Monomial curves of homogeneous type

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- Let $\underline{\mathbf{n}} := 0 < n_1 < \cdots < n_d$ be a family of positive integers.
- Let $S = \langle n_1, \ldots, n_d \rangle = \{r_1 n_1 + \cdots + r_d n_d; r_i \geq 0\}$ be the semigroup generated by the family $\underline{\mathbf{n}}$.
- Let $\mathbb{k}$ be a field and $\mathbb{k}[S] = \mathbb{k}[t^{n_1}, \ldots, t^{n_d}] \subseteq \mathbb{k}[t]$ be the semigroup ring defined by $\underline{\mathbf{n}}$.

We may consider the presentation

$$0 \longrightarrow I(S) \longrightarrow \mathbb{k}[x_1, \ldots, x_d] \xrightarrow{\varphi} \mathbb{k}[S] \longrightarrow 0$$

given by $\varphi(x_i) = t^{n_i}$.

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- Set $R := \mathbb{k}[x_1, \ldots, x_d]$.

For any $i \geq 0$, the i-th (total) Betti number of $I(S)$ is

$$\beta_i(I(S)) = \dim_{\mathbb{k}} \mathrm{Tor}_i^R(I(S), \mathbb{k}).$$

-We call the Betti numbers of $I(S)$ as the Betti numbers of S.

For any $j \geq 0$ we consider the shifted family

$$\underline{\mathbf{n}} + j := 0 < n_1 + j < \cdots < n_d + j$$

and the semigroup

$$S + j := \langle n_1 + j, \ldots, n_d + j \rangle$$

that we call the j-th shifting of S.

Conjecture (J. Herzog and H. Srinivasan)

The Betti numbers of $S + j$ are eventually peridic on j with period $n_d - n_1$.

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The Betti numbers of $S + j$ are eventually peridic on j with period $n_d - n_1$.

- Let $S = \langle 4, 7 \rangle$. Then $S + 2 = \langle 6, 9 \rangle$ is not a numerical semigroup ($\gcd(6, 9) > 1$).
- Let $S = \langle 4, 10, 11 \rangle$, then $S + 2 = \langle 6, 12, 13 \rangle = \langle 6, 13 \rangle$.

If $j > n_d - 2n_1$, then $S + j$ is minimally generated by d elements.

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If $j > n_d - 2n_1$, then $S + j$ is minimally generated by d elements.

The conjecture has been proven to be true for:

- $d = 3$ (A. V. Jayanthan and H. Srinivasan, 2013).
- d=4 (particular cases, A. Marzullo, 2013).
- Arithmetic sequences, $n_i - n_{i-1} = n_{i+1} - n_i$, (P. Gimenez, I. Senegupta, and H. Srinivasan, 2013).

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Remark

- the bound N depends on the Castelnuovo-Mumford regularity of $J(S)$, the ideal generated by the homogeneous elements in $I(S)$.

-The proof of Vu is based on a careful study of the simplicial complex defined byA. Campillo and C. Marijuan, 1991 (later extended by J. Herzog and W. Bruns, 1997) whose homology provides the Betti numbers of the defining ideal of a monomial curve.

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The other main ingredient of the proof by Vu is the following technical result:

Theorem

There exists an integer N such that for all $j > N$, any minimal binomial inhomogeneous generator of $I(S)$ is of the form

$$x_1^{\alpha} u - v x_d^{\beta}$$

where $\alpha, \beta > 0$, and where u and v are monomials in the variables $x_2, \ldots, x_{d-1}$ with

$$\deg x_1^{\alpha} u > \deg v x_d^{\beta}.$$

Let $I^*(S)$ be the initial ideal of $I(S)$, that is, the ideal generated by the initial forms of the elements of $I(S)$.
$I^*(S) \subseteq \mathbb{k}[x_1, \ldots, x_d]$ is an homogeneous ideal. It is the definition ideal of the tangent cone of S:

$$G(S) := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1},$$

$\mathfrak{m} = (x_1, \ldots, x_d)$.

Turning around the above result by Vu, J. Herzog and D. I. Stamate, 2014, have shown that for any $j > N$,

$$\beta_i(I(S+j)) = \beta_i(I^*(S+j)) \text{ for all } i \geq 0$$

In particular, for any $j > N$, $G(S)$ is Cohen-Macaulay.

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In particular, for any $j > N$, $G(S)$ is Cohen-Macaulay.

The condition $\beta_i(I(S+j)) = \beta_i(I^*(S+j))$ for all $i \geq 0$ corresponds to the definition of varieties of homogeneous type.

So what Herzog-Stamate have shown is that for a given monomial curve defined by a numerical semigroup S, all the monomial curves defined by $S+j$ are of homogeneous type for $j \gg 0$.

Homogeneous semigroups and semigroups of homogeneous type

For each vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}_0^d$, let

$$s(\mathbf{a}) := \sum_{i=1}^{d} a_i n_i \in S.$$

For each element $s = \sum_{i=1}^{d} a_i n_i$, the vector $\mathbf{a}$ is called a factorization of s and the set of all factorizations of s is denoted by $\mathcal{F}(s)$.

- $I(S)$ is a binomial ideal.
- $x^{\mathbf{a}} - x^{\mathbf{b}} \in I(S)$ if and only if $s(\mathbf{a}) = s(\mathbf{b})$.

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For an element $s \in S$, the Apéry set of S with respect to s is defined as
$$\mathrm{AP}(S, s) = \{x \in S \mid x - s \notin S\}.$$

$s \in \mathrm{AP}(S, n_i)$ if and only if $a_i = 0$ for all $\mathbf{a} = (a_1, \ldots, a_d) \in \mathcal{F}(s)$

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Given $0 \neq s \in S$, the set of lengths of s in S is defined as

$$\mathcal{L}(s) = \{\sum_{i=1}^{d} r_i \mid s = \sum_{i=1}^{d} r_i n_i, r_i \geq 0\}.$$

Definition

A subset $T \subset S$ is called homogeneous if either it is empty or $\mathcal{L}(s)$ is singleton for all $0 \neq s \in T$.

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The numerical semigroup S is called homogeneous, when the Apéry set $\mathrm{AP}(S, n_1)$ is homogeneous.

In each of the following cases, S is homogeneous.

- $d = 2$, then $\mathrm{AP}(S, n_1) = \{0, n_2, \ldots, (n_1 - 1)n_2\}$.
- $d = e$ (maximal embedding dimension) or $d = e - 1$ (almost maximal embedding dimension).
- S is minimally generated by a generalized arithmetic sequence n_0, $n_i = hn_0 + it$, where t and h are positive integers, $\gcd(n_0, t) = 1$, $i = 1, \ldots, d$.
- Let $b > a > 3$ be coprime integers. Then, the semigroup

$$H_{a,b} = \langle a, b, ab - a - b\rangle$$

is a Frobenius semigroup (it is obtained from $\langle a, b\rangle$ by adding its Frobenius number). Then, $H_{a,b}$ is homogeneous.

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(On can see that in this case, the tangent cone $G(H_{a,b})$ is never Cohen-Macaulay.)

AP(S, n) is homogeneous for some $n \in S$, precisely when the binomials

$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I(S) \text{ with } s(\mathbf{a}) = s(\mathbf{b}) \in \mathrm{AP}(S, n),$$

are homogeneous in standard grading of the polynomial ring.

A family of elements of $I(S)$ such that their initial forms generate $I^*(S)$ is called a standard basis.

Any standard basis is system of generators of $I(S)$ (but not conversely).

And finding minimal systems of generators of $I(S)$ which are also a standard basis is not easy.

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Proposition

The following statements are equivalent.

1. S is homogeneous and $G(S)$ is Cohen-Macaulay.
2. There exists a minimal set of binomial generators E for $I(S)$ such that for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.
3. There exists a minimal set of binomial generators E for $I(S)$ which is a standard basis and for all $x^{\mathbf{a}} - x^{\mathbf{b}} \in E$ with $|\mathbf{a}| > |\mathbf{b}|$, we have $a_1 \neq 0$.
4. There exists a minimal Gröbner basis G for $I(S)$ with respect to $<_{ds}$, such that x_1 belongs to the support of all non-homogeneous elements of G and x_1 does not divide $\mathrm{lm}_{<_{ds}}(f)$, for all $f \in G$.

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Example

Let $S = \langle 8, 10, 12, 25\rangle$. Then

$$\mathrm{AP}(S, 8) = \{25, 10, 35, 12, 37, 22, 47\},$$

$$G_1 = \{x_1^3 - x_3^2, x_2^5 - x_4^2, x_1x_3 - x_2^2\}$$

is a minimal generating set for $I(S)$. We can change $x_2^5 - x_4^2$ by the two binomials $x_1x_2^3x_3 - x_2^5$ and $x_1x_2^3x_3 - x_4^2$. Then, the set

$$G_2 = \{x_1^3 - x_3^2, x_1x_2^3x_3 - x_2^5, x_1x_2^3x_3 - x_4^2, x_1x_3 - x_2^2\}$$

is a generating set. Removing the superfluous generator $x_1x_2^3x_3 - x_2^5$ we get the minimal generating set

$$G_3 = \{x_1^3 - x_3^2, x_1x_2^3x_3 - x_4^2, x_1x_3 - x_2^2\}$$

that satisfies the properties (3) and (5).

- Let $I^*(S)$ be the initial ideal of $I(S)$ i.e. the ideal generated by the initial forms of the elements of $I(S)$.
- $I^*(S) \subset K[x_1, \cdots, x_d]$ is an homogeneous ideal. It is the definition ideal of the tangent cone $G(S)$.

By a general result due to Robbiano, for all $i \geq 0$

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Let S be a homogeneous numerical semigroup with Cohen-Macaulay tangent cone. Then S is of homogeneous type.

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Assume that $G(S)$ is a complete intersection. Then S is also a complete intersection and both S and $G(S)$ have the same number of minimal generators. So we have that S is of homogeneous type.

Example

Let $S := \langle 15, 21, 28 \rangle$. Then

$$I(S) = (x_2^4 - x_3^3, x_1^7 - x_2^5)$$

is minimally generated by a standard basis of two elements. Hence $G(S)$ is complete intersection and so S is of homogeneous type, but it is not homogeneous, since

$$3 \times 28 = 4 \times 21 = 84 \in \mathrm{AP}(S, 15).$$

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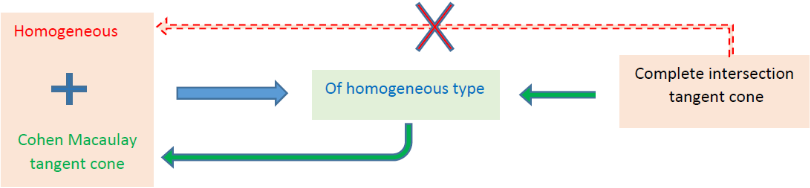
Homogeneous
+
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Of homogeneous type
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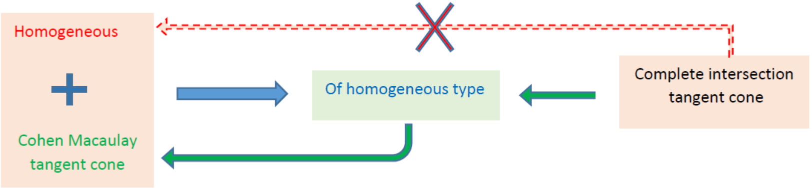

Of homogeneous type with d=3 = Homogeneous with Cohen Macaulay tangent cone

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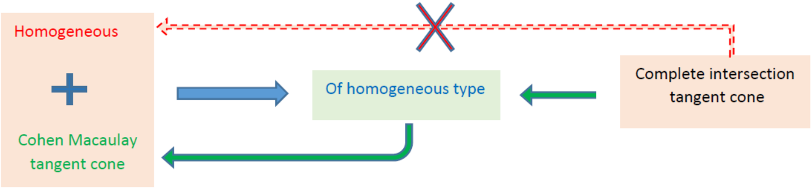

Of homogeneous type with d=3 = Homogeneous with Cohen Macaulay tangent cone or Complete intersection tangent cone

Theorem

Let S be a numerical semigroup with $d = 3$. Then TFAE

1. S is of homogeneous type.
2. $\beta_0(I(S)) = \beta_0(I^*(S))$.
3. $G(S)$ is Cohen-Macaulay, and either S is homogeneous or $I^*(S)$ is generated by pure powers of x_2 and x_3.
4. Either S is homogeneous with Cohen-Macaulay tangent cone, or $G(S)$ is complete intersection.

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Let S be a numerical semigroup with $d = 4$. Then TFAE

1. $\mathrm{AP}(S, n_i)$ is homogeneous.
2. $\{c_j n_j \mid j \neq i\} \cap \mathrm{AP}(S, n_i)$ is a homogeneous set.

$$c_i = \min\{r \geq 1 \ ; \ rn_i \in \langle n_1, \ldots, \widehat{n_i}, \ldots, n_d \rangle\}$$

Corollary

Let S be a numerical semigroup with $d = 4$. Then TFAE

1. S is homogeneous.
2. $c_2 n_2$ and $c_4 n_4$ are not in $\mathrm{AP}(S, n_1)$ and, if $c_3 n_3 \in \mathrm{AP}(S, n_1)$, then $\{c_3 n_3\}$ is homogeneous.

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Asymptotic behavior under shifting

- Let $m_i := n_d - n_i$, for all $1 \leq i \leq d$.
- Let $g := gcd(m_1, \ldots, m_{d-1})$ and $T := \langle \frac{m_1}{g}, \ldots, \frac{m_{d-1}}{g} \rangle$.
- Let

$$L := m_1 m_2 (\frac{gc + dm_1}{m_{d-1}} + d) - n_d$$

where c is the conductor of T.

Proposition

Let $j > L$ and $s \in S + j$. If $\mathbf{a}, \mathbf{a}'$ are two factorizations of s with $|\mathbf{a}| > |\mathbf{a}'|$, then there exists a factorization $\mathbf{b}$ of s such that $|\mathbf{a}| = |\mathbf{b}|$ and $b_1 \neq 0$.

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Corollary

For any $j > L$, the j-th shifted numerical semigroup $S + j$ is homogeneous and $G(S + j)$ is Cohen-Macaulay. In particular, $S + j$ is of homogeneous type.

Proof:

Take E any system of binomials generators of $I(S + j)$. By the previous Proposition, for any binomial $x^{\mathbf{a}} - x^{\mathbf{a}'} \in E$ such that $|\mathbf{a}| > |\mathbf{a}'|$, there exists a binomial $x^{\mathbf{a}} - x^{\mathbf{b}}$ such that $|\mathbf{a}| = |\mathbf{b}| > |\mathbf{a}'|$ and $b_1 \neq 0$. Then, substituting $x^{\mathbf{a}} - x^{\mathbf{a}'}$ by $x^{\mathbf{a}} - x^{\mathbf{b}}$ and $x^{\mathbf{b}} - x^{\mathbf{a}'}$ and then refining to a minimal system of generators, we get that $S + j$ fulfills condition (2) in the first Proposition and so we are done.

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Remark:

The bound L is not optimal.

For instance, for a given numerical semigroup:

$$S_k = \langle k, k+a, k+b \rangle$$

D. Stamate, 2015, has found the bound

$$k_{a,b} = \max\{b(\frac{b-a}{g} - 1), b\frac{a}{g}\}$$

such that S_k is of homogeneous type if $k > k_{ab}$. Compared with ours, this is a better bound.

Now, we may consider the differences $s_i = n_d - n_{d-i}$ for all $1 \leq \cdots \leq i \leq \cdots \leq d-1$.

Then, the sequence of integers **n** only depends on these differences and n_1.

We call these differences the shifting type of **n**.

Taking $n_1 = 1$ we obtain the sequence with smallest n_1 among those with the same shifting type. In this case, the bound L only depends on the shifting type.

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Hence, for any numerical semigroup S with this shifting type and multiplicity $e > L$, S is homogeneous and $G(S)$ is Cohen-Macaulay.

On the other hand, the width of a numerical semigroup S is defined as the difference $\mathrm{wd}(S) = n_d - n_1$.

It is clear that for a given width, there only exist a finite number of possible shifting types for a numerical semigroup having this width. So we may conclude that:

Proposition

Let $w \geq 2$. Then, there exists a positive integer W such that all numerical semigroups S with $wd(S) \leq w$ and multiplicity $e \geq W$, are homogeneous and $G(S)$ is Cohen-Macaulay.

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SOME REFERENCES

- P. Gimenez, I. Sengupta, and H. Srinivasan, Minimal graded free resolutions for monomial curves defined by arithmetical sequences, J. Algebra **388** (2013), 249–310.
- J. Herzog and D.I. Stamate, On the defining equations of the tangent cone of a numerical semigroup ring. J. Algebra **418** (2014), 8–28.
- A. V. Jayanthan and H. Srinivasan, Periodic occurrence of complete intersection monomial curves, Proc. Amer. Math. Soc. **141** (2013), 4199-4208.
- T. Vu, Periodicity of Betti numbers of monomial curves. J. Algebra **418** (2014), 66–90.