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MAXIMAL SUBRINGS OF AFFINE RINGS

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ABSTRACT. We prove that if $R = F[\alpha_1, \dots, \alpha_n]$ is an affine integral domain over a field F, then R has only finitely many maximal subrings if and only if F has only finitely many maximal subrings and each α_i is algebraic over F, which is similar to the celebrated Zariski's Lemma.

All rings in this note are commutative with $1 \neq 0$. All subrings, ring extensions, homomorphisms and modules are unital. A proper subring S of a ring R is called a maximal subring if S is maximal with respect to inclusion in the set of all proper subrings of R. In this paper $RgMax(R)$ is the set of all maximal subrings of a ring R.

Theorem 1.1. [3, Corollary 1.9]. *Let R be a ring, then either R has infinitely many maximal subrings or R is a Hilbert ring.*

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Theorem 1.2. *Let E be a field. Then the following conditions are equivalent:*

1. *E has only finitely many maximal subrings.*
2. *E has a subfield F which has no maximal subring and $[E : F]$ is finite.*
3. *every descending chain $\cdots \subset R_2 \subset R_1 \subset R_0 = E$, where each R_i is a maximal subring of R_{i-1} for $i \geq 1$, is finite.*

Moreover, if one of the above conditions holds then F is unique and all chains in (3) have the same length, m say, and $R_m = F$.

Corollary 1.3. *Let $E \subseteq K$ be a finite extension of fields. Then E has only finitely many maximal subrings if and only if K has only finitely many maximal subrings.*

Affine Rings

Let us recall the important Zariski's Lemma (which play a key role in the proof of Hilbert's Nullstellensatz Theorem, see [15]) which say an affine integral domain $R = F[\alpha_1, \ldots, \alpha_n]$ over a field F is a field if and only if each α_i is algebraic over F.

One can easily see that in fact this lemma is also valid if instead of assuming that R is a field we just assume that R is semilocal (i.e., an affine integral domain $R = F[\alpha_1, \ldots, \alpha_n]$ over a field F is semilocal if and only if each α_i is algebraic over F and therefore R is a field too).

More generally, in the light of [11, Theorem 22], one also can prove that if T is an integral domain and $T = R[\alpha_1, \ldots, \alpha_n]$, then T is a G-domain (field) if and only if R is a G-domain and each α_i is algebraic over R.

Theorem 2.1. *Let $F \subseteq E$ be an extension of fields and $\alpha_1, \ldots, \alpha_n \in E$. Then*

(1) *$K = F(\alpha_1, \ldots, \alpha_n)$ has only finitely many maximal subrings if and only if F has only finitely many maximal subrings and K/F is finite.*
(2) *$R = F[\alpha_1, \ldots, \alpha_n]$ has only finitely many maximal subrings if and only if F has only finitely many maximal subrings and each α_i is algebraic over F (i.e., R/F is a finite extension of fields).*

Corollary 2.2. *Let F be an algebraically closed field and R be an affine integral domain over F. Then R has only finitely many maximal subrings if and only if $R = F = \bar{F}_p$ for some prime number p. In particular in this case R has no maximal subrings.*

Proposition 2.3. *Let F be a field and $R = F[\alpha_1, \ldots, \alpha_n]$ be a reduced F-algebra. If R has only finitely many maximal subrings, then the following statements hold:*

(1) *F has only finitely many maximal subrings.*
(2) *$R \cong K_1 \times \cdots \times K_m$, where each K_i is a finite field extension of F (therefore each K_i has only finitely many maximal subrings). Moreover, if K_i is infinite, then $K_i \ncong K_j$ for each $j \neq i$.*

Corollary 2.4. *Let F be a field and V be an affine variety in $A^n(F)$. If the coordinate ring $F[V]$ of V has only finitely many maximal subrings, then V is finite. Moreover in this case either $F[V]$ is finite or $F[V] = F$ (and therefore $|V| = 1$).*

Corollary 2.5. [6, Proposition V.1]. *Let R be a ring with nonzero characteristic which has only finitely many subrings, then R is finite.*

By the above corollary and [3, Proposition 2.1], one can easily deduce that if R is a zero-dimensional ring with only finitely many subrings, then R is a finite ring.

Lemma 2.6. *Let K be a field and x be an indeterminate over K. Then any subring R, where $K \subsetneq R \subsetneq K[x]$ is affine over K (thus R is noetherian) and $K[x]$ is integral over R. Moreover, R has a maximal subring $T \neq K$. Consequently, there exists an infinite chain $K \subsetneq \cdots \subset R_1 \subset R_0 = K[x]$, where each R_i is a maximal subring of R_{i-1} and $K[x]$ is integral over each R_i, for $i \geq 1$.*

Corollary 2.7. *Let R be a ring and x be an indeterminate over R. Then there exists an infinite chain $\cdots \subset R_1 \subset R_0 = R[x]$, where each R_i is a maximal subring of R_{i-1} and $R[x]$ is integral over each R_i, for $i \geq 1$.*

Proposition 2.8. *Let K be an algebraically closed field and R be an K-algebra. Then either R has infinitely many maximal subrings or $K = \bar{F}_p$, for some prime number p, in which case R is a zero dimensional ring with unique prime ideal M such that $R/M \cong K$ and R is integral over F_p. In particular, if R is an integral domain then $R = K$.*

Theorem 2.9. *Let $R \subseteq T$ be an extension of rings and $T = R[\alpha_1, \ldots, \alpha_n]$. Assume that T has only finitely many maximal subrings. Then the following statements hold:*

(1) *R is zero-dimensional if and only if T is zero-dimensional.*
(2) *R is semilocal (resp. artinian) if and only if T is semilocal (resp. artinian).*

Moreover, in any case T is a finitely generated R-module and for each prime ideal P of R, the ring R/P has only finitely many maximal subrings. Furthermore, in case (2), $R/N(R)$ has only finitely many maximal subrings up to isomorphism.

Example 2.10. Let K be an infinite field without maximal subrings which is not algebraically closed.

(1) Assume that $R = K \times K$, then by [3, Corollary 3.5], R has infinitely many maximal subrings. Now let α and β be elements of algebraic closure of K with different degrees over K. Hence $K[\alpha] \ncong K[\beta]$ and therefore by [3, Corollary 3.7], the ring $T = K[\alpha] \times K[\beta]$ has only finitely many maximal subrings It is clear that $T = R[(\alpha, \beta)]$.
(2) Assume that $R = K$ and $T = K \times K$. Clearly $T = R[(1,0)]$; as we see in (1), T has infinitely many maximal subrings but R has no maximal subrings.

Let K be a field, then in [8, Lemma 1.2] it is shown that the minimal ring extensions of K, up to K-algebra, isomorphism are as follow:

(1) a finite minimal field extension E.
(2) $K \times K$.
(3) $K[x]/(x^2)$.

Conversely, in [3, Theorem 3.4], it is proved that R is a maximal subring of $K \times K$ if and only if R satisfies in exactly one of the following conditions:

(1) $R = S \times K$ or $R = K \times S$, for some $S \in RgMax(K)$.
(2) $R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$, where $\sigma_i \in Aut(K)$ for $i = 1, 2$.

In the next theorem we determine exactly maximal subrings of $K[x]/(x^2)$. We recall that if $\sigma \in Aut(K)$, then the additive map $\delta : K \to K$ is called a σ-derivation of K if for each $x, y \in K$, we have $\delta(xy) = \sigma(x)\delta(y) + \sigma(y)\delta(x)$. One can easily see that for each nonzero element x of K we have $\delta(x^{-1}) = -\delta(x)\sigma(x)^{-2}$. In [8], it is shown that if R is a maximal subring of T, then $(R : T) := \{x \in T \mid Tx \subseteq R\}$ is a prime ideal of R. Moreover, T is integral over R if and only if $(R : T) \in Max(R)$; and otherwise (i.e., R is integrally closed in T) we have $(R : T) \in Spec(T)$. Now the following is in order.

Theorem 2.11. *Let K be a field and $T = K[x]/(x^2)$ $(= K[\alpha]$, where $\alpha = x+(x^2))$. Then R is a maximal subring of T if and only if R satisfies in exactly one of the following conditions:*

(1) $R = S + K\alpha$, *for* $S \in RgMax(K)$.
(2) $R = \{\sigma(x) + \delta(x)\alpha \mid x \in K\}$, *where* $\sigma \in Aut(K)$ *and* δ *is a* σ-*derivation of* K.

Now assume that K is a field, $\sigma \in Aut(K)$ and δ is a σ-derivation of K. If F is the prime subfield of K, then one can easily see that for each $x \in F$, we have $\delta(x) = 0$ (note, $\delta(1) = 0$). Moreover, if $x \in K$ is algebraic over F, then it is not hard to see that $\delta(x) = 0$. Thus if K is algebraic over its prime subfield then the only σ-derivation of K is 0. Now the following immediate corollaries are in order.

Corollary 2.12. *Let K be a field which is algebraic over its prime subfield and $T = K[\alpha]$, where $\alpha^2 = 0$. Then R is a maximal subring of T if and only if either $R = K$ or $R = S + K\alpha$ where $S \in RgMax(K)$.*

Corollary 2.13. *Let K be a field and $T = K[\alpha]$, where $\alpha^2 = 0$. Then T has finitely many maximal subrings if and only if K has only finitely many maximal subrings. Moreover in this case we have $|RgMax(T)| = 1 + |RgMax(K)|$.*

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