_{\Delta^\alpha}) = \mathrm{pd}(S/I_\Delta) + s_1 + \cdots + s_n - n$
- $\mathrm{depth}\,(S^\alpha/I_{\Delta^\alpha}) = \mathrm{depth}\,(S/I_\Delta)$

Theorem. [Moradi,Rahmati-Asghar] Let Δ be a simplicial complex on $[n]$ and $\alpha = (s_1, \ldots, s_n)$. Then

$$\mathrm{reg}(I_{\Delta^\alpha}) \leq \mathrm{reg}(I_\Delta) + r$$

where $r = |\{i : s_i > 1\}|$.

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Outline of proof. First by induction on s_i we show that

$$\mathrm{reg}(I_{\Delta^{(1,\ldots,1,s_i,1,\ldots,1)}}) \leq \mathrm{reg}(I_\Delta) + 1$$

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$$\mathrm{reg}(\mathrm{I}_{\Delta^{(1,\ldots,1,s_i,1,\ldots,1)}}) \leq \max\{\mathrm{reg}(\mathrm{I}_{\mathrm{lk}_{\Delta}(x_i)})+1, \mathrm{reg}(\mathrm{I}_{\Delta^{(1,\ldots,1,s_i-1,1,\ldots,1)}})\}.$$

Then from the equality

$$\Delta^{(s_1,\ldots,s_n)} = (\Delta^{(s_1,\ldots,s_{n-1},1)})^{(1,\ldots,1,s_n)}$$

we have

$$\mathrm{reg}(\mathrm{I}_{\Delta^{(s_1,\ldots,s_n)}}) \leq \mathrm{reg}(\mathrm{I}_{\Delta^{(s_1,\ldots,s_{n-1},1)}}) + 1$$

and one can get the result by induction on n.

Thanks!