Covering techniques in representation theory

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The talk is based on a joint work with H. Asashiba and R. Hafezi

Overview

The idea of representing a complex mathematical object by a simpler one is as old as mathematics itself. It is particularly useful in classification problems.

Covering theory is one of these ideas to present a technique for the computation of the indecomposable modules over a representation-finite algebra.

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Covering techniques in representation theory have become important after the work of Bongartz-Gabriel, Gabriel and Riedtmann.

- K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. **65** (1982) 331-378.
- P. Gabriel, *The universal cover of a representation-finite algebra*, in: Lecture Notes in Math., vol. **903**, Springer-Verlag, Berlin/New York, 1981, 68-105.
- C. Riedtmann, *Algebren, Darstellungskocher, Uberlagerungen und zuruck*, Comment. Math. Helv. **55** (1980) 199-224.

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Riedtmann introduce coverings of the Auslander-Reiten quiver Γ_Λ of a representation-finite algebra Λ.

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Riedtmann introduce coverings of the Auslander-Reiten quiver Γ_Λ of a representation-finite algebra Λ.

Bongartz and Gabriel developed this notion to provide concrete algorithms which enable us to construct the Auslander-Reiten quivers for plenty of algebras.

One of the most important results in this theory is the following theorem which is proved by Gabriel and then completed by Martinez and De le Peña:

Main Theorem

let $\mathcal{C}$ be a locally bounded $\mathbb{k}$-category over a field $\mathbb{k}$ and let a group G act freely on $\mathcal{C}$. Then $\mathcal{C}$ is locally representation-finite if and only if $\mathcal{C}/G$ is so.

- R. Martinez, J. A. De le Peña, *Automorphisms of representation-finite algebras*, Invent. Math. **72** (1983), 359-362.

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Asashiba brought this point of view to the derived equivalence classification problem of algebras. He investigated that when does a derived equivalence between categories $\mathcal{C}$ and $\mathcal{C}'$ yield a derived equivalence between orbit categories $\mathcal{C}/G$ and $\mathcal{C}'/H$.

Asashiba generalized the covering technique for an arbitrary $\mathbb{k}$-category to apply covering techniques to usual additive categories such as the homotopy category $\mathbb{K}(\text{Prj-}\mathcal{C})$ of projectives.

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Our Aim

Using this generalization, we plan to give a classification of algebras of finite Cohen-Macaulay type.

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- $s(\alpha)$ is the source of α
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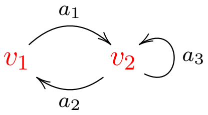

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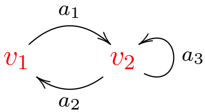

- A path of length $n \geq 1$ in a quiver $\mathcal{Q}$ is $\rho = \alpha_1 \cdots \alpha_n$ where $\alpha_i \in E$ and $t(\alpha_i) = s(\alpha_{i+1})$ for all $i \in \{1, \cdots, n-1\}$.

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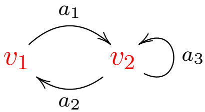

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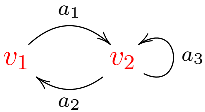

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Example

- $a_2 a_1 a_3$ is a path of length 3.
- v_1 and v_2 are paths of length 0.

Path Algebra

Let $\mathcal{Q}$ be a quiver and $\Bbbk$ a field. The path $\Bbbk$-algebra of the quiver $\mathcal{Q}$, denoted by $\Bbbk\mathcal{Q}$, is the algebra obtained as follows:

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The Jordan quiver 

- Basis as $\mathbb{k}$-vector space is $\{v, \alpha, \alpha^2, \alpha^3, \cdots\}$.
- Multiplication: $v\alpha^n = \alpha^n = \alpha^n v$.
- $\mathbb{k}\mathcal{Q} \cong \mathbb{k}[x]$.

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We denote by $\mathrm{rep}_{\Bbbk}(\mathcal{Q})$ the category of all finite dimensional representations of $\mathcal{Q}$.

Admissible ideal

An ideal I of $\Bbbk Q$ is called admissible, if there exists $n \in \mathbb{Z}$ such that $R_Q^n \subset I \subset R_Q^2$, where R_Q^n is the ideal of $\Bbbk Q$ generated, as a $\Bbbk$-vector space, by the set of all paths of length $\geq n$.

Let $\mathcal{Q}$ be a quiver and I be an admissible ideal of $\Bbbk\mathcal{Q}$. A representation $\mathcal{M} = (\mathcal{M}_v, \mathcal{M}_\alpha)$ of $\mathcal{Q}$ is called bound by I, if we have $\mathcal{M}_\alpha = 0$, for all relations $\alpha \in I$.

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We denote by $\mathrm{rep}_{\Bbbk}(\mathcal{Q}, I)$ the category of all finite dimensional representations of $\mathcal{Q}$ bound by I.

Theorem

Let $\mathcal{Q}$ be a finite connected quiver and $\Lambda = \Bbbk\mathcal{Q}/I$, where I is an admissible ideal of $\Bbbk\mathcal{Q}$. Then there exists a $\Bbbk$-linear equivalence of categories

$$F : \text{mod-}\Lambda \xrightarrow{\sim} \mathrm{rep}_{\Bbbk}(\mathcal{Q}, I).$$

Notations

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  - $\mathcal{C}$ is called a $\mathbb{k}$-category, if
    - $\mathcal{C}(x, y)$ is a $\mathbb{k}$-module
    - $\cdot : \mathcal{C}(y, z) \times \mathcal{C}(x, y) \longrightarrow \mathcal{C}(x, z)$ is $\mathbb{k}$-bilinear

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- G: a group

Locally Bounded Categories

Definition

- $\mathcal{C}$ is a spectroid if
  1. $x \neq y \Longrightarrow x \ncong y, \forall\, x, y \in \mathcal{C}$ ($\mathcal{C}$ is basic);
  2. $\mathcal{C}(x, x)$ is a local $\Bbbk$-algebra $\forall\, x \in \mathcal{C}$ ($\mathcal{C}$ is semiperfect);
  3. $\dim_{\Bbbk} \mathcal{C}(x, y) < \infty, \forall\, x, y \in \mathcal{C}$.

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  3. $\dim_{\Bbbk} \mathcal{C}(x,y) < \infty$, $\forall\, x, y \in \mathcal{C}$.
- A spectroid $\mathcal{C}$ is called locally bounded, if

$$\forall\, x \in \mathcal{C}, \{y \in \mathcal{C} \mid \mathcal{C}(x,y) \neq 0 \;\&\; \mathcal{C}(y,x) \neq 0\} \text{ is finite.}$$

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$ax := A(a)_x, \forall\ a \in G, x \in \mathcal{C}$.

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Trivial G-action

For every $\mathbb{k}$-category $\mathcal{C}$ and every group G, we set
$\Delta(\mathcal{C}) := (\mathcal{C}, 1)$, where

$$\begin{aligned} 1 : G &\longrightarrow \mathrm{Aut}(\mathcal{C}) \\ a &\mapsto \mathrm{id}_{\mathcal{C}} \end{aligned}$$

G-Actions

Let $\mathcal{C} = (\mathcal{C}, A)$ be a G-category.

- The G-action A is called free, if $ax \neq x$, for every $a \neq 1$ and $x \in \mathcal{C}$, i.e. the map surjective map

$$\begin{aligned} G &\longrightarrow Gx := \{ax \mid a \in G\} \\ a &\mapsto ax \end{aligned}$$

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- The G-action A is called locally bounded, if for every $x, y \in \mathcal{C}$,

$$\{a \in G \mid \mathcal{C}(ax, y) \neq 0\}$$

  is finite.

Galois G-Covering

$\mathcal{C}$, $\mathcal{B}$: Spectroids
$\mathcal{C} = (\mathcal{C}, A)$ with A: free, locally bounded
$F : \mathcal{C} \longrightarrow \mathcal{B}$: a $\Bbbk$-functor

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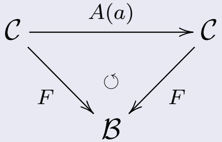
Strictly G-invariant

The $\mathbb{k}$-functor F is called strictly G-invariant, if $F = FA(a)$, for every $a \in G$, i.e.



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A strictly G-invariant F is called a Galois G-precovering, if

- $F^{-1}(Fx) = Gx$, i.e. the map

$$\begin{aligned} G &\longrightarrow F^{-1}(Fx) \\ a &\mapsto ax \end{aligned}$$

is bijection.

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- $F^{-1}(Fx) = Gx$,
- F induces $\Bbbk$-module isomorphisms

$$\bigoplus_{a\in G} \mathcal{C}(ax, y) \longrightarrow \mathcal{B}(Fx, Fy)$$
$$(f_a)_{a\in G} \mapsto \sum_{a\in G} F(f_a)$$

$$\bigoplus_{b\in G} \mathcal{C}(x, by) \longrightarrow \mathcal{B}(Fx, Fy)$$
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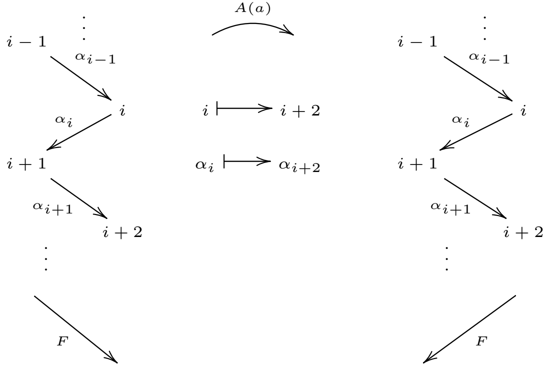
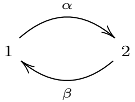
The $\mathbb{k}$-functor F is a Galois G-covering, if F is a Galois G-precovering and, in addition $F : \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{B})$ is serjective.

$\mathcal{C} := \Bbbk[\widetilde{\mathcal{Q}}]/\langle \alpha_{i+2}\alpha_{i+1}\alpha_i \rangle$, $G := \langle a \rangle$,



$\mathcal{B} := \Bbbk[\mathcal{Q}]/\langle \alpha\beta\alpha, \beta\alpha\beta \rangle$



$$F(i) := \begin{cases} 1 & i \notin 2\mathbb{Z} \\ 2 & i \in 2\mathbb{Z} \end{cases}, \quad F(\alpha_i) := \begin{cases} \alpha & i \notin 2\mathbb{Z} \\ \beta & i \in 2\mathbb{Z} \end{cases}$$

Orbit Category

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- $\forall u, v \in \mathrm{Obj}(\mathcal{C}/G)$,

$$(\mathcal{C}/G)(u,v) := \{(f_{yx})_{\substack{y\in v\\ x\in u}} \in \prod_{\substack{y\in v\\ x\in u}} \mathcal{C}(x,y) \mid af_{yx} = f_{ax,ay}, \quad \forall a \in G\}$$

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- $\forall f = (f_{yx}) : u \longrightarrow v, g = (g_{zy}) : v \longrightarrow w$ in $\mathcal{C}/G$,

$$gf := \left(\sum_{y\in v} g_{zy} f_{yx}\right)_{\substack{y\in v\\ x\in u}}$$

Canonical Functor

$\mathcal{C}$: Spectroid

$\mathcal{C} = (\mathcal{C}, A)$ with A: free, locally bounded

The canonical functor $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ is defined as follows:

$$\begin{array}{ccc} x & & P_x := Gx \\ \Big\downarrow f & \longmapsto & \Big\downarrow P_f := (\delta_{ab} af)_{\substack{by \in Gy \\ ax \in Gx}} \\ y & & P_y := Gy \end{array}$$

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Proposition

The canonical functor $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ is a Galois G-covering.

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- P is strictly G-invariant;
- for every strictly G-invariant functor $E : \mathcal{C} \longrightarrow \mathcal{B}$, $\exists!\ H : \mathcal{C}/G \longrightarrow \mathcal{B}$ s.t. $E = HP$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{E} & \mathcal{B} \\ {\scriptstyle P}\downarrow & \nearrow{\scriptstyle H} & \\ \mathcal{C}/G & & \end{array}$$

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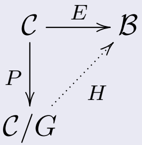

Thus, E is a Galois G-covering iff H is an isomorphism.

Mod-$\mathcal{B}$

$\mathcal{B}$: small $\Bbbk$-category

The category of right $\mathcal{B}$-modules, Mod-$\mathcal{B}$

- Objects: additive contravariant functors $\mathcal{B} \longrightarrow \text{Mod-}\Bbbk$
- Morphisms: natural transformations of functors
- Composition law: usual composition of natural transformations

Mod-$\mathcal{B}$

$\mathcal{B}$: small $\mathbb{k}$-category

The category of right $\mathcal{B}$-modules, Mod-$\mathcal{B}$

- Objects: additive contravariant functors $\mathcal{B} \longrightarrow \text{Mod-}\mathbb{k}$
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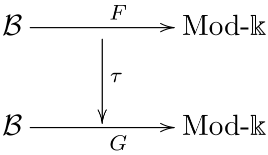

mod-$\mathcal{B}$

A $\mathcal{B}$-module M is called finitely generated, if
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The full subcategory of mod-$\mathcal{B}$ consisting of projective $\mathcal{B}$-modules is denoted by prj-$\mathcal{B}$.

G-action on Mod-$\mathcal{C}$

Let $\mathcal{C} = (\mathcal{C}, A)$ be a G-category.

Mod-$\mathcal{C} = (\text{Mod-}\mathcal{C}, \bar{A})$ turns out to be a G-category by defining:

$$\bar{A} : G \longrightarrow \text{Aut}(\text{Mod-}\mathcal{C}) \quad \text{as}$$

$\bar{A}_{(a)}(M) = M \circ A(a^{-1}), \forall a \in G, M \in \text{Mod-}\mathcal{C}$

${}^{a}M := \bar{A}_{(a)}(M)$

Pullup and pushdown functors

$\mathcal{C}$: a spectoid G-category, $P : \mathcal{C} \longrightarrow \mathcal{C}/G$
We have the functor

$$P^{\cdot} : \text{Mod-}\mathcal{C}/G \longrightarrow \text{Mod-}\mathcal{C}$$

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- **On Objects:** $M \in \text{Mod-}\mathcal{C}$, $u, v \in \text{Obj}(\mathcal{C}/G)$, $f : u \longrightarrow v$

$$\begin{array}{ccc} (P_{\bullet}M)_v : & = & \oplus_{y \in v} M(y) \\ \Big\downarrow (P_{\bullet}M)_{(f)} & & \Big\downarrow (M(f_{yx}))_{\substack{x \in u \\ y \in v}} \\ (P_{\bullet}M)_u : & = & \oplus_{x \in u} M(x) \end{array}$$

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- **On Morphisms:** $\alpha : M \longrightarrow M'$ in $\text{Mod-}\mathcal{C}$

$$\begin{array}{ccc} (P_{\bullet}M)_u & \xrightarrow{(P_{\bullet}\alpha)_u} & (P_{\bullet}M')_u \\ \| & & \| \\ \oplus_{x \in u} M(x) & \overset{\oplus_{x \in u}\alpha_x}{\dashrightarrow} & \oplus_{x \in u} M'(x) \end{array}$$

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- $P_{\cdot}$ preserves finitely generated, i.e.

$$P_{\cdot} : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}\mathcal{C}/G$$

Main Theorem

$\mathcal{C}$: a locally bounded spectroid

$\mathcal{C}$: a G-category with a free and locally bounded action

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A locally representation finite category is a locally bounded category $\mathcal{C}$ such that the number of $M \in$ ind-$\mathcal{C}$ satisfying $M(x) \neq 0$ is finite for each $x \in \mathcal{C}$.

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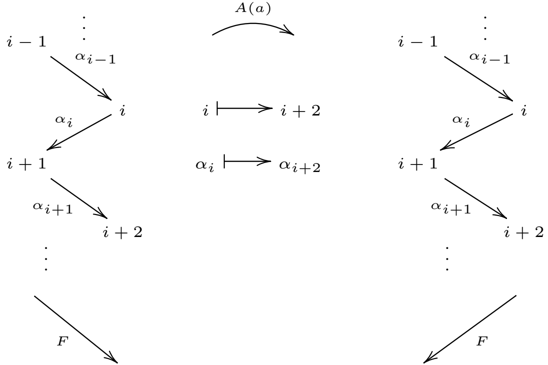
- $\mathcal{C}$: locally representation finite $\Longrightarrow P_{\bullet} :$ ind-$\mathcal{C} \longrightarrow$ ind-$\mathcal{C}/G$ is a Galois G-covering.
- $P_{\bullet}$ induces an isomorphism

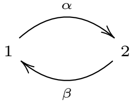
$$(\text{ind-}\mathcal{C})/G \simeq \text{ind-}(\mathcal{C}/G)$$

So, $\mathcal{C}$ is locally representation finite if and only if $\mathcal{C}/G$ is so.

$\mathcal{C} := \Bbbk[\widetilde{Q}]/\langle \alpha_{i+2}\alpha_{i+1}\alpha_i \rangle$, $G := \langle a \rangle$,



$\mathcal{B} := \Bbbk[Q]/\langle \alpha\beta\alpha, \beta\alpha\beta \rangle$



$$F(i) := \begin{cases} 1 & i \notin 2\mathbb{Z} \\ 2 & i \in 2\mathbb{Z} \end{cases}, \quad F(\alpha_i) := \begin{cases} \alpha & i \notin 2\mathbb{Z} \\ \beta & i \in 2\mathbb{Z} \end{cases}$$

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- $\mathbb{K}^{\mathrm{b}}(\text{prj-}R)$
  - It is not semiperfect.
  - If we construct the full subcategory of indecomposable objects, then we destroy additional structures like a structure of a triangulated category and the basic property.

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H. Asashiba generalized the covering technique to remove all these assumptions.

- H. Asashiba, *A generalization of Gabriels Galois covering functors and derived equivalences*, J. Algebra **334** (2011), 109-149.

G-invariants

$\mathcal{C}$: a skeletally small $\Bbbk$-category equipped with an action of a group G

Definition

A functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is called G-invariant, if $\exists\ \varphi := (\varphi_\alpha)_{\alpha \in G}$ of natural isomorphisms $\varphi_\alpha : F \longrightarrow FA_\alpha$ such that for every $\alpha, \beta \in G$, the following diagram is commutative

$$\begin{array}{ccc} F & \xrightarrow{\varphi_\alpha} & FA_\alpha \\ & {\scriptstyle \varphi_{\beta\alpha}} \searrow & \downarrow {\scriptstyle \varphi_\beta A_\alpha} \\ & & FA_{\beta\alpha} = FA_\beta A_\alpha. \end{array}$$

The family $\varphi := (\varphi_\alpha)_{\alpha \in G}$ is called an invariance adjuster of F.

G-COVERINGS

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Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a G-invariant functor.

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- F is called a G-precovering if for every $x, y \in \mathcal{C}$ the following two $\mathbb{k}$-homomorphisms are isomorphisms

$$F^{(1)}_{x,y} : \bigoplus_{\alpha \in G} \mathcal{C}(\alpha x, y) \longrightarrow \mathcal{C}'(Fx, Fy), \quad (f_\alpha)_{\alpha \in G} \mapsto \sum_{\alpha \in G} F(f_\alpha).\varphi_{\alpha,x};$$

$$F^{(2)}_{x,y} : \bigoplus_{\beta \in G} \mathcal{C}(x, \beta y) \longrightarrow \mathcal{C}'(Fx, Fy), \quad (f_\beta)_{\beta \in G} \mapsto \sum_{\beta \in G} \varphi_{\beta^{-1},\beta y}.F(f_\beta).$$

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- If, in addition, F is dense, then F is called a G-covering.

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$$\{(f_{\beta,\alpha})_{(\alpha,\beta)} \in \prod_{(\alpha,\beta)\in G\times G} \mathcal{C}(\alpha x, \beta y) \mid f \ {}_{\substack{\text{is row finite and column finite and} \\ f_{\gamma\beta,\gamma\alpha}=\gamma(f_{\beta,\alpha}),\forall\gamma\in G}}\}.$$

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- Composition low:
  For two composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{C}/G$, we set

$$gf := (\sum_{\gamma\in G} g_{\beta,\gamma} f_{\gamma,\alpha})_{(\alpha,\beta)\in G\times G}.$$

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- $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ is universal among G-invariant functors starting from $\mathcal{C}$.

The Pushdown functor P.

$\mathcal{C}$: a skeletally small $\mathbb{k}$-category with a G-action

The Pushdown functor $P_{\cdot}$

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- The canonical functor $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ induces a functor

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- [Asashiba's result] The pushdown $P_{\cdot} : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}(\mathcal{C}/G)$ is a G-precovering.

G-action on $\mathbb{K}(\text{prj-}\mathcal{C})$

- The G-action on Mod-$\mathcal{C}$ can be canonically extended to the G-action on $\mathbb{K}(\text{prj-}\mathcal{C})$, resp. $\mathbb{K}^{\rm b}(\text{prj-}\mathcal{C})$.

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That is, for every complex $X := (X^i, d^i)_{i\in\mathbb{Z}}$ and every $\alpha \in G$, ${}^{\alpha}X := ({}^{\alpha}X^i, {}^{\alpha}d^i)_{i\in\mathbb{Z}}$.

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- [Asashiba] The pushdown functor $P_{\cdot} : \mathbb{K}^{\mathrm{b}}(\text{prj-}\mathcal{C}) \longrightarrow \mathbb{K}^{\mathrm{b}}(\text{prj-}(\mathcal{C}/G))$ is a G-precovering.

Totally acyclic complexes

- A complex $\mathbf{X}$ in $\mathbb{C}(\text{prj-}\mathcal{C})$ is called totally acyclic of projectives if for every projective object $P \in \text{prj-}\mathcal{C}$, the induced complexes $\text{Hom}_{\mathcal{C}}(\mathbf{X}, P)$ and $\text{Hom}_{\mathcal{C}}(P, \mathbf{X})$ of abelian groups are acyclic.

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- The full subcategory of $\mathbb{K}(\text{prj-}\mathcal{C})$ consisting of totally acyclic complexes of projective is denoted by $\mathbb{K}_{\text{tac}}(\text{prj-}\mathcal{C})$.

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Proposition

The pushdown functor $P_{.} : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}(\mathcal{C}/G)$ induces a functor

$$P_{.} : \mathbb{K}_{\text{tac}}(\text{prj-}\mathcal{C}) \longrightarrow \mathbb{K}_{\text{tac}}(\text{prj-}(\mathcal{C}/G)).$$

$\mathcal{G}p$-$\mathcal{C}$

An object G in mod-$\mathcal{C}$ is called Gorenstein projective if G is a syzygy of a totally acyclic complex of finitely generated projective $\mathcal{C}$-modules, i.e.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & C^{-1} & \longrightarrow & C^{0} & & \longrightarrow & & C^{1} & \longrightarrow & C^{2} & \longrightarrow & \cdots \\ & & & & & \searrow & & \nearrow & & & & & \\ & & & & & & G & & & & & & \end{array}$$

$\mathcal{G}p$-$\mathcal{C}$

An object G in mod-$\mathcal{C}$ is called Gorenstein projective if G is a syzygy of a totally acyclic complex of finitely generated projective $\mathcal{C}$-modules, i.e.

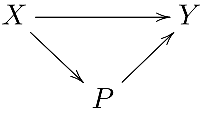
$$\begin{array}{ccccccccc} \cdots \longrightarrow & C^{-1} & \longrightarrow & C^{0} & & \longrightarrow & & C^{1} & \longrightarrow C^{2} \longrightarrow \cdots \\ & & & & \searrow & & \nearrow & & \\ & & & & & G & & & \end{array}$$

We denote the full subcategory of mod-$\mathcal{C}$ consisting of all Gorenstein projective objects in mod-$\mathcal{C}$ by $\mathcal{G}p$-$\mathcal{C}$.

The stable category $\underline{\mathcal{G}p}$-$\mathcal{C}$

Let $X, Y \in \mathcal{G}p$-$\mathcal{C}$.

$\mathcal{P}(X, Y)$: the subgroup of morphisms belong to $\mathcal{G}p$-$\mathcal{C}(X, Y)$ such that factor through a projective $P \in$ prj-$\mathcal{C}$.



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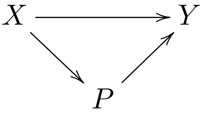

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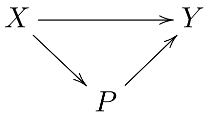

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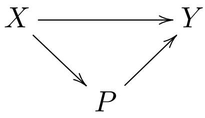

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- $\text{Obj}(\underline{\mathcal{G}p}\text{-}\mathcal{C}) = \text{Obj}(\mathcal{G}p\text{-}\mathcal{C})$;
- $\underline{\mathcal{G}p}\text{-}\mathcal{C}(X, Y) = \mathcal{G}p\text{-}\mathcal{C}(X, Y)/\mathcal{P}(X, Y)$.

One can easily show that there is a triangle equivalence

$$\underline{\mathcal{G}p}\text{-}\mathcal{C} \simeq \mathbb{K}_{\text{tac}}(\text{prj-}\mathcal{C}),$$

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THEOREM

Let $\mathcal{C}$ be a small G-category. Then the pushdown functor

$$P_{\bullet} : \underline{\mathcal{G}p}\text{-}\mathcal{C} \longrightarrow \underline{\mathcal{G}p}\text{-}(\mathcal{C}/G)$$

is a G-precovering.

Locally support finite

$\mathcal{C}$: a $\mathbb{k}$-category
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We denote by Supp-M the support of M, i.e., the full subcategory of $\mathcal{C}$ consisting of all objects x of $\mathcal{C}$ such that $M(x) \neq 0$.

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- A locally bounded $\Bbbk$-category $\mathcal{C}$ is called locally support finite if for every $x \in \mathcal{C}$, $\mathcal{C}_x$ is finite.

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A locally Cohen-Macaulay finite category is a locally support finite category $\mathcal{B}$ such that the number of $M \in \text{ind-}(\underline{\mathcal{G}p}\text{-}\mathcal{B})$ satisfying $M(x) \neq 0$ is finite for each $x \in \mathcal{B}$.

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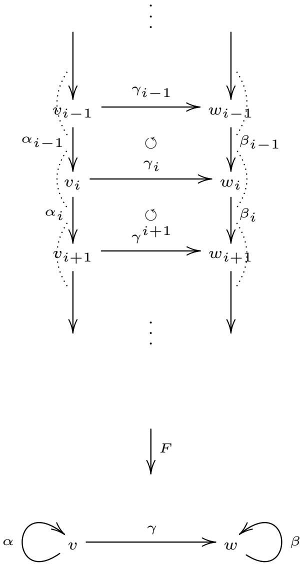
- $\mathcal{C}$ is locally Cohen-Macaulay finite if and only if $\mathcal{C}/G$ is so.



$$\begin{cases} F(v_i) = v \\ F(w_i) = w \end{cases}, \begin{cases} F(\alpha_i) = \alpha \\ F(\gamma_i) = \gamma \\ F(\beta_i) = \beta \end{cases}$$

$$I = \langle \alpha^2, \beta^2, \alpha\gamma - \gamma\beta \rangle$$

Acknowledgement

Thank you all for your attention!