Semistar operations on graded integral domains

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Assumptions

Let D be a commutative **integral domain** **not necessarily Noetherian**, with non-zero identity and **with quotient field** K.

Star operations

The star operations are defined by axioms selected by Krull among properties satisfy by some classical operations, such as the v-operation, the t-operation and the completion (or the integral closure of ideals).

Star operations

Star operations have shown to be an essential tool in multiplicative ideal theory, allowing a new approach for characterizing several classes of integral domains. For example,

An integrally closed domain D is a Prüfer domain if and only if $I^t = I$ for each nonzero ideal I of D (that is $t =$identity).

A domain D is a Krull domain if and only if $(II^{-1})^t = D$, for each nonzero ideal I of D.

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Semistar operations

Semistar operations were introduced in 1994 by Okabe and Matsuda in order to generalize the classical concept of star operation, as described in the book by Gilmer, and hence the related classical theory of ideal systems based on the works of W. Krull, E. Noether, H. Prüfer and P. Lorenzen from 1930's.

Semistar operations

Let $\overline{\mathcal{F}}(D)$ denote the set of all nonzero D-submodules of K. Let $\mathcal{F}(D)$ be the set of all nonzero fractional ideals of D. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of D. Obviously,

$$f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D).$$

A semistar operation on D is a map $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(D)$:

$\star_1$: $(xE)^\star = xE^\star$;

$\star_2$: $E \subseteq F$ implies that $E^\star \subseteq F^\star$;

$\star_3$: $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

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$\star_f$

Let $\star$ be a semistar operation on the domain D.

For every $E \in \overline{\mathcal{F}}(D)$, put

$$E^{\star_f} := \bigcup F^{\star},$$

where the union is taken over all finitely generated $F \subseteq E$. It is easy to see that $\star_f$ is a semistar operation on D, and $\star_f$ is called the semistar operation of finite type associated to $\star$. Note that $(\star_f)_f = \star_f$.

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Quasi-$\star$-ideals

We say that a nonzero ideal I of D is a quasi-$\star$-ideal of D if $I^\star \cap D = I$.

A quasi-$\star$-prime ideal of D is a prime ideal which is also a quasi-$\star$-ideal. The set of all quasi-$\star$-prime ideals is denoted by $\mathrm{QSpec}^\star(D)$.

An ideal I is called a quasi-$\star$-maximal ideal of D, if I is maximal in the set of all proper quasi-$\star$-ideals of D. The set of all quasi-$\star$-maximal ideals is denoted by $\mathrm{QMax}^\star(D)$.

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$\widetilde{\star}$

It has become standard to say that a semistar operation $\star$ is stable if $(E \cap F)^\star = E^\star \cap F^\star$ for all $E, F \in \overline{\mathcal{F}}(D)$.

Given a semistar operation $\star$ on D, it is possible to construct a semistar operation $\widetilde{\star}$, which is stable and of finite type defined as follows: for each $E \in \overline{\mathcal{F}}(D)$,

$$E^{\widetilde{\star}} := \bigcup \{(E :_K J) | J \subseteq D, J \in f(D), J^\star = D^\star\}.$$

It is well known that

$$E^{\widetilde{\star}} := \bigcap \{ED_P | P \in \mathrm{QMax}^{\star_f}(D)\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

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Classical examples

1. The map v_D defined by $E^{v_D} := (E^{-1})^{-1}$, with $E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}$.
2. $t_D := (v_D)_f$.
3. $w_D := \widetilde{v_D}$.
4. b_D-operation defined by $E^{b_D} := \bigcap \{EV | V \text{ is a valuation overring of } D\}$.
5. d_D the identity operation on D.

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An order on semistar operations

Definition: If $\star_1$ and $\star_2$ are semistar operations on D, one says that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\mathcal{F}}(D)$.

It is easy to see that for a given semistar operation $\star$ on D we have

$$\tilde{\star} \leq \star_f \leq \star.$$

If $\star$ is a semistar operation on D such that $D^\star = D$ then we have

$$d_D \leq \star \leq v_D.$$

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Graded integral domains

Let $D = \bigoplus_{\alpha\in\Gamma} D_\alpha$ be a graded (commutative) integral domain graded by an arbitrary grading torsionless monoid Γ, that is Γ is a commutative cancellative monoid (written additively). Let $\langle\Gamma\rangle = \{a - b | a, b \in \Gamma\}$, be the quotient group of Γ, which is a torsionfree abelian group.

Let H be the saturated multiplicative set of nonzero homogeneous elements of D. Then $D_H = \bigoplus_{\alpha\in\langle\Gamma\rangle}(D_H)_\alpha$, called the homogeneous quotient field of D, is a graded integral domain whose nonzero homogeneous elements are units.

A fractional ideal I of D is homogeneous if sI is an integral homogeneous ideal of D for some $s \in H$ (thus $I \subseteq D_H$).

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Homogeneous elements of $\mathrm{QSpec}^{\star}(D)$

The homogeneous elements of $\mathrm{QSpec}^{\star}(D)$, is denoted by h-$\mathrm{QSpec}^{\star}(D)$.

Lemma

Let $D = \bigoplus_{\alpha\in\Gamma} D_\alpha$ be a graded integral domain, $\star$ a finite type semistar operation on D, which sends homogeneous fractional ideals to homogeneous ones, and such that $D^{\star} \subsetneq D_H$. If I is a proper homogeneous quasi-$\star$-ideal of D, then I is contained in a proper homogeneous quasi-$\star$-prime ideal.

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h-QMax$^\star(D)$

Let h-QMax$^\star(D)$ denotes the set of ideals of D which are maximal in the set of all proper homogeneous quasi-$\star$-ideals of R. The above lemma shows that, if $D^\star \subsetneq D_H$ and $\star = \star_f$ sends homogeneous fractional ideals to homogeneous ones, then h-QMax$^{\star_f}(D)$ is nonempty.

For any such semistar operation, if I is a homogeneous ideal of D, we have $I^{\star_f} = D^\star$ if and only if $I \not\subseteq Q$ for each $Q \in h$-QMax$^{\star_f}(D)$.

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$\widetilde{\star}$ is homogeneous preserving

Proposition

Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on D such that $D^\star \subsetneq D_H$. Then, $\widetilde{\star}$ sends homogeneous fractional ideals to homogeneous ones. In particular h-QMax$^{\widetilde{\star}}(D) \neq \emptyset$, and $D^{\widetilde{\star}}$ is a homogeneous overring of D.

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A counterexample

(1) Let $R := k[X, Y]$ (k be a field).

(2) Set $M := (X, Y + 1)$ which is maximal non-homogeneous.

(3) Let (T, N) be a DVR dominating the local ring R_M.

(4) So that, $R_H \nsubseteq T$.

(5) Define a semistar operation $\star$ by $E^\star = ET \cap ER_H$.

(6) Then clearly $\star = \star_f$ and $R^\star \subsetneq R_H$.

(7) $\mathrm{QSpec}^{\star_f}(R) = \{M\} \cup \{P \in \mathrm{Spec}(R) | P \neq 0 \text{ and } P \cap H = \emptyset\}$.

(8) $\mathrm{QSpec}^{\widetilde{\star}}(R) = \{Q \in \mathrm{Spec}(R) | 0 \neq Q \subseteq M\} \cup \{P \in \mathrm{Spec}(R) | P \neq 0 \text{ and } P \cap H = \emptyset\}$.

(9) $h\text{-}\mathrm{QSpec}^{\star_f}(R) = h\text{-}\mathrm{QMax}^{\star_f}(R) = \emptyset$.

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A counterexample

(1) Let $R := k[X, Y]$ (k be a field).

(2) Set $M := (X, Y + 1)$ which is maximal non-homogeneous.

(3) Let (T, N) be a DVR dominating the local ring R_M.

(4) So that, $R_H \nsubseteq T$.

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Kronecker function ring

Definition

For $f \in D_H$, let $C(f)$ denote the fractional ideal of D generated by the homogeneous components of f. For a fractional ideal I of D with $I \subseteq D_H$, let $C(I) = \sum_{f \in I} C(f)$.

Theorem

Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$ be a graded integral domain with a unit of nonzero degree, $\star$ a semistar operation on D which sends homogeneous fractional ideals to homogeneous ones, and

$$Kr(D, \star) := \left\{ \frac{f}{g} \;\middle|\; \begin{array}{c} f, g \in D, g \neq 0, \text{ and there is } 0 \neq h \in D \\ \text{such that } C(f)C(h) \subseteq (C(g)C(h))^\star \end{array} \right\}.$$

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P$\star$MDs

Definition

An integral domain D called a Prüfer domain if for every nonzero finitely generated ideal I of D, $II^{-1} = D$.

Definition

Assume that $\star$ is a semistar operation on D. Then D is called a Prüfer $\star$-multiplication domain (for short a P$\star$MD) if for every nonzero finitely generated ideal I of D, $(II^{-1})^{\star_f} = D^\star$.

Note that P$\star$MD=P$\star_f$MD=P$\tilde{\star}$MD.

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Assume that $\star$ is a semistar operation on $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$. Then D is called a graded Prüfer $\star$-multiplication domain (for short a graded P$\star$MD) if for every nonzero finitely generated homogeneous ideal I of D, $(II^{-1})^{\star_f} = D^\star$.

When $\star = v$ we recover the classical notion of a graded PvMD.It is known that D is a graded PvMD if and only if D is a PvMD.Also when $\star = d$, a graded PdMD is called a graded Prüfer domain.

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Graded P$\star$MDs

Theorem

Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on D such that $D^\star \subsetneq D_H$. Then, the following statements are equivalent:

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(3) $D_{H \setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QMax}^{\tilde{\star}}(D)$.

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Theorem

Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$ be a graded integral domain with a unit of nonzero degree, and $\star$ be a semistar operation on D such that $D^\star \subsetneq D_H$. Then D is a graded P$\star$MD if and only if

$$(C(f)C(g))^{\widetilde{\star}} = C(fg)^{\widetilde{\star}}$$

for all $f, g \in D_H$.

Graded P⋆MDs

Let $N_\star(H) = \{f \in D | C(f)^\star = D^\star\}$. It is well-known that $N_\star(H)$ is a saturated multiplicative subset of D.

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(2) Every ideal of $D_{N_\star(H)}$ is extended from a homogeneous ideal of D.

(3) $D_{N_\star(H)}$ is a Prüfer domain.

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(5) $D_{N_\star(H)} = Kr(D, \widetilde{\star})$.

(6) $Kr(D, \widetilde{\star})$ is a quotient ring of D.

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Graded PvMDs

Corollary

Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$ be a graded integral domain with a unit of nonzero degree. Then, the following statements are equivalent:

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(4) $D_{N_v(H)}$ is a Bézout domain.

(5) $D_{N_v(H)} = Kr(D, w)$.

(6) $Kr(D, w)$ is a quotient ring of D.

(7) $Kr(D, w)$ is a flat D-module.

Graded PvMDs

Corollary

Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$ be a graded integral domain with a unit of nonzero degree. Then, the following statements are equivalent:

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THANKS FOR YOUR ATTENTION