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(الف)

IPM
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Annihilators of local. cohomology

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R Commutative ring Noeth., $I \subset R$ an ideal.

Known results.

(Schenzel, 82): R local, $I \subset R$ ideal $I = (f_1, \dots, f_t)$, $A_i := (0 :_R H_I^i(R))$
Then $A_0 \cdots A_t = 0$

(Eghbali and Schenzel 2012 / Lynch 2012):
$(R, \underline{m})$ local ring, $\dim R = d$, $H_I^d(R) \neq 0$. then

$$(0 :_{\hat{R}} H_I^d(R)) = \bigcap \{ Q \mid Q \text{ primary Component of } \hat{R},\ \dim \frac{\hat{R}}{Q} = d,\ \dim \frac{\hat{R}}{I + Q} = 0 \}$$

If R is unmixed, ~~(complete)~~ and $\sqrt{I} = \underline{m}$, then $(0 :_R H_I^d(R)) = 0$

(Huneke-Koh, 91) $(R, \underline{m})$ Noeth. Comm. ring regular Containing a field of positive characteristic.

$$H_I^i(R) \neq 0 \quad \text{iff.} \quad (0 :_R H_I^i(R)) = 0$$

(Lyubeznik, 93) $(R, \underline{m})$ Noeth. Comm. ring regular Containing a field of characteristic Zero.

$$H_I^i(R) \neq 0 \quad \text{iff.} \quad (0 : H_I^i(R)) = 0$$

(دو)

Question: $(R, \underline{m})$ Noeth. Comm. ring, $I \subset R$ an ideal.
Is it true that:

$$H^i_I(R) \neq 0 \text{ iff } (0 :_R H^i_I(R)) = 0 \text{ ?}$$

Ans: No, In general.

plan:

I) characteristic free proof of the results of Huneke-Koh and Lyubeznik.

II) Answer question in mixed characteristic case. (positive answer)

III) positive answer to the question in characteristic $p>0$, Case for Strongly F-regular domains.

IV) positive answer via flat endomorphism.

V) positive answer in Normal rings and Cohen-Macaulay Cases.

(سه)

0. Preliminaries.

Let A Comm. ring, B (not necessarily Comm) ring.
$A \xrightarrow{\phi} B$ ring homomorphism. M left B-mod.
and also is a left A-mod

Lemma 1:
(1) $(0 :_B M) = \{b \in B \mid bm = 0 \ \forall m \in M\}$ is a two sided ideal of B.
(2) $(0 :_A M) = (0 :_B B) \cap \phi^{-1}(B)$
(3) $(0 :_B M) = 0$ Then $(0 :_A M) = 0$.

Lemma 2: Let $A \longrightarrow A'$ faithfully flat homomorphism of Comm. Noeth. rings, I ideal of A, J' ideal of A'.
Suppose $(0 :_{A'} H^i_{IA'}(A')) \subseteq J'$, then $(0 :_A H^i_I(A)) \subseteq J' \cap A$;
in particular, $(0 :_{A'} H^i_{IA'}(A')) = 0$, then $(0 :_A H^i_I(A)) = 0$.

Thm 3. A Comm. ring, B (not necessarily Comm.) ring s.t. $A \subseteq B$, M left B-mod. If B is Simple then $(0 :_A M) = 0$.

Definition 4 K field. B a Noeth K-algebra.
B is geometrically regular, if for any finite field extension B is again regular.
Nt: K a perfect field regular rings over $K \equiv$ g. regular.

Def 5. A Comm ring R Commutative A-algebra
R is smooth A-algebra if R is a finitely presented, flat A-algebra, and $\forall \mathfrak{p} \in \operatorname{Spec}(A)$; $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is geometrically regular over $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$

Def. 6 R Comm. ring, $k \subset R$ a subring
A k-linear differential operator of order 0 is a map $\tilde{r}: R \longrightarrow R$, $x \longmapsto rx$. A k-linear diff operator of order $\leq n$ is a k-linear map $\delta: R \longrightarrow R$; s.t $\forall r \in R$, $\delta \circ \tilde{r} - \tilde{r} \circ \delta$ is a diff. operator of order $\leq n-1$.

$$D^n(R,k) = \{k\text{-linear diff operators of order } \leq n\}$$
(R-mod) $\delta \in D^n(R,k), \delta' \in D^m(R,k) \Rightarrow \delta \circ \delta' \in D^{n+m}(R,k)$

$D_k(R) := \bigcup_n D^n(R,k)$ ring of diff. operators.

1) $R \hookrightarrow D(R,k)$ is a subring
$r \longmapsto \tilde{r}: R \longrightarrow R$

2) R is a $D_k(R)$-mod. and is every localization of it

3) $H^i_I(R)$ is $D_k(R)$-mod.

Lemma 7. (B. Bhatt, M. Blickle, G. Lyubeznik, A.K. Singh, W. Zhang).
A Comm. ring, R is either a polynomial ring over A
or formal power Series ring over A
or Smooth A-algebra

Then $D_A(R) \otimes_A B \cong D_B(R \otimes_A B)$

I) Characteristic free proof.

Thm 8. K field. R Comm. Noeth. ring.
(Boix-Eghbali) R is either regular ring, affine over K or
formal power Series ring over K, or
R regular affinoid over K, K is a complete valued field

$\dfrac{K\{x_1, \ldots, x_n\}}{I}$

Then $D_K(R)$ is a Simple ring.

Thm 9 (Boix-Eghbali) K field. R Comm. Noeth. ring.
R is regular, affine or affinoid over K or
R regular local ring with K as residue field.
Then $H^i_I(R) \neq 0$ iff $(0 : H^i_I(R)) = 0$

proof: – for affine or affinoid use Thm 8, Thm 3, Lemma 1.
– for regular local ring; pass to Completion (Lemma 2)
use Cohen's structure Thm and Thm 8, Thm 3.

II) Mixed characteristic.

Def. 10. A char. zero ring is of mixed characteristic if for some ideal I of it, its quotient is of positive characteristic.

Exp. $\mathbb{Z}$

local PID of dim 1.

Thm 11 (Boix-Eghbali) Let (V, uV) be a DVR of mixed char. R a V-algebra is either smooth over V or a formal power series ring over V.

(i) If the multiplication by u, $H^i_I(R) \xrightarrow{\cdot u} H^i_I(R)$ is not injective, then $(0:_R H^i_I(R)) \subseteq uR$ and equality holds iff $\xrightarrow{\cdot u}$ is the zero map.

(ii) If the multiplication by u, $H^i_I(R) \xrightarrow{\cdot u} H^i_I(R)$ is not surjective, then $(0:_R H^i_I(R)) \subseteq uR$.

proof: $M = \ker u$ is non-zero left $D_V(R)$-mod. and so $\frac{D_V(R)}{uD_V(R)}$-mod.

$\xrightarrow[\text{Lemma 7}]{\text{by}} D_{\frac{V}{uV}}(\frac{R}{uR})$-mod. $\xRightarrow{\text{Thm 9}} D_{V/uV}(\frac{R}{uR})$ is simple (R/uR is smooth)

then $(0:_{R/uR} M) = 0$. Hence $(0:_R H^i_I(R)) \subseteq uR$

Def. 12. R Comm. $\mathbb{Z}$-algebra. R verifies star-assumption if for any infinite set of primes (integers) $\{p_i\}_{i\in I}$, one has $\bigcap_{i\in I} p_i R = (0)$

Exp. $\mathbb{Z}$, $\mathbb{Z}[x_1,\dots,x_n]$

Thm 13. Let R be either Smooth $\mathbb{Z}$-algebra or a formal power series ring over $\mathbb{Z}$, p prime number.

(i) If $H^i_I(R) \xrightarrow{\cdot p} H^i_I(R)$ is not injective then $(0 :_R H^i_I(R)) \subseteq pR$, equality $\Rightarrow \xrightarrow{\cdot p}$, zero map.

(ii) If $H^i_I(R) \xrightarrow{\cdot p} H^i_I(R)$ is not surjective, then $(0 : H^i_I(R)) \subseteq pR$

(iii) If R verifies star-assumption, there are infinitely many prime integers p s.t. mult by p on $H^i_I(R)$ is not surjective, then

$$H^i_I(R) \neq 0 \quad \text{iff} \quad (0 :_R H^i_I(R)) = 0.$$

Exp. E = elliptic curve in $\mathbb{P}^2_{\mathbb{Q}}$, Consider Segre embed. $E\times\mathbb{P}^1_{\mathbb{Q}}$ in $\mathbb{P}^5_{\mathbb{Q}}$

I = lifting ideal of an ideal $\underline{a}$ in $R = \mathbb{Z}[x_0,\dots,x_5]$. i.e.

$$\operatorname{proj}\left(\frac{R}{\underline{a}} \otimes_{\mathbb{Z}} \mathbb{Q}\right) = E\times\mathbb{P}^1_{\mathbb{Q}}.$$

By Hartshorne-Speiser and Lyubeznik: $\begin{cases} H^4_I\left(\frac{R}{pR}\right) = 0 & \text{for inf. } p \\ \text{and} & \\ H^4_I\left(\frac{R}{pR}\right) \neq 0 & \text{for inf. } p. \end{cases}$

So $H^4_I(R) \xrightarrow{p} H^4_I(R)$ is not surjective for inf. p.

(هشت)

III) Annihilator via flat endomorphism + Normal rings.

R Comm. ring ~~Cont. 2 & 2~~ Noeth. with a flat endomorphism $\varphi: R \to R$.

$\varphi_* R$ = An (R,R)-bimodule $\forall r, r_1, r_2 \in R$; ($M$ R-mod)

$$r_1 \cdot (\varphi_* r) \cdot r_2 := \varphi_*(\varphi(r_1) r r_2)$$

$\Phi(-)$ functor of R-Mods. $\Phi(M) := \varphi_* R \otimes_R M$

$\Phi^t(M) := \varphi_* R \otimes_R \Phi^{t-1}(M)$ $t \geq 1$

Singh-Walter: $\Phi(H^i_I(R)) \cong H^i_{\varphi(I)}(R) \cong H^i_I(R)$ $\forall i \geq 0$, whenever $\underbrace{\{\varphi^t(I)R\}_t}_{\text{decreasing chain}}$ cofinal with $\{I^t\}_t$.

EXP: 1) Frobenius endomorphism in positive charact.

2) $\varphi(x_i) = x_i^t$ of $K[x_1, \dots, x_n]$ examples of φ.

Thm 14. (Boix-Eghbali) R Comm. Noeth. ring. either $\left\langle \begin{array}{l} \text{domain} \\ \text{local} \end{array} \right.$ or. $\varphi: R \to R$ flat, $\forall J$ ideal of R, $\varphi^t(J)R \subseteq J^t$ $\forall t \geq$,

$\{\varphi^t(J)R\}_t$ decereasing chain of ideals cofinal with $\{J^t\}_t$.

$$H^i_I(R) \neq 0 \text{ iff } (0 : H^i_I(R)) = 0$$

proof. $0 \neq x \in H^i_I(R) \neq 0$, $J := (0 :_R x)$ $\frac{R}{J} \hookrightarrow H^i_I(R)$. Apply Φ^t

$$\Phi^t(R/J) \cong \frac{R}{\varphi^t(J)R} \longrightarrow \Phi^t(H^i_I(R)) \cong H^i_I(R).$$

$$\xrightarrow[\text{intersection}]{\text{Krull's}} (0 : H^i_I(R)) \subset \bigcap_{t \geq} \varphi^t(J)R \subseteq \bigcap J^t = 0$$

prop. 15 (Boix–Eghbali) R Noeth. normal reduced ring
$(0 : H^i_I(R/\mathfrak{p})) = 0$ for at least one minimal prime $\mathfrak{p}$ of R, then
$(0 : H^i_I(R)) = 0$.

V) C-M Case.

Thm 16 (Lynch 2012): R complete Cohen-MaCaulay ring, UFD, $\dim R \leq 4$.

$$H^i_I(R) \neq 0 \iff (0 :_R H^i_I(R)) = 0$$

Def 17. R Comm. ring, Noeth. $x_1, \ldots, x_d \in R$ an R-regular sequence contained in Jacobson radical of R. $r \in R$ is a monomial w.r.t. $x_1, \ldots, x_d$ if $r = x^{\alpha} := x_1^{a_1} \cdots x_d^{a_d}$ $\exists\, \alpha = (a_1, \ldots, a_d) \in \mathbb{N}^d$.
$I \trianglelefteq R$ is a mon. ideal w.r.t. $x_1, \ldots, x_d$ if it is generated by monomials

Thm 18 (Boix–Eghbali) R C-M local containing a field K.
I monomial ideal w.r.t. a system of parameter for R. Then
$H^i_I(R) \neq 0$ iff $(0 : H^i_I(R)) \cap A = 0$; $A = K[X_1, \ldots, X_d]$.

proof: $x_1, \ldots, x_d$ s.o.p. for R, I mon. ideal w.r.t. $x_1, \ldots, x_d$.

① K-alg. hom. $A : K[X_1, \ldots, X_d] \xrightarrow{\phi} R$ is flat.
$$X_j \longmapsto x_j$$

② $A \xrightarrow{\phi} R$
$\phi^{-1}(I) = J \leftarrow I$
monomial

③ $H^i_I(R) \cong H^i_J(A)$ A-mods

④ $(0 :_A H^i_J(A)) = (0 :_R H^i_I(R)) \cap A$