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DYNAMICAL SYSTEM AND CROSSED PRODUCTS

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Dynamical systems:

Let A be a C^*-algebra and G be locally compact group. Let

$$Aut(A) = \{\varphi : A \to A \mid 1-1,\ onto,\ * - homo.\}.$$

Definition: A C^*-dynamical system is a triple (A, G, α) consisting of a C^*-algebra A, a locally compact, Hausdorff group G and a strongly continuous homomorphism $\alpha : G \to Aut(A)$.

This means that $g \to \alpha(g)(a)$ is continuous, and we have

$$\alpha(g_1 g_2)(a) = \alpha(g_1)[\alpha(g_2)(a)]$$
$$\alpha(g)^{-1}(a) = \alpha(g^{-1})(a)$$
$$\alpha(g)(a)^* = \alpha(g)(a^*).$$

Example: Let G be a locally compact group. Let

$$\pi_l : G \to Aut(C_\circ(G)) \quad s.t \quad \pi_l(x)(f)(y) = f(x^{-1}y)$$

for each $x, y \in G$. Obviously $(C_\circ(G), \pi_l, G)$ is a dynamical system.

More generally, let X locally compact space and G be locally compact group with the action $G \times X \to X$, let $\varphi_g \in homeo(X)$ be defined by $\varphi_g(x) = g.x$, for $g \in G$ and $x \in X$, then for

$$\pi_l : G \to Aut(C_\circ(X)), \qquad \pi_l(g)(f)(y) = f(g^{-1}.y),$$

$(C_\circ(X), \pi_l, G)$ is a dynamical system.

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Conversely, let X locally compact space, G be locally compact group and $\alpha : G \to Aut(C_\circ(X))$ be an action. For each $g \in G$ and $x \in X$, we have a $*$-homomorphism

$$C_\circ(X) \xrightarrow{\alpha(g)} C_\circ(X) \xrightarrow{\Lambda_x} \mathbb{C}$$
$$f \to \alpha(g)(f) \to \alpha(g)(f)(x).$$

There is $h(g) \in Homeo(X)$ such that $\alpha(g)(f)(x) = f(h(g)(x))$ and

$$\begin{aligned}\alpha(g_1g_2)(f)[x] &= f[h(g_1g_2)(x)]\\ &= \alpha(g_1)(\alpha(g_2)(f))[x] = \alpha(g_2)(f)[h(g_1)(x)]\\ &= f[h(g_2)(h(g_1)(x))].\end{aligned}$$

Thus we define the action of G on X by $g.x := h(g^{-1})(x)$ and we have $\alpha(g)(f)(x) = f(g^{-1}.x)$ for each $g \in G$ and $x \in X$. Now, we show that

$$\begin{aligned}&G \times X \to X\\ &(g, x) \to g.x = h(g^{-1})(x)\end{aligned}$$

is continuous. Let $O_{g.x}$ be an open subset of X such that $g.x \in O_{g.x}$. By Urysohn Lemma, there is $f \in C_c(X)^+$ such that $g.x \prec f \prec O_{g.x}$. As $1 = f(g.x) = \alpha(g^{-1})(f)(x)$ and α is sot-continuous, for $\epsilon = 1/2$ there is an open subset $O_x \subseteq X$ such that $x \in O_x$ and for each $y \in O_x$ we have $\|\alpha(g^{-1})(f)(x) - \alpha(g^{-1})(f)(y)\| < \epsilon$. On the other hand, α is sot-continuous, thus there is an open subset $O_g \subseteq G$ such that, for each $g' \in O_g$, we have $\|\alpha(g^{-1})(f) - \alpha(g'^{-1})(f)\| < \epsilon$. Therefore, for each $g' \in O_g$ and $y \in O_x$ we have

$$|f(g.x) - f(g'.y)| = \|\alpha(g^{-1})(f)(x) - \alpha(g'^{-1})(f)(y)\| < 1.$$

This means that $g'.y \in O_{g.x}$ and $O_g.O_x \subseteq O_{g.x}$. Thus the action of G on X is continuous.

Representation on groups: Let G be a locally compact group and $\pi : G \to U(H_\pi)$ be sot-continuous $*$-representation. That means, $g \to \pi(g)h$ is continuous and

$$\pi(g_1g_2)[h] = \pi(g_1)[\pi(g_2)h]$$
$$\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1}).$$

For π, there is an extension

$$\pi : C_c(G) \to B(H_\pi) \quad s.t \quad \pi(f) = \int_G f(x)\pi(x)dx$$

such that dx is left Haar measure of G. As

$$\|\pi(f)\| = \|\int_G f(x)\pi(x)dx\| \leq \int_G |f(x)|\|\pi(x)\|dx = \|f\|_1,$$

there is a lift of π on $L^1(G)$,

$$\pi : L^1(G) \to B(H_\pi) \quad s.t \quad \pi(f) = \int_G f(x)\pi(x)dx.$$

Consider the universal representation

$$\pi_u := \oplus_\pi \pi : L^1(G) \to \oplus_\pi B(H_\pi) \subseteq B(H_u := \oplus_\pi H_\pi)$$

defined by

$$\pi_u(f) = (\pi(f))_\pi = \begin{bmatrix} \ddots & & & & 0 \\ & \int_G f(x)\pi(x)dx & & & \\ & & \ddots & & \\ & & & \int_G f(x)\pi'(x)dx & \\ 0 & & & & \ddots \end{bmatrix}.$$

We define

$$C^*(G) = \overline{C_c(G)}^{\|.\|} \subseteq B(H_u).$$

We want $C^*(G)$ to be a C^*-algebra, so we give its product and involution.

Product: Let $f, g \in C^*(G)$, we would like to have $\pi(f.g) = \pi(f)\pi(g)$. We have

$$\begin{aligned}
\pi(f.g) &= \int_G f.g(x)\pi(x)dx \\
&= \pi(f)\pi(g) = \int_G f(y)\pi(y)dy \int_G g(x)\pi(x)dx \\
&= \int_G \int_G f(y)g(x)\pi(yx)dxdy = \int_G \int_G f(y)g(y^{-1}x)\pi(x)dxdy \\
&= \int_G [\int_G f(y)g(y^{-1}x)dy]\pi(x)dx.
\end{aligned}$$

Thus we define the convolution by

$$f * g(x) := f.g(x) = \int_G f(y)g(y^{-1}x)dy.$$

Involution: We would like to have $\pi(f^*) = \pi(f)^*$, that is,

$$\begin{aligned}\pi(f^*) &= \int_G f^*(x)\pi(x)dx \\ &= \pi(f)^* = [\int_G f(x)\pi(x)dx]^* = \int_G \overline{f(x)}\pi(x^{-1})dx \\ &= \int_G \overline{f(x^{-1})}\Delta(x^{-1})\pi(x)dx\end{aligned}$$

where Δ is the modular function of the left Haar measure of G. Thus for each $x \in G$, we define

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}).$$

Crossed Products:

Let A be a C^*-algebra and G a locally compact group. Let $\alpha : G \to Aut(A)$ be an action. The representations

$$\rho : A \to B(H_\rho)$$

and

$$U : G \to U(H_\rho)$$

are called α-covariant representations, if for each $a \in A$ and $x \in G$,

$$\rho(\alpha(x)(a)) = U(x)\rho(a)U(x^{-1}).$$

We define

$$\begin{aligned}\rho \rtimes U :& A \otimes C_c(G) \to B(H_\rho) \\ & a \otimes f \to \rho(a)U(f) = \rho(a)\int_G f(x)U(x)dx = \int_G \rho(f(x)a)U(x)dx.\end{aligned}$$

As $A \otimes C_c(G) \subseteq C_c(G, A)$, then we may extend $\rho \rtimes U$ to

$$\begin{aligned}\rho \rtimes U :& C_c(G, A) \to B(H_\rho) \\ & F \to \rho \rtimes U(F) = \int_G \rho(F(x))U(x)dx.\end{aligned}$$

Example: Consider $\alpha = \pi_l : G \to Aut(C_\circ(G))$ and representations

$$\rho : C_\circ(G) \to B(L^2(G)) \quad s.t \quad \rho(f)(g) = fg$$

and

$$U = \pi_l : G \to B(L^2(G)) \quad s.t \quad \pi_l(x)(g)(y) = g(x^{-1}y)$$

for $g \in L^2(G)$ and $x, y \in G$. We have

$$\rho[\pi_l(x)(f)](g)(y) = [\pi_l(x)(f)g](y) = \pi_l(x)(f)(y)g(y) = f(x^{-1}y)g(y).$$

On the other hand,

$$\begin{aligned}[U(x)\rho(f)U(x^{-1})](g)(y) &= U(x)[\rho(f)\pi_l(x^{-1})(g)](y) = [\rho(f)\pi_l(x^{-1})(g)](x^{-1}y) \\ &= [f\pi_l(x^{-1})(g)](x^{-1}y) = f(x^{-1}y)\pi_l(x^{-1})(g)(x^{-1}y) \\ &= f(x^{-1}y)g(y).\end{aligned}$$

That means,

$$\rho(\pi_l(x)(f))(g) = U(x)\rho(f)U(x^{-1})(g)$$

and (ρ, U) is α-covariant pair.

Question: Does there exist an α-covariant pair for each dynamical system (A, G, α)?

Let $A \subseteq B(H)$. As the action $\alpha : G \to Aut(A)$ is sot-continuous, we have the embedding

$$\begin{aligned}\rho : A &\to C_b(G, A) \\ a &\to \rho(a)(x) := \alpha(x^{-1})(a).\end{aligned}$$

On the other hand,

$$\begin{aligned}C_b(G, A) &\subseteq M(C_0(G, A)) = M(C_0(G) \otimes_{min} A) \\ &\subseteq B(L^2(G) \otimes_2 H) = B(L^2(G, H))\end{aligned}$$

where $M(C_0(G) \otimes_{min} A)$ is the multiplier algebra of $C_0(G) \otimes_{min} A$. We define

$$\rho : A \to B(L^2(G, H)) \quad s.t \quad \rho(a)(F)(x) := \alpha(x^{-1})(a)(F(x))$$

and

$$U = \pi_l : G \to U(L^2(G, H)) \quad s.t \quad U(x)(F)(y) := F(x^{-1}y).$$

In this case,

$$\rho(\alpha(x)(a))[F](y) = \alpha(y^{-1}x)(a)[F(y)].$$

On the other hand,

$$\begin{aligned}U(x)\rho(a)U(x^{-1})[F](y) &= \rho(a)[U(x^{-1})F](x^{-1}y) \\ &= \alpha(y^{-1}x)(a)[U(x^{-1})F](x^{-1}y) \\ &= \alpha(y^{-1}x)(a)[F(y)].\end{aligned}$$

This means that (ρ, U) is an α-covariant pair. For the universal representation

$$\oplus_{(\rho,U)}\ \rho \rtimes U : C_c(G, A) \to \oplus_{(\rho,U)} B(H_\rho) \subseteq B(\oplus_{(\rho,U)} H_\rho)$$

defined by

$$f \hookrightarrow \begin{bmatrix} \ddots & & 0 \\ & \int_G \rho(f(x))U(x)dx & \\ & \ddots & \\ & \int_G \rho'(f(x))U'(x)dx & \\ 0 & \ddots & \end{bmatrix},$$

we define the crossed product

$$A \rtimes_\alpha G := \overline{C_c(G, A)}^{\|\cdot\|} \subseteq B(\oplus_{(\rho,U)} H_\rho).$$

Same as for $C^*(G)$, we would like $A \rtimes_\alpha G$ to become a C^*-algebra. Thus we should find its product and involution.

Product: Let $f, g \in C_c(G, A)$. For $f.g$ in $A \rtimes_\alpha G$,

$$\begin{aligned} \rho \rtimes U(f.g) &= \int_G \rho((f.g)(x))U(x)dx \\ &= \rho \rtimes U(f)\ \rho \rtimes U(g) \\ &= \int_G \rho(f(y))U(y)dy \int_G \rho(g(x))U(x)dx \\ &= \int_G \int_G \rho(f(y))U(y)\rho(g(x))U(x)dxdy \\ &= \int_G \int_G \rho(f(y))\rho[\alpha(y)(g(x))]U(y)U(x)dxdy \\ &= \int_G \int_G \rho[f(y)\alpha(y)(g(x))]U(yx)dxdy \\ &= \int_G \int_G \rho[f(y)\alpha(y)(g(y^{-1}x))]U(x)dxdy \\ &= \int_G \rho[\int_G f(y)\alpha(y)(g(y^{-1}x))dy]U(x)dx. \end{aligned}$$

Thus we define

$$f *_\alpha g(x) := f.g(x) = \int_G f(y)\alpha(y)(g(y^{-1}x))dy.$$

Involution: We would like $\rho \rtimes U(f^*) = [\rho \rtimes U(f)]^*$, for each α-covariant pair (ρ, U), that is,

$$\begin{aligned}\rho \rtimes U(f^*) &= \int_G \rho(f^*(x))U(x)dx \\ &= [\rho \rtimes U(f)]^* = [\int_G \rho(f(x))U(x)dx]^* \\ &= \int_G U(x)^* \rho(f(x))^* dx = \int_G U(x^{-1})\rho(f(x)^*)dx \\ &= \int_G \rho[\alpha(x^{-1})(f(x))^*]U(x^{-1})dx \\ &= \int_G \rho[\alpha(x)(f(x^{-1}))^* \Delta(x^{-1})]U(x)dx.\end{aligned}$$

Thus for each $f \in A \rtimes_\alpha G$, we define

$$f^*(x) = \alpha(x)(f(x^{-1})^*)\Delta(x^{-1}).$$

This means that, for α-covariant pair (ρ, U), we have a $*$-representation

$$\rho \rtimes U : A \rtimes_\alpha G \to B(H_\rho)$$

such that

$$\rho \rtimes U(f) = \int_G \rho(f(x))U(x)dx.$$

□

Now we want to characterize all representations of $A \rtimes_\alpha G$.

As an example, let $A = \mathbb{C}$, and $G = \mathbb{R}$ be the real line. Here $\alpha = id$ and $(\mathbb{C}, \mathbb{R}, id)$ is a dynamical system. Let

$$\rho : \mathbb{C} \to B(H_\rho) \quad \text{and} \quad U : \mathbb{R} \to U(H_\rho)$$

be id-covariant representations. As $\rho = I$, for each unitary representation U of $G = \mathbb{R}$, (ρ, U) is a id-covariant pair. Thus

$$A \rtimes_\alpha G = \overline{C_c(\mathbb{R}, \mathbb{C})}^{\|\cdot\|} = C^*(\mathbb{R}) \cong C_\circ(\hat{\mathbb{R}}) \cong C_\circ(\mathbb{R})$$

which is not a unital C^*-algebra. In this case, $A = \mathbb{C}I$ can not be embedded into $A \rtimes_\alpha G \cong C_\circ(\mathbb{R})$.

In general one could embed A and G into the multiplier algebra of $A \rtimes_\alpha G$. As each element $f \in C_c(G, A)$ in $A \rtimes_\alpha G$, given all values $\rho \rtimes U(f)$ for α-covariant pairs (ρ, U), we define

$$i_A : A \to M(A \rtimes_\alpha G)$$

such that

$$\begin{aligned} i_A(a)(f) &= i_A(a)[(\rho \rtimes U(f))_{(\rho,U)}] \\ &:= [\rho(a)\rho \rtimes U(f)]_{(\rho,U)} \\ &= [\rho(a) \int_G \rho(f(x))U(x)dx)]_{(\rho,U)} \\ &= [\int_G \rho(af(x))U(x)dx)]_{(\rho,U)}, \end{aligned}$$

and

$$i_A(a)(f)(x) = af(x).$$

Define

$$i_G : G \to M(A \rtimes_\alpha G)$$

such that

$$\begin{aligned} i_G(x)(f) &= i_G(x)[(\rho \rtimes U(f))_{(\rho,U)}] \\ &:= [U(x)\rho \rtimes U(f)]_{(\rho,U)} \\ &= [U(x) \int_G \rho(f(y))U(y)dy)]_{(\rho,U)} \\ &= [\int_G \rho[\alpha(x)(f(y))]U(xy)dy)]_{(\rho,U)} \\ &= [\int_G \rho[\alpha(x)(f(x^{-1}y))]U(y)dy)]_{(\rho,U)}, \end{aligned}$$

and

$$i_G(x)(f)(y) = \alpha(x)(f(x^{-1}y)).$$

Now let

$$\sigma : A \rtimes_\alpha G \to B(H_\sigma)$$

be a non-degenerate representation. Define

$$\rho_\sigma : A \to B(H_\sigma)$$

such that

$$\begin{aligned} \rho_\sigma(a)(\sigma[(\rho \rtimes U(f))_{(\rho,U)}]h) &:= \sigma[(\rho(a)\ \rho \rtimes U(f))_{(\rho,U)}]h \\ &= \sigma[(\rho \rtimes U(af))_{(\rho,U)}]h \\ &= \sigma[\rho \rtimes U(i_A(a)f))_{(\rho,U)}]h \end{aligned}$$

and define

$$U_\sigma : G \to U(H_\sigma)$$

such that

$$\begin{aligned} U_\sigma(x)(\sigma[\rho \rtimes U(f))_{(\rho,U)}]h) &:= \sigma[U(x)\ \rho \rtimes U(f))_{(\rho,U)}]h \\ &= \sigma[(\rho \rtimes U(i_G(x)f))_{(\rho,U)}]h. \end{aligned}$$

Thus

$$\begin{aligned} \rho_\sigma(\alpha(x)(a))[\sigma[(\rho \rtimes U(f))_{(\rho,U)}]h] &= \sigma[(\rho(\alpha(x)(a))\ \rho \rtimes U(f))_{(\rho,U)}]h \\ &= \sigma[(U(x)\rho(a)U(x^{-1})\ \rho \rtimes U(f))_{(\rho,U)}]h \\ &= U_\sigma(x)\rho_\sigma(a)U_\sigma(x^{-1})[\sigma[(\rho \rtimes U(f))_{(\rho,U)}]h]. \end{aligned}$$

This means that,

$$\rho_\sigma(\alpha(x)(a)) = U_\sigma(x)\rho_\sigma(a)U_\sigma(x^{-1})$$

and (ρ_σ, U_σ) is an α-covariant pair. We have

$$\begin{aligned} \sigma[\rho \rtimes U(f))_{(\rho,U)}](\sigma[\rho \rtimes U(g))_{(\rho,U)}]h) &= \sigma[(\rho \rtimes U(f)\rho \rtimes U(g))_{(\rho,U)}]h \\ &= \sigma[(\int_G \rho(f(x))U(x)dx\ \rho \rtimes U(g))_{(\rho,U)}]h \\ &= \int_G \rho_\sigma(f(x))U_\sigma(x)dx\ \sigma[\rho \rtimes U(g))_{(\rho,U)}]h. \end{aligned}$$

Thus

$$\sigma[f] = \sigma[(\rho \rtimes U(f))_{(\rho,U)}] = \int_G \rho_\sigma(f(x))U_\sigma(x)dx$$

and

$$\sigma = \rho_\sigma \rtimes U_\sigma.$$

Example: (i) Let $G = \mathbb{Z}_2 = \{0, 1\}$ and A be any non-degenerate C^*-subalgebra of $B(H)$. Let $\alpha : \mathbb{Z}_2 \to Aut(A)$ be an action. Since $\alpha(0) = I$, α can be characterized by $\alpha(1)$. Let (ρ, U) be the α-covariant pair given by

$$\begin{aligned} \rho :&A \to B(\ell^2(\mathbb{Z}) \otimes_2 H) = B(\mathbb{C}^2 \otimes_2 H) \cong B(H^2) \\ &\rho(a)(F)(x) = \alpha(x^{-1}(a))(F(x)) \end{aligned}$$

and left regular representation

$$U = \pi_l : G \to B(\ell^2(\mathbb{Z}) \otimes_2 H) \cong B(H^2).$$

Let

$$F \in C_c(\mathbb{Z}_2, A) = C(\mathbb{Z}_2, A) = C(\mathbb{Z}_2) \otimes A = \ell_2^\infty \otimes A \cong \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

Then $F = a \otimes \delta_0 + b \otimes \delta_1$, and

$$\rho \rtimes U(a \otimes \delta_0 + b \otimes \delta_1) = \rho(a)U(0) + \rho(b)U(1) = \rho(a) + \rho(b)U(1).$$

As $\rho(a) \in B(H^2)$, $\rho(a) = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ such that $\{T_i\}_{i=1}^4 \subseteq B(H)$. For each $h \in H$, there is $h_1, h_2 \in H$ such that

$$\rho(a) \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

or

$$\rho(a)(h \otimes \delta_0) = h_1 \otimes \delta_0 + h_2 \otimes \delta_1.$$

We have

$$h_1 = \rho(a)(h \otimes \delta_0)(0) = [\alpha(0)(a)h]\delta_0(0) = ah,$$

and

$$h_2 = \rho(a)(h \otimes \delta_0)(1) = [\alpha(1)(a)h]\delta_0(1) = 0.$$

Thus $T_1 = a$ and $T_3 = 0$. Similarly, for each $g \in H$, there are $g_1, g_2 \in H$ such that

$$\rho(a) \begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

or

$$\rho(a)(g \otimes \delta_1) = g_1 \otimes \delta_0 + g_2 \otimes \delta_1.$$

We have

$$g_1 = \rho(a)(g \otimes \delta_1)(0) = [\alpha(0)(a)h]\delta_1(0) = 0,$$

and

$$g_2 = \rho(a)(g \otimes \delta_1)(1) = [\alpha(1)(a)g]\delta_1(1) = \alpha(1)(a)g.$$

Thus $T_2 = 0$ and $T_4 = \alpha(1)(a)$, and

$$\rho(a) = \begin{bmatrix} a & 0 \\ 0 & \alpha(1)(a) \end{bmatrix}.$$

Similarly,

$$U(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus

$$\rho \times U : C_c(\mathbb{Z}_2, A) = C(\mathbb{Z}_2, A) = \ell_2^\infty \otimes A \to B(H^2)$$

is given by

$$\rho \rtimes U(a \otimes \delta_0 + b \otimes \delta_1) = \rho(a) + \rho(b)U(1) = \begin{bmatrix} a & b \\ \alpha(1)(b) & \alpha(1)(a) \end{bmatrix}.$$

As $\rho \times U$ is injective,

$$A \rtimes_\alpha \mathbb{Z}_2 \cong \{ \begin{bmatrix} a & b \\ \alpha(1)(b) & \alpha(1)(a) \end{bmatrix} : a, b \in A\}.$$

$\square$.

(ii) Let $G = \{x_i\}_{i=0}^{n-1}$ be a finite group and $\pi_l : G \to Aut(C_\circ(G))$ be an action. Consider the α-covariant pair (ρ, U) with

$$\begin{aligned} \rho : & C_\circ(G) = l_n^\infty \to B(l^2(G)) = B(l_n^2) = M_n(\mathbb{C}) \\ & f \to \rho(f)(g) = fg \end{aligned}$$

for $g \in l_n^2$, and the left regular representation

$$U = \pi_l : G \to B(l^2(G)) = M_n(\mathbb{C}).$$

By definition,

$$\rho(f) = \begin{bmatrix} f(x_0) & 0 & 0 & \dots \\ 0 & f(x_1) & 0 & \dots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & f(x_{n-1}) \end{bmatrix}_{n \times n}$$

and each $\pi_l(x_i)$ is a shift operator in $M_n(\mathbb{C})$. Let $x_0 = e$, then

$$\pi_l(x_0) = I_n = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix}_{n \times n}$$

is the identity of $M_n(\mathbb{C})$. Let $\delta_0 \in C_\circ(G) = l_n^\infty$. Then

$$\rho \rtimes U(\mathbb{C}\delta_0 \rtimes x_o) = \mathbb{C}\rho(\delta_0)U(x_o) = \begin{bmatrix} \mathbb{C} & 0 & 0 & \dots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}_{n \times n}$$

For each i, $\pi_l(x_i)$ is a shift operator in $M_n(\mathbb{C})$. In each row, there is one coordinate 1 and the rest are 0. Thus

$$span\{\rho \rtimes U(\mathbb{C}\delta_0 \rtimes x_i)\}_{i=0}^{n-1} = span\{\mathbb{C}\rho(\delta_0)U(x_i)\}_{i=0}^{n-1} = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \dots & \mathbb{C} \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}_{n \times n}.$$

Similarly,

$$span\{\rho \rtimes U(\mathbb{C}\delta_1 \rtimes x_i)\}_{i=0}^{n-1} = span\{\mathbb{C}\rho(\delta_1)U(x_i)\}_{i=0}^{n-1} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \mathbb{C} & \mathbb{C} & \dots & \mathbb{C} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}_{n\times n}.$$

Since

$$\rho \rtimes U : C_c(G, C_\circ(G)) = C(G) \otimes C(G) \to B(\ell_n^2) = M_n(\mathbb{C})$$

is a faithful representation,

$$\rho \rtimes U(C_c(G, C_\circ(G)) = \rho(C(G))U(C(G)) = span\{\rho(\delta_i)U(x_j)\}_{i=1,j=0}^{n,n-1} = M_n(\mathbb{C}).$$

Second session: Sunday, 1393/10/07

Some important Crossed Products:

Let $\alpha = id : G \to Aut(A)$ be a dynamical system. Let

$$\rho : A \to B(H_\rho)$$

and

$$U : G \to U(H_\rho)$$

be α-covariant representations, then

$$U(x)\rho(a) = \rho(\alpha(x)(a))U(x) = \rho(a)U(x).$$

We can extend the unitary representation U to

$$U : C^*(G) \to B(H_\rho)$$

and we have a $*$-representation

$$\rho \otimes U : A \otimes_{max} C^*(G) \to B(H_\rho).$$

On the other hand, $A \otimes_{max} C^*(G)$ can be characterized by all its non-degenerate $*$-representations

$$\pi \otimes W : A \otimes_{max} C^*(G) \to B(H_\pi),$$

where the values of π and U are commuting. We may define

$$W : G \to B(H_\pi)$$

such that

$$W(x)[\sum_{i=1}^{n} \pi(a_i)W(f_i)] = W(x)[\sum_{i=1}^{n} W(f_i)\pi(a_i)] = \sum_{i=1}^{n} W(\pi_l(x)(f_i))\pi(a_i)$$
$$= \sum_{i=1}^{n} \pi(a_i)W(\pi_l(x)(f_i)).$$

then (π, W) is an id-covariant pair. Thus for each

$$\sum_{i=1}^{n} a_i \otimes f_i \in A \otimes C_c(G) \subseteq C_c(G, A),$$

$$\begin{aligned}\|\sum_{i=1}^{n} a_i \otimes f_i\|_{A\rtimes_\alpha G} &= sup_{(\rho,U)}\|\sum_{i=1}^{n}\rho(a_i)U(f_i)\| \\ &\leq \|\sum_{i=1}^{n} a_i \otimes f_i\|_{A\otimes_{max}C^*(G)} = sup_{(\pi,W)}\|\sum_{i=1}^{n}\pi(a_i)W(f_i)\| \\ &\leq \|\sum_{i=1}^{n} a_i \otimes f_i\|_{A\rtimes_\alpha G}.\end{aligned}$$

This means that

$$A \rtimes_\alpha G \cong A \otimes_{\max} C^*(G).$$

$\square$.

Let B be any C^*-algebra. We show that

$$(A \rtimes_\alpha G) \otimes_{max} B \cong (A \otimes_{max} B) \rtimes_{\alpha\otimes id} G$$

where

$$\alpha \otimes id : G \to Aut(A \otimes_{max} B)$$

is defined by

$$(\alpha \otimes id)(x)(a \otimes b) = \alpha(x)(a) \otimes b$$

for each $x \in G$, $a \in A$ and $b \in B$. Let

$$\rho : A \otimes_{max} B \to B(H_\rho)$$

and

$$U : G \to B(H_\rho)$$

be $(\alpha \otimes id)$-covariant representations. We may write $\rho = \rho_1 \otimes \rho_2$ such that

$$\rho(a \otimes b) = \rho_1(a)\rho_2(b) = \rho_2(b)\rho_1(a).$$

We have

$$\begin{aligned}\rho[\alpha(x)(a) \otimes b] &= \rho[(\alpha(x) \otimes id)(a \otimes b)] \\ &= U(x)\rho[a \otimes b]U(x^{-1}) \\ &= U(x)\rho_1(a)\rho_2(b)U(x^{-1}).\end{aligned}$$

On the other hand,

$$\rho[\alpha(x)(a) \otimes b] = \rho_1[\alpha(x)(a)]\rho_2(b)$$

and

$$\rho_1[\alpha(x)(a)]\rho_2(b) = U(x)\rho_1(a)\rho_2(b)U(x^{-1}).$$

Using a bounded approximate identity of A, we get

$$\rho_2(b) = U(x)\rho_2(b)U(x^{-1})$$

and

$$U(x)\rho_2(b) = \rho_2(b)U(x)$$

for each $b \in B$ and $x \in G$. This means that ρ_2 and U have commuting values. Same way, by bounded approximate identity of B, we have

$$\rho_1(\alpha(x)(a)) = U(x)\rho_1(a)U(x^{-1}).$$

That means (ρ_1, U) is α-covariant representation. Thus

$$\rho_1 \rtimes U : A \rtimes_\alpha G \to B(H_\rho)$$

and

$$\rho_2 : B \to B(H_\rho)$$

are commuting representations such that

$$\begin{aligned}\rho_1 \rtimes U(a \otimes f)\rho_2(b) &= \rho_1(a)U(f)\rho_2(b) = \rho_2(b)\rho_1(a)U(f)\\ &= \rho_2(b)(\rho_1 \rtimes U)(a \otimes f).\end{aligned}$$

Thus,

$$(\rho_1 \rtimes U) \otimes \rho_2 : (A \rtimes_\alpha G) \otimes_{max} B \to B(H_\rho).$$

Conversely, let

$$(\pi_1 \rtimes U) \otimes \pi_2 : (A \rtimes_\alpha G) \otimes_{max} B \to B(H_\pi)$$

be a non-degenerate $*$-representation. Then

$$[(\pi_1 \rtimes U) \otimes \pi_2][(a \otimes f) \otimes b] = [\pi_2 \otimes (\pi_1 \rtimes U)][b \otimes (a \otimes f)]$$

and

$$\pi_1(a)U(f)\pi_2(b) = \pi_2(b)\pi_1(a)U(f).$$

Since $\{\varphi_r\}_r \subseteq C_c(G)$, using a bounded approximate identity for $(L^1(G), *)$,

$$\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$$

and

$$\pi = \pi_1 \otimes \pi_2 : A \otimes_{max} B \to B(H_\pi).$$

Similarly, using a bounded approximate identity for A and $\{\pi_l(x)(\varphi_r)\}_r$,

$$\pi_2(b)U(x) = U(x)\pi_2(b).$$

Thus

$$\begin{aligned}\pi[(\alpha(x) \otimes id)(a \otimes b)] &= \pi[(\alpha(x)(a) \otimes b)] = \pi_1(\alpha(x)(a))\pi_2(b)\\ &= U(x)\pi_1(a)U(x^{-1})\pi_2(b) = U(x)\pi_1(a)\pi_2(b)U(x^{-1})\\ &= U(x)\pi(a \otimes b)U(x^{-1}).\end{aligned}$$

Therefore (π, U) is an $(\alpha \otimes id)$-covariant pair and

$$\pi \rtimes U : (A \otimes_{max} B) \rtimes_\alpha G \to B(H_\pi).$$

Let $\{a_i\}_{i=1}^n \subseteq A$, $\{b_i\}_{i=1}^n \subseteq B$ and $\{f_i\}_{i=1}^n \subseteq C_c(G)$, then

$$\begin{aligned}
\|\sum_{i=1}^n (a_i \otimes b_i) \otimes f_i\|_{(A\otimes_{max}B)\rtimes_{\alpha\rtimes id}G} &= sup_{(\rho,U)}\|\sum_{i=1}^n \rho(a_i \otimes b_i)U(f_i)\| \\
&= sup_{(\rho,U)}\|\sum_{i=1}^n \rho_1(a_i)\rho_2(b_i)U(f_i)\| \\
&= sup_{(\rho,U)}\|\sum_{i=1}^n \rho_1(a_i)U(f_i)\rho_2(b_i)\| \\
&= sup_{(\rho_1\rtimes U,\rho_2)}\|\sum_{i=1}^n \rho_1 \rtimes U(a_i \otimes f_i)\rho_2(b_i)\| \\
&= \|\sum_{i=1}^n (a_i \otimes b_i) \otimes f_i\|_{(A\rtimes_\alpha G)\otimes_{max}B}.
\end{aligned}$$

This means that

$$(A \otimes_{max} B) \rtimes_{\alpha\otimes id} G \cong (A \rtimes_\alpha G) \otimes_{max} B.$$

$\square$.

Rotation algebra: Let θ be an irrational number. Let

$$\pi_\theta : \mathbb{Z} \to Aut(C(\mathbf{T})) \quad \text{s.t} \quad \pi_\theta(n)(f)(x) = f(e^{-2\pi in\theta}x)$$

for each $n \in \mathbb{Z}$ and $x \in \mathbf{T}$. Obviously, $(C(\mathbf{T}), \mathbb{Z}, \pi_\theta)$ is a dynamical system. Let

$$\rho : C(\mathbf{T}) \to B(H_\rho)$$

and

$$U : \mathbb{Z} \to U(H_\rho)$$

be any π_θ-covariant representations. Let $f_\circ \in C(\mathbf{T})$ be such that $f_\circ(z) = z$, for $z \in \mathbf{T}$. Then, $C(\mathbf{T}) = C^*(f_\circ)$. Thus

$$\rho(\sum_{i=-n}^n \lambda_i z^i) = \sum_{i=-n}^n \lambda_i \rho(f_\circ)^i$$

and ρ can be characterized by $\rho(f_\circ)$. On the other hand, one can extend U to

$$U : \ell^1(\mathbb{Z}) \to U(H_\rho)$$

such that

$$U(\sum_{i=-n}^{n}\lambda_i\delta_i)=\sum_{i=-n}^{n}\lambda_iU(\delta_i)=\sum_{i=-n}^{n}\lambda_iU(\delta_1^n)=\sum_{i=-n}^{n}\lambda_iU(\delta_1)^n.$$

Thus U can be characterized by $U(\delta_1)$. Since,

$$\rho\rtimes U(\sum_{-n}^{n}\lambda_i z^{m_i}\otimes\delta_i)=\sum_{-n}^{n}\lambda_i\rho(f_\circ)^{m_i}U(\delta_1)^i,$$

$C(\mathbf{T})\rtimes_\theta\mathbb{Z}$ can be characterized by $\rho(f_\circ)$ and $U(\delta_1)$ such that

$$U(\delta_1)\rho(f_\circ)=\rho(\alpha_\theta(1)(f_\circ))U(\delta_1),$$

and

$$\alpha_\theta(1)(f_\circ)(x)=f_\circ(e^{-2\pi i\theta}x)=e^{-2\pi i\theta}f_\circ(x),$$

and

$$U(\delta_1)\rho(f_\circ)=e^{-2\pi i\theta}\rho(f_\circ)U(\delta_1).$$

This means that

$$C(\mathbf{T})\rtimes_\theta\mathbb{Z}=C^*(\rho(f_\circ),U(\delta_1))$$

such that $\sigma(f_\circ)=\mathbf{T}$ and

$$U(\delta_1)\rho(f_\circ)=e^{-2\pi i\theta}\rho(f_\circ)U(\delta_1).$$

Now, let U and V be unitaries in $B(H)$ such that $UV=e^{-2\pi i\theta}VU$. Note that

$$\begin{aligned}\lambda\in\sigma(V)&\Leftrightarrow V-\lambda I\quad\text{is not invertible}\\&\Leftrightarrow U^n(V-\lambda I)\quad\text{is not invertible}\\&\Leftrightarrow(e^{-2\pi in\theta}V-\lambda I)U^n\quad\text{is not invertible}\\&\Leftrightarrow V-e^{2\pi in\theta}\lambda I\quad\text{is not invertible}\\&\Leftrightarrow e^{2\pi in\theta}\lambda\in\sigma(V)\quad\text{is not invertible.}\end{aligned}$$

As $\theta\in\mathbb{Q}^c$, we have $\sigma(V)=\mathbf{T}$ and $C(\mathbf{T})\cong C^*(V)$. Consider the $*$-representation

$$\begin{gathered}\rho:C(\mathbf{T})\to C^*(V)\subseteq B(H)\\ f\to\rho(f)=f(V)\end{gathered}$$

and unitary

$$U':\mathbb{Z}\to U(H)\quad s.t.\quad U'(n)=U^n.$$

We have

$$\rho(\pi_\theta(n)(f_\circ)) = \rho(e^{-2\pi in\theta} f_\circ) = e^{-2\pi in\theta}\rho(f_\circ) = e^{-2\pi in\theta}V$$
$$= U^nVU^{-n} = U'(n)\rho(f_\circ)U'(n)^*.$$

Therefore, for each $f \in C(\mathbf{T})$,

$$\rho(\pi_\theta(n)(f)) = U'(n)\rho(f)U'(n)^*$$

and (ρ, U') is a π_θ-covariant pair. Thus

$$\rho \rtimes U' : C(\mathbf{T}) \rtimes_{\pi_\theta} \mathbb{Z} \to B(H)$$

is a $*$-representation. By [1, Proposition 2.56], $C(\mathbf{T}) \rtimes_{\pi_\theta} \mathbb{Z}$ is a simple C^*-algebra and $\rho \rtimes U'$ is a faithful representation. This means that

$$C(\mathbf{T}) \rtimes_{\pi_\theta} \mathbb{Z} \cong C^*(V, U).$$

□.

Let I be an ideal of A. Let $\alpha : G \to Aut(A)$ be an action such that $\alpha(G)(I) \subseteq I$. Note that $C_c(G, I)$ sits in $C_c(G, A)$ as a $*$-closed two-sided ideal. Therefore, its closure $Ex(I)$ is a closed ideal.

Now, $\alpha_I : G \to Aut(I)$ is dynamical system, and we show that

$$I \rtimes_{\alpha_I} G = Ex(I).$$

Let

$$\pi : I \to B(H_\pi) \quad \text{and} \quad W : G \to B(H_\pi)$$

be a non-degenerate α_I-covariant representation. As I is an ideal in A, there is an extension

$$\tilde{\pi} : A \to B(H_\pi)$$

such that

$$\tilde{\pi}(a)(\pi(b)h) := \pi(ab)h.$$

We have

$$\begin{aligned}\tilde{\pi}(\alpha(x)(a))\pi(b)h &= \pi[\alpha(x)(a)b]h \\ &= \pi[\alpha(x)(a\alpha(x^{-1})(b))]h \\ &= W(x)\pi[a\alpha(x^{-1})(b)]W(x^{-1})h \\ &= W(x)\tilde{\pi}(a)\pi(\alpha(x^{-1})(b)W(x^{-1})h \\ &= W(x)\tilde{\pi}(a)W(x^{-1})\pi(b)h.\end{aligned}$$

Thus

$$\tilde{\pi}(\alpha(x)(a)) = W(x)\tilde{\pi}(a)W(x^{-1}).$$

That is, $(\tilde{\pi}, W)$ is an α-covariant pair. Let $f \in C_c(G, I) \subseteq C_c(G, A)$. Then,

$$\begin{aligned} \|f\|_{A \rtimes_\alpha G} &= sup\|\rho \rtimes U(f)\| = sup\|\rho_I \rtimes U(f)\| \\ &\leq \|f\|_{I \rtimes_\alpha G} = sup\|\pi \rtimes W(f)\| = sup\|\tilde{\pi} \rtimes W(f)\| \\ &\leq \|f\|_{A \rtimes_\alpha G}, \end{aligned}$$

which means that

$$\|f\|_{A \rtimes_\alpha G} = \|f\|_{I \rtimes_\alpha G}$$

and

$$I \rtimes_{\alpha_I} G = Ex(I).$$

Third session: Monday, 1393/10/8

Hilbert C^*-modules:

Let H and K be Hilbert spaces. Consider the inner product

$$
\begin{aligned}
{}_{B(H)}\langle , \rangle : B(K,H) \times B(K,H) &\to B(H) \\
(S_1, S_2) \to {}_{B(H)}\langle S_1, S_2 \rangle &:= S_1 S_2^*.
\end{aligned}
$$

For each $T \in B(H)$ and $S_1, S_2 \in B(K,H)$,

(1) ${}_{B(H)}\langle S_1, S_2 \rangle^* = (S_1 S_2^*)^* = S_2 S_1^* = {}_{B(H)}\langle S_2, S_1 \rangle$,

(2) ${}_{B(H)}\langle T S_1, S_2 \rangle = T S_1 S_2^* = T\ {}_{B(H)}\langle S_1, S_2 \rangle$

(3) ${}_{B(H)}\langle S_1, T S_2 \rangle = S_1 S_2^* T^* = {}_{B(H)}\langle S_1, S_2 \rangle T^*$

and for each $S \in B(K,H)$,

$$\|S\|^2 = \|SS^*\| = \|{}_{B(H)}\langle S, S \rangle\|.$$

This is an example of a left Hilbert C^*-module. In general, let V be a Banach space and A be a C^*-algebra. Let V be left A Banach module, that is,

$$
\begin{aligned}
A \times V &\to V \\
(a, T) &\to a.T
\end{aligned}
$$

is a continuous bilinear mapping. Instead of $a.T$, we write aT. The Banach space V is called a left Hilbert A-module, if there is an inner product

$$ {}_A\langle , \rangle : V \times V \to A $$

such that

(1) ${}_A\langle T, T \rangle \geq 0$ for each $T \in V$,

(2) ${}_A\langle T, T \rangle = 0$ if and only if $T = 0$,

(3) ${}_A\langle aT, S \rangle = a\langle T, S \rangle$ for each $T, S \in V$, $a \in A$,

(4) ${}_A\langle T, S \rangle^* = {}_A\langle S, T \rangle$,

(5) V is complete by the norm $\|T\|^2 = \|{}_A\langle T, T \rangle\|$.

The module is called full if ${}_A\langle V, V \rangle$ is dense in A. In the rest of this note, the modules are assumed to be full.

Let $\mathbb{B}(V)$ be the set of adjointable maps $\varphi : V \to V$, that is, the set of maps $\varphi : V \to V$ such that there exists a map $\varphi^* : V \to V$ with

$$ {}_A\langle \varphi(T), S\rangle = {}_A\langle T, \varphi^*(S)\rangle, $$

for each $T, S \in V$. Note that in general, $\varphi : V \to V$ is not adjointable. For example, let $C[0,1]$ be the left $C[0,1]$ Hilbert C^*-module with inner product $\langle f, g\rangle = f\bar{g}$. Let

$$ \varphi : C[0,1] \to C[0,1] \quad s.t \quad \varphi(f) = f(0)1. $$

If φ is adjointable, there is $\varphi^* : C[0,1] \to C[0,1]$ such that

$$ {}_A\langle \varphi(f), g\rangle = {}_A\langle f, \varphi^*(g)\rangle $$

and

$$ \varphi(f)\overline{g} = f\overline{\varphi^*(g)}. $$

For $f = 1$,

$$ \overline{g} = f(0)\overline{g} = \varphi(f)\overline{g} = {}_A\langle \varphi(f), g\rangle = {}_A\langle f, \varphi^*(g)\rangle = f\overline{\varphi^*(g)} = \overline{\varphi^*(g)}. $$

Thus $\varphi^* = id$. For $f(x) = x$ and $g = 1$,

$$ 0 = f(0)\overline{g} = {}_A\langle \varphi(f), g\rangle = {}_A\langle f, \varphi^*(g)\rangle = f\overline{\varphi^*(g)} = f, $$

which is a contradiction.

Same as inner product of Hilbert spaces,

$$ \|{}_A\langle T, S\rangle\| \le \|T\|\|S\| $$

for each $T, S \in V$. Thus, for each $\varphi \in \mathbb{B}(V)$,

$$ \begin{aligned} \|\varphi\|^2 &= sup\|\varphi(T)\|^2 = sup\|{}_A\langle \varphi(T), \varphi(T)\rangle\| = sup\|{}_A\langle \varphi^*\varphi(T), T\rangle\| \\ &\le sup\|\varphi^*\varphi(T)\| = \|\varphi^*\varphi\| = sup\|{}_A\langle \varphi^*\varphi(T), S\rangle\| \\ &= sup\|{}_A\langle \varphi(T), \varphi(S)\rangle\| \\ &\le \|\varphi\|^2 \end{aligned} $$

where the sup is on the unit ball of V. Thus, $\|\varphi\|^2 = \|\varphi^*\varphi\|$ and $\mathbb{B}(V)$ is a C^*-algebra. For $T, S \in V$, we define $\varphi_{T\otimes S}$ by

$$ \varphi_{T\otimes S}(S') = {}_A\langle S', S\rangle T. $$

Let $S_1, S_2 \in V$, then

$$ \begin{aligned} {}_A\langle(\varphi_{T\otimes S}(S_1), S_2\rangle &= {}_A\langle {}_A\langle S_1, S\rangle T, S_2\rangle = {}_A\langle S_1, S\rangle\ {}_A\langle T, S_2\rangle = {}_A\langle S_1,\ {}_A\langle S_2, T\rangle S\rangle \\ &= \langle S_1, \varphi_{S\otimes T} S_2\rangle. \end{aligned} $$

That is,

$$ \varphi^*_{T\otimes S} = \varphi_{S\otimes T} $$

and $\varphi_{T\otimes S} \in \mathbb{B}(V)$. We write $T \otimes S$ instead of $\varphi_{T\otimes S}$. Define $\mathbb{K}(V)$ to be the subspace of $\mathbb{B}(V)$ generated by all $T \otimes S$, for $T, S \in V$.

We define an inner product on $\begin{bmatrix} A \\ V \end{bmatrix}$ making it a left Hilbert A-module. We have $\begin{bmatrix} A \\ V \end{bmatrix}$ is a left A-module by

$$a \begin{bmatrix} a' \\ T \end{bmatrix} = \begin{bmatrix} aa' \\ aT \end{bmatrix}$$

and we define the inner product by

$$_A\langle \begin{bmatrix} a \\ T \end{bmatrix}, \begin{bmatrix} a' \\ T' \end{bmatrix} \rangle = aa'^* + \ _A\langle T, S\rangle.$$

It is easy to check that $\begin{bmatrix} A \\ V \end{bmatrix}$ is a left Hilbert C^*-module. We have

$$\begin{bmatrix} A & V \\ V^* & \mathbb{K}(V) \end{bmatrix} : \begin{bmatrix} A \\ V \end{bmatrix} \to \begin{bmatrix} A \\ V \end{bmatrix}$$

such that

$$\begin{bmatrix} a & T_1 \\ T_2^* & T_3 \otimes T_4 \end{bmatrix} \begin{bmatrix} b \\ S \end{bmatrix} = \begin{bmatrix} ab + \ _A\langle T_1, S\rangle \\ T^*b + \ _A\langle S, T_4\rangle T_3 \end{bmatrix}.$$

Obviously,

$$\mathcal{L}(V) := \begin{bmatrix} A & V \\ V^* & \mathbb{K}(V) \end{bmatrix} \subseteq \mathbb{B}(\begin{bmatrix} A \\ V \end{bmatrix})$$

is a C^*-algebra such that V is its corner. The algebra $\mathcal{L}(V)$ is called the linking C^*-algebra of V.

$\square$.

Now let $C \subseteq B(H)$ be a C^*-algebra and $p \in B(H)$ be a projection. Let $p^\perp = I - p$, then

$$C \cong \begin{bmatrix} pCp & pCp^\perp \\ p^\perp Cp & p^\perp Cp^\perp \end{bmatrix}.$$

Let $A := pCp$, $B := p^\perp Cp^\perp$ and $V := pCp^\perp$. Then

$$\begin{aligned} _A\langle , \rangle : & V \times V \to A \\ & (T, S) \to TS^*. \end{aligned}$$

Obviously, $V = pCp^\perp$ is a left Hilbert A-module and

$$C \cong \mathcal{L}(V) = \begin{bmatrix} A & V \\ V^* & B \end{bmatrix}.$$

That means, each left Hilbert C^*-module is the corner of some C^*-algebra. Thus, for each T_1, T_2 and $T_3 \in V$,

$$\begin{aligned}\begin{bmatrix}0 & T_1T_2^*T_3\\0 & 0\end{bmatrix} &= \begin{bmatrix}0 & T_1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & T_2\\0 & 0\end{bmatrix}^*\begin{bmatrix}0 & T_3\\0 & 0\end{bmatrix}\\ &= (\begin{bmatrix}0 & T_1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & T_2\\0 & 0\end{bmatrix}^*)\begin{bmatrix}0 & T_3\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & T_1T_2^*\\0 & 0\end{bmatrix}\begin{bmatrix}0 & T_3\\0 & 0\end{bmatrix}\\ &= \begin{bmatrix}0 & T_1\\0 & 0\end{bmatrix}(\begin{bmatrix}0 & T_2\\0 & 0\end{bmatrix}^*\begin{bmatrix}0 & T_3\\0 & 0\end{bmatrix}) = \begin{bmatrix}0 & T_1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & T_2^*T_3\end{bmatrix}.\end{aligned}$$

That means,

$$_A\langle T_1, T_2\rangle T_3 = T_1T_2^*T_3 = T_1\langle T_2, T_3\rangle_B$$

Motivated by the above idea, we say that the C^*-algebras A and B are Morita equivalent if there is a Banach space V such that V is a left A Hilbert C^*-module and a right B Hilbert C^*-module and

$$_A\langle T_1, T_2\rangle T_3 = T_1\langle T_2, T_3\rangle_B,$$

for each T_1, T_2 and $T_3 \in V$. In this case, we have $\mathbb{K}(V) = B$ and

$$\mathcal{L}(V) = \begin{bmatrix}A & V\\V^* & B\end{bmatrix}$$

is a C^*-algebra.

□.

Induced ideals: Let A and B be Morita equivalent by V. Then $\begin{bmatrix}A & V\\V^* & B\end{bmatrix}$ is a C^*-algebra. Let I be ideal of A. Then

$$C^*(\begin{bmatrix}0 & IV\\0 & 0\end{bmatrix}) = \begin{bmatrix}I & IV\\(IV)^* & V^*IV\end{bmatrix}$$

and V^*IV is an ideal of B. We call $ind_A^B(I) = V^*IV$ the induced representation of I. On the other hand, V^*IV is ideal of B, and

$$C^*(\begin{bmatrix}0 & V(V^*IV)\\0 & 0\end{bmatrix}) = C^*(\begin{bmatrix}0 & AIV\\0 & 0\end{bmatrix}) = C^*(\begin{bmatrix}0 & IV\\0 & 0\end{bmatrix}) = \begin{bmatrix}I & IV\\(IV)^* & V^*IV\end{bmatrix}.$$

Thus,

$$ind_B^A(ind_A^B(I)) = ind_B^A(V^*IV) = I.$$

Let $b \in B$ be such that $Vb \subseteq IV$. Then $V^*Vb \subseteq V^*IV$ and $b \subseteq V^*IV$. If $b \in V^*IV$, then

$$Vb \subseteq VV^*IV = AIV = IV.$$

That means,

$$ind_A^B(I) = V^*IV = \{b \in B : Vb \subseteq IV\}.$$

$\square$.

Induced representations: Let A and B be Morita equivalent by V, that is, $\begin{bmatrix} A & V \\ V^* & B \end{bmatrix}$ be a C^*-algebra. Let $\rho : B \to B(H_\rho)$ be a non-degenerate $*$-representation. We find some Hilbert space K such that $\begin{bmatrix} A & V \\ V^* & B \end{bmatrix}$ can act on $\begin{bmatrix} K \\ H_\rho \end{bmatrix}$. As the action of V on H_ρ must give an element of K, and the action is bilinear, the best candidate for K is $V \otimes H_\rho$ with inner product

$$\langle T \otimes h, S \otimes g \rangle = \langle \rho(S^*T)h, g \rangle.$$

Also,

$$\begin{aligned} \langle Tb \otimes h, S \otimes g \rangle &= \langle \rho(S^*Tb)h, g \rangle \\ &= \langle \rho(S^*T)\rho(b)h, g \rangle \\ &= \langle T \otimes \rho(b)h, S \otimes g \rangle. \end{aligned}$$

In the above inner product, $Tb \otimes h = T \otimes \rho(b)h$, and $V \otimes_B H_\rho$ is a Hilbert space. Thus we have the action

$$\begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \begin{bmatrix} V \otimes_B H_\rho \\ H_\rho \end{bmatrix}.$$

Now we define

$$ind_B^A(\rho) : A \to B(V \otimes_B H_\rho),$$

then by definition,

$$ind_B^A(\rho)(a)[\sum_{i=1}^n T_i \otimes h_i] = \sum_{i=1}^n aT_i \otimes h_i.$$

On the other hand, $ind_B^A(\rho) : A \to B(V \otimes_B H_\rho)$. Next, we find a Hilbert space K' such that

$$\begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \begin{bmatrix} V \otimes_B H_\rho \\ K' \end{bmatrix}.$$

As V^* acts bilinearly on $V \otimes_B H_\rho$, we should define $K' := V^* \otimes (V \otimes_B H_\rho)$ with inner product

$$\begin{aligned}\langle T_1^* \otimes (S_1 \otimes h_1), T_2^* \otimes (S_2 \otimes h_2)\rangle &= \langle ind_B^A(\rho)(T_2 T_1^*)(S_1 \otimes h_1), S_2 \otimes h_2\rangle \\ &= \langle T_2 T_1^* S_1 \otimes h_1, S_2 \otimes h_2\rangle \\ &= \langle \rho(S_2^* T_2 T_1^* S_1) h_1, h_2\rangle \\ &= \langle \rho(T_1^* S_1) h_1, \rho(T_2^* S_2)) h_2\rangle.\end{aligned}$$

Thus there is isometric surjection

$$\begin{aligned}V^* \otimes_A (V \otimes_B H_\rho) &\to H_\rho \\ T^* \otimes (S \otimes h) &\to \rho(T^* S) h\end{aligned}$$

and

$$ind_A^B(ind_B^A(\rho)) : B \to B(V^* \otimes_A (V \otimes_B H_\rho)) \to B(H_\rho)$$

such that

$$ind_A^B(ind_B^A(\rho))(b)[T^* \otimes (S \otimes h)] = bT^* \otimes (S_1 \otimes h) \to \rho(bT^* S)h = \rho(b)\rho(T^* S)h.$$

That means,

$$ind_A^B(ind_B^A(\rho)) = \rho.$$

□.

Let A, B and C be C^*-algebras such that A and B and also B and C are Morita equivalent. Let V and W be Banach spaces such that

$$\begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B & W \\ W^* & C \end{bmatrix}$$

are C^*-algebras. We want to find (?), such that

$$L = \begin{bmatrix} A & V & ? \\ V^* & B & W \\ ?^* & W^* & C \end{bmatrix}$$

is a C^*-algebra. If L is a C^*-algebra, then

$$\begin{bmatrix} 0 & T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & T.S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in L.$$

As the product is bilinear, the best candidate for (?) is $V \otimes W$. On the other hand, the product of L is associative

$$\left(\begin{bmatrix} 0 & T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Tb & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Tb \otimes S \\ 0 & 0 & 0 \end{bmatrix}.$$

which is equal to

$$\begin{bmatrix} 0 & T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Sb \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T \otimes bS \\ 0 & 0 & 0 \end{bmatrix}.$$

That means, $Tb \otimes S = T \otimes bS$ and

$$\begin{bmatrix} A & V & V \otimes_B W \\ V^* & B & W \\ W^* \otimes_B V^* & W^* & C \end{bmatrix}$$

is C^*-algebra. Let

$$p = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

The C^*-algebra pLp is spatially isomorphic to

$$\begin{bmatrix} A & V \otimes_B W \\ W^* \otimes_B V^* & C \end{bmatrix}$$

and A and C are Morita equivalent.

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crossed product of linking C^*-algebras:

Let V be an A-B Hilbert C^*-module. Let

$$\alpha : G \to Aut(A) \quad \text{and} \quad \beta : G \to Aut(B)$$

be actions and

$$\varphi : G \to inv(V)$$

be an sot-continuous map, where $inv(V)$ is the family of all invertible mappings on V. We say that φ is α-β-compatible if

$$\begin{aligned} &(1) \quad \varphi(x)(aT) = \alpha(x)(a)\varphi(x)(T),\\ &(2) \quad \varphi(x)(Tb) = \varphi(x)(T)\beta(x)(b),\\ &(3) \quad \langle \varphi(x)(T), \varphi(x)(S)\rangle_B = \beta(x)(\langle T, S\rangle_B). \end{aligned}$$

Then

$$\begin{aligned} {}_A\langle \varphi(x)(T), \varphi(x)(S)\rangle \ \varphi(x)(S') &= \varphi(x)(T) \ \langle \varphi(x)(S), \varphi(x)(S')\rangle_B \\ &= \varphi(x)(T) \ \beta(x)(\langle S, S'\rangle_B) \\ &= \varphi(x)(T\langle S, S'\rangle_B) \\ &= \varphi(x)({}_A\langle T, S\rangle \ S') \\ &= \alpha(x)({}_A\langle T, S\rangle) \ \varphi(x)(S'). \end{aligned}$$

That means,

$${}_A\langle \varphi(x)(T), \varphi(x)(S)\rangle = \alpha(x)({}_A\langle T, S\rangle).$$

As φ is sot-continuous,

$$\Phi := \begin{bmatrix} \alpha & \varphi \\ \varphi^* & \beta \end{bmatrix} : G \to Aut(\begin{bmatrix} A & V \\ V^* & B \end{bmatrix})$$

is a dynamical system,

$$\Phi(x)(\begin{bmatrix} a & T \\ S^* & b \end{bmatrix}) = \begin{bmatrix} \alpha(x)(a) & \varphi(x)(T) \\ \varphi(x)(S)^* & \beta(x)(b) \end{bmatrix}.$$

Thus we can construct the crossed product

$$\begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \rtimes_\Phi G.$$

Let
$$\rho : \begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \to B(H_\rho)$$
and
$$U : G \to B(H_\rho)$$
be Φ-covariant representations. We can extend ρ to the unitization
$$\begin{bmatrix} A + \mathbb{C}I & 0 \\ 0 & B + \mathbb{C}I \end{bmatrix}.$$
Let
$$H := \rho(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix})H_\rho \quad \text{and} \quad K := \rho(\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix})H_\rho.$$
We have $H_\rho = H \oplus K$ and
$$\rho = \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_2^* & \rho_3 \end{bmatrix} : \begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \to B(H \oplus K)$$
$$\rho(\begin{bmatrix} a & T \\ S^* & b \end{bmatrix}) = \begin{bmatrix} \rho_1(a) & \rho_2(T) \\ \rho_2(S)^* & \rho_3(b) \end{bmatrix}.$$
Also,
$$U = \begin{bmatrix} U_1 & U' \\ U'^* & U_2 \end{bmatrix} : G \to B(H \oplus K).$$
As we have
$$\begin{aligned} \begin{bmatrix} \rho_1(\alpha(x)(a)) & 0 \\ 0 & 0 \end{bmatrix} &= \rho(\Phi(x)(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix})) \\ &= U(x)\rho(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix})U(x^{-1}) \\ &= \begin{bmatrix} U_1(x)\rho_1(a)U_1(x^{-1}) & U_1(x)\rho_1(a)U'(x^{-1}) \\ U'(x)\rho_1(a)U_1(x^{-1}) & U'(x)\rho_1(a)U'(x^{-1}) \end{bmatrix}, \end{aligned}$$
we get $U_1(x)\rho_1(a)U'(x^{-1}) = 0$, and using a bounded approximate identity of A and $x = e$, we get $U' = 0$, and
$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}.$$
Therefore, (ρ_1, U_1) is an α-covariant pair and (ρ_3, U_2) is a β-covariant pair.

Now, let
$$\rho : B \to B(H_\rho) \quad \text{and} \quad U : G \to B(H_\rho)$$

be β-covariant representations. For the linking C^*-algebra

$$\begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \begin{bmatrix} V \otimes_B H_\rho \\ H_\rho \end{bmatrix}$$

we have the action

$$ind_B^A(\rho) : A \to B(V \otimes_B H_\rho)$$

such that

$$ind_B^A(\rho)(a)(\sum_{i=1}^n T_i \otimes h_i) = \sum_{i=1}^n aT_i \otimes h_i.$$

Also we have

$$ind_B^V(\rho) : V \to B(H_\rho, V \otimes_B H_\rho)$$

$$ind_B^V(\rho)(T)(h) = T \otimes h.$$

By definition of inner product,

$$\begin{aligned} \langle ind_B^V(\rho)(T)(h), S \otimes g\rangle &= \langle T \otimes h, S \otimes g\rangle \\ &= \langle h, \rho(T^*S)g\rangle. \end{aligned}$$

Thus there is

$$ind_B^V(\rho)^* : V^* \to B(V \otimes_B H_\rho, H_\rho)$$

such that

$$ind_B^V(\rho)^*(T^*)(S \otimes h) = \rho(TS^*)h$$

which is the adjoint of $ind_B^V(\rho)$. Therefore we have an $*$-representation

$$ind(\rho) := \begin{bmatrix} ind_B^A(\rho) & ind_B^V(\rho) \\ ind_B^V(\rho)^* & \rho \end{bmatrix} : \begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \to B(\begin{bmatrix} V \otimes_B H_\rho \\ H_\rho \end{bmatrix}).$$

Next,

$$\langle ind_B^A(\rho)[\alpha(x)(a)](T \otimes h), S \otimes g\rangle = \langle \alpha(x)(a)T \otimes h, S \otimes g\rangle$$

$$\begin{aligned}
&= \langle \rho[\langle S, \alpha(x)(a)T\rangle_B]h, g\rangle \\
&= \langle \rho[\langle \varphi(x)(\varphi(x^{-1})(S), \varphi(x)(a\varphi(x^{-1})(T)\rangle_B)]\ h, g\rangle \\
&= \langle \rho[\beta(x)(\langle \varphi(x^{-1})(S), a\varphi(x^{-1})(T)\rangle_B)]\ h, g\rangle \\
&= \langle U(x)\rho[\langle \varphi(x^{-1})(S), a\varphi(x^{-1})(T)\rangle_B]\ U(x^{-1})h, g\rangle \\
&= \langle \rho[\langle \varphi(x^{-1})(S), a\varphi(x^{-1})(T)\rangle_B]\ U(x^{-1})h, U(x^{-1})g\rangle \\
&= \langle a\varphi(x^{-1})(T) \otimes U(x^{-1})h, \varphi(x^{-1})(S) \otimes U(x^{-1})g\rangle \\
&= \langle ind_B^A(\rho)(a)[\varphi(x^{-1}) \otimes U(x^{-1})](T \otimes h), [\varphi(x^{-1}) \otimes U(x^{-1}](S \otimes g)\rangle \\
&= \langle [\varphi(x) \otimes U(x)]ind_B^A(\rho)(a)[\varphi(x^{-1}) \otimes U(x^{-1})](T \otimes h), S \otimes g\rangle.
\end{aligned}$$

That means,

$$ind_B^A(\rho)[\alpha(x)(a)] = [\varphi(x) \otimes U(x)]ind_B^A(\rho)(a)[\varphi(x^{-1}) \otimes U(x^{-1})]$$

On the other hand,

$$\begin{aligned}
\langle ind_B^V(\rho)(\varphi(x)(T))(h), S \otimes g\rangle &= \langle \varphi(x)(T) \otimes h, S \otimes g\rangle \\
&= \langle \rho(\langle S, \varphi(x)(T)\rangle_B)h, g\rangle \\
&= \langle \rho(\beta(x)(\langle \varphi(x^{-1})(S), T\rangle_B)h, g\rangle \\
&= \langle U(x)\rho(\langle \varphi(x^{-1})(S), T\rangle_B)U(x^{-1})h, g\rangle \\
&= \langle \rho(\langle \varphi(x^{-1})(S), T\rangle_B)U(x^{-1})h, U(x^{-1})g\rangle \\
&= \langle\langle T \otimes U(x^{-1})h, \varphi(x^{-1})(S) \otimes U(x^{-1})g\rangle \\
&= \langle\langle \varphi(x) \otimes U(x))ind_B^V(T)U(x^{-1})(h), S \otimes g\rangle
\end{aligned}$$

That means,

$$ind_B^V(\rho)(\varphi(x)(T)) = (\varphi(x) \otimes U(x))ind_B^V(T)U(x^{-1}).$$

Thus

$$ind(U) := \begin{bmatrix} \varphi \otimes U & 0 \\ 0 & U \end{bmatrix} : G \to B(\begin{bmatrix} V \otimes_B H_\rho \\ H_\rho \end{bmatrix})$$

is a unitary representation such that $(ind(\rho), ind(U))$ is Φ-covariant pair. Let $f \in C_c(G, B)$, then

$$\begin{aligned}
\| \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} \|_{\mathcal{L}(V) \rtimes_\Phi G} &= sup\|\rho \rtimes U(\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix})\| = sup\| \begin{bmatrix} 0 & 0 \\ 0 & \rho_3 \rtimes U_2(f) \end{bmatrix})\| \\
&= sup\|\rho_3 \rtimes U_2(f)\| \\
&\leq \|f\|_{B \rtimes_\beta G} = sup_{(\pi,W)}\|\pi \rtimes_\beta W(f)\| \\
&= sup\|ind(\pi) \rtimes ind(W)(\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix})\| \\
&\leq \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} \|_{\mathcal{L}(V) \rtimes_\Phi G}.
\end{aligned}$$

Therefore,

$$\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} \|_{\mathcal{L}(V) \rtimes_\Phi G} = \|f\|_{B \rtimes_\beta G}$$

and

$$\begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \rtimes_\Phi G = \begin{bmatrix} A \rtimes_\alpha G & V \rtimes_\varphi G \\ (V \rtimes_\varphi G)^* & B \rtimes_\beta G \end{bmatrix}.$$

Next, we want to find the appropriate product. Let $f \in C_c(G, A)$ and $g \in C(c(G, V)$. Then

$$\begin{bmatrix} 0 & f.g \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} A \rtimes_\alpha G & V \rtimes_\varphi G \\ (V \rtimes_\varphi G)^* & B \rtimes_\beta G \end{bmatrix}$$

which can be characterized by its covariant representations. Let

$$\rho = \begin{bmatrix} \rho_1 & \rho' \\ \rho'^* & \rho_2 \end{bmatrix} : \begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \to B(H \oplus K)$$

and

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} : G \to B(H \oplus K)$$

be Φ-covariant representations. Then

$$\Phi \rtimes U(\begin{bmatrix} 0 & f.g \\ 0 & 0 \end{bmatrix}) = \Phi \rtimes U(\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}) \Phi \rtimes U(\begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix}).$$

Thus

$$\begin{aligned}
\int_G \rho(\begin{bmatrix} 0 & f.g(x) \\ 0 & 0 \end{bmatrix}) U(x) dx &= \int_G \rho(\begin{bmatrix} f(y) & 0 \\ 0 & 0 \end{bmatrix}) U(y) dy \int_G \rho(\begin{bmatrix} 0 & g(x) \\ 0 & 0 \end{bmatrix}) U(x) dx \\
&= \int_G \int_G \rho(\begin{bmatrix} f(y) & 0 \\ 0 & 0 \end{bmatrix}) \rho(\Phi(y)(\begin{bmatrix} 0 & g(x) \\ 0 & 0 \end{bmatrix})) U(yx) dx dy \\
&= \int_G \int_G \rho(\begin{bmatrix} f(y) & 0 \\ 0 & 0 \end{bmatrix}) \rho(\begin{bmatrix} 0 & \varphi(y)(g(x)) \\ 0 & 0 \end{bmatrix}) U(yx) dx dy \\
&= \int_G \int_G \rho(\begin{bmatrix} f(y) & 0 \\ 0 & 0 \end{bmatrix}) \rho(\begin{bmatrix} 0 & \varphi(y)(g(y^{-1}x)) \\ 0 & 0 \end{bmatrix}) U(x) dx dy \\
&= \int_G \rho(\begin{bmatrix} 0 & \int_G f(y)\varphi(y)(g(y^{-1}x)) dy \\ 0 & 0 \end{bmatrix}) U(x) dx.
\end{aligned}$$

Therefore,

$$f.g(x) = \int_G f(y)\varphi(y)(g(y^{-1}x)) dy.$$

Now we want to characterize the involution. Let $g \in C_c(G,V)$, then

$$\begin{aligned}
\begin{bmatrix} 0 & g^* \\ 0 & 0 \end{bmatrix} \leftrightarrow \rho \rtimes U(\begin{bmatrix} 0 & g^* \\ 0 & 0 \end{bmatrix}) &= \int_G \rho(\begin{bmatrix} 0 & g^*(x) \\ 0 & 0 \end{bmatrix})U(x)dx \\
&= [\rho \rtimes U(\begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix})]^* = [\int_G \rho(\begin{bmatrix} 0 & 0 \\ g(x) & 0 \end{bmatrix})U(x)dx]^* \\
&= \int_G U(x)^* \rho(\begin{bmatrix} 0 & 0 \\ g(x) & 0 \end{bmatrix})^* dx \\
&= \int_G U(x^{-1}) \rho(\begin{bmatrix} 0 & g(x)^* \\ 0 & 0 \end{bmatrix}) dx \\
&= \int_G \rho(\Phi(x^{-1})(\begin{bmatrix} 0 & g(x)^* \\ 0 & 0 \end{bmatrix}))U(x^{-1})dx \\
&= \int_G \rho(\Phi(x)(\begin{bmatrix} 0 & g(x^{-1})^* \\ 0 & 0 \end{bmatrix}))U(x)\Delta(x^{-1})dx \\
&= \int_G \rho(\begin{bmatrix} 0 & \varphi(x)(g(x^{-1}))^* \Delta(x^{-1}) \\ 0 & 0 \end{bmatrix})U(x)dx
\end{aligned}$$

and

$$g^*(x) = \varphi(x)(g(x^{-1})^*)\Delta(x^{-1}).$$

Now, let $g_1, g_2 \in C_c(G,V)$. Then

$$\begin{aligned}
\begin{bmatrix} 0 & 0 \\ 0 & g_1^*.g_2 \end{bmatrix} \leftrightarrow \rho \rtimes U(\begin{bmatrix} 0 & 0 \\ 0 & g_1^*.g_2 \end{bmatrix}) &= \int_G \rho(\begin{bmatrix} 0 & 0 \\ 0 & g_1^*.g_2(x) \end{bmatrix})U(x)dx \\
&= (\rho \rtimes U)(\begin{bmatrix} 0 & 0 \\ g_1^* & 0 \end{bmatrix})(\rho \rtimes U)(\begin{bmatrix} 0 & g_2 \\ 0 & 0 \end{bmatrix}) \\
&= \int_G \rho(\begin{bmatrix} 0 & 0 \\ g_1^*(y) & 0 \end{bmatrix})U(y)dy \int_G \rho(\begin{bmatrix} 0 & g_2(x) \\ 0 & 0 \end{bmatrix})U(x)dx \\
&= \int_G \int_G \rho(\begin{bmatrix} 0 & 0 \\ g_1^*(y) & 0 \end{bmatrix})U(y)\rho(\begin{bmatrix} 0 & g_2(x) \\ 0 & 0 \end{bmatrix})U(x)dxdy \\
&= \int_G \int_G \rho(\begin{bmatrix} 0 & 0 \\ g_1^*(y) & 0 \end{bmatrix} \rho(\Phi(y)(\begin{bmatrix} 0 & g_2(x) \\ 0 & 0 \end{bmatrix}))U(yx)dxdy \\
&= \int_G \int_G \rho(\begin{bmatrix} 0 & 0 \\ g_1^*(y) & 0 \end{bmatrix} \rho(\begin{bmatrix} 0 & \varphi(y)(g_2(x)) \\ 0 & 0 \end{bmatrix})U(yx)dxdy \\
&= \int_G \int_G \rho(\begin{bmatrix} 0 & 0 \\ g_1^*(y) & 0 \end{bmatrix} \rho(\begin{bmatrix} 0 & \varphi(y)(g_2(y^{-1}x)) \\ 0 & 0 \end{bmatrix})U(x)dxdy \\
&= \int_G \int_G \begin{bmatrix} 0 & 0 \\ 0 & g_1^*(y)\varphi(y)(g_2(y^{-1}x)) \end{bmatrix} U(x)dxdy \\
&= \int_G \int_G \begin{bmatrix} 0 & 0 \\ 0 & \varphi(y)(g_1(y^{-1}))^*\Delta(y^{-1})\varphi(y)(g_2(y^{-1}x)) \end{bmatrix} U(x)dxdy \\
&= \int_G \int_G \begin{bmatrix} 0 & 0 \\ 0 & \langle \varphi(y)(g_1(y^{-1}))\Delta(y^{-1}), \varphi(y)(g_2(y^{-1}x))\rangle_B \end{bmatrix} U(x)dxdy \\
&= \int_G \int_G \begin{bmatrix} 0 & 0 \\ 0 & \beta(y)[\langle g_1(y^{-1})\Delta(y^{-1}), g_2(y^{-1}x)\rangle_B] \end{bmatrix} U(x)dxdy \\
&= \int_G \int_G \begin{bmatrix} 0 & 0 \\ 0 & \beta(y^{-1})[\langle g_1(y), g_2(yx)\rangle_B] \end{bmatrix} U(x)dxdy \\
&= \int_G \begin{bmatrix} 0 & 0 \\ 0 & \int_G \beta(y^{-1})[\langle g_1(y), g_2(yx)\rangle_B]dy \end{bmatrix} U(x)dx.
\end{aligned}$$

That means,

$$g_1^*.g_2(x) = \int_G \beta(y^{-1})[\langle g_1(y), g_2(yx)\rangle_B]dy.$$

$\square$.

finally, we want to discuss Morita equivalence of crossed product C^*-algebras. Let

$$\varphi : G \to inv(V)$$

be an

$$\alpha : G \to Aut(A) \quad \text{and} \quad \beta : G \to Aut(B)$$

compatible mapping. And

$$\psi : G \to inv(W)$$

be a

$$\beta : G \to Aut(B) \quad \text{and} \quad \gamma : G \to Aut(C)$$

compatible mapping. Then

$$\begin{bmatrix} A & V \\ V^* & B \end{bmatrix} \rtimes_\Phi G = \begin{bmatrix} A \rtimes_\alpha G & V \rtimes_\varphi G \\ (V \rtimes_\varphi G)^* & B \rtimes_\beta G \end{bmatrix},$$

and

$$\begin{bmatrix} B & W \\ W^* & C \end{bmatrix} \rtimes_\Psi G = \begin{bmatrix} B \rtimes_\beta G & W \rtimes_\psi G \\ (W \rtimes_\psi G)^* & C \rtimes_\gamma G \end{bmatrix}$$

are C^*-algebras. Consider the action

$$\Theta := \begin{bmatrix} \alpha & \varphi & \varphi \otimes \psi \\ \varphi^* & \beta & \psi \\ \psi^* \otimes \varphi^* & \psi^* & \gamma \end{bmatrix} : G \to Aut(\begin{bmatrix} A & V & V \otimes_B W \\ V^* & B & W \\ W^* \otimes_B V^* & W^* & C \end{bmatrix}).$$

Then

$$\begin{bmatrix} A & V & V \otimes_B W \\ V^* & B & W \\ W^* \otimes_B V^* & W^* & C \end{bmatrix} \rtimes_\Theta G$$

$$= \begin{bmatrix} A \rtimes_\alpha G & V \rtimes_\varphi G & (V \otimes_B W) \rtimes_{\varphi \otimes \psi} G \\ (V \rtimes_\alpha G)^* & B \rtimes_\beta G & W \rtimes_\psi G \\ (W^* \otimes_B V^*) \rtimes_{\psi^* \otimes \varphi^*} G & (W \rtimes_\psi G)^* & C \rtimes_\gamma G. \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} A \rtimes_\alpha G & (V \otimes_B W) \rtimes_{\varphi \otimes \psi} G \\ (W^* \otimes_B V^*) \rtimes_{\psi^* \otimes \varphi^*} G & C \rtimes_\gamma G \end{bmatrix}$$

is a C^*-algebra. This means that, the C^*-algebras $A \rtimes_\alpha G$ and $C \rtimes_\gamma G$ are Morita equivalent.

□

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