FALTINGS' LOCAL-GLOBAL PRINCIPLE FOR THE FINITENESS OF LOCAL COHOMOLOGY MODULES

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This talk will describe some interesting results on the Faltings' local-global principle for the finiteness of local cohomology modules over commutative Noetherian rings. Let R denote a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module. An important theorem in local cohomology is G. Faltings' Local-global Principle for the finiteness of local cohomology modules $H^i_{\mathfrak{a}}(M)$ (cf. [5, Satz 1]), which states that *for a positive integer r, the $R_{\mathfrak{p}}$-module $H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is finitely generated for all $i \leq r$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$ if and only if the R-module $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i \leq r$.*

Another formulation of the Faltings' local-global principle is in terms of the *finiteness dimension*, $f_{\mathfrak{a}}(M)$, of M relative to $\mathfrak{a}$, where

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\},$$

with the usual convention that the infimum of the empty set of integers is interpreted as ∞. We can restate Faltings' local-global principle in the form

$$f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}. \qquad (\dagger)$$

Now, let $\mathfrak{b}$ be a second ideal of R. Recall that the $\mathfrak{b}$-*finiteness dimension* of M relative to $\mathfrak{a}$ is defined by

$$\begin{aligned} f^{\mathfrak{b}}_{\mathfrak{a}}(M) &:= \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b} \not\subseteq \operatorname{Rad}(0 :_R H^i_{\mathfrak{a}}(M))\} \\ &= \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b}^n H^i_{\mathfrak{a}}(M) \neq 0 \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

So it is rather to ask whether Faltings' local-global principle, as stated in ($\dagger$), generalizes in the obvious way to the invariants $f^{\mathfrak{b}}_{\mathfrak{a}}(M)$. In other words, is the statement

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}, \qquad (\dagger\dagger)$$

true?

Brodmann et al. in [3, Corollary 3.12] proved that the statement ($\dagger\dagger$) holds, whenever $\dim R \leq 4$ (see also [4, Corollary 2.13]).

The first our main result is that the statement ($\dagger\dagger$) holds, when the set $\operatorname{Ass}_R(H^{f^{\mathfrak{b}}_{\mathfrak{a}}(M)}_{\mathfrak{a}}(M))$ is finite or $f_{\mathfrak{a}}(M) \neq c^{\mathfrak{b}}_{\mathfrak{a}}(M)$, where $c^{\mathfrak{b}}_{\mathfrak{a}}(M)$ denotes the first non $\mathfrak{b}$-cofiniteness of local cohomology module $H^i_{\mathfrak{a}}(M)$. As a consequence of this, we provide a short proof of the Faltings' local-global principle for finiteness dimensions. More precisely, we shall show that:

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Theorem 1.1. (cf. [1].) *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Let M be a finitely generated R-module such that the set* $\operatorname{Ass}_R(H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M)}(M))$ *is finite or* $f_{\mathfrak{a}}(M) \neq c_{\mathfrak{a}}^{\mathfrak{b}}(M)$. *Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})|\ \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

In particular,

$$f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})|\ \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Our method is based on the nth $\mathfrak{b}$-finiteness dimension of M relative to $\mathfrak{a}$ (resp. the nth $\mathfrak{b}$-minimum $\mathfrak{a}$-adjusted depth of M)

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)_n = \inf\{i \in \mathbb{N}_0|\ \dim \operatorname{Supp} \mathfrak{b}^t H_{\mathfrak{a}}^i(M) \geq n \text{ for all } t \in \mathbb{N}_0\}.$$

(resp. $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)_n = \inf\{\lambda_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})|\ \mathfrak{p} \in \operatorname{Spec}(R), \dim R/\mathfrak{p} \geq n\}$), and the upper nth $\mathfrak{b}$-finiteness dimension of M relative to $\mathfrak{a}$

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M)^n = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})|\ \mathfrak{p} \in \operatorname{Spec}(R), \dim R/\mathfrak{p} \geq n\}.$$

Note that

$$f_{\mathfrak{a}}^{\mathfrak{a}}(M)^n = f_{\mathfrak{a}}^n(M) := \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}(M/\mathfrak{a}M) \text{ and } \dim R/\mathfrak{p} \geq n\},$$

is the nth finiteness dimension of M relative to $\mathfrak{a}$ and $f_{\mathfrak{a}}^0(M) = f_{\mathfrak{a}}(M)$ (cf. [2]).

By a *weakly Laskerian* module, we mean an R-module M such that the set $\operatorname{Ass}_R M/N$ is finite, for each submodule N of M. Also, an R-module M is said to be *minimax*, if there exists a finitely generated submodule N of M, such that M/N is Artinian. The second our main result is the following:

Theorem 1.2. *Let $\mathfrak{a}$ be an ideal of R and let M be a finitely generated R-module. Then*

(i) $f_{\mathfrak{a}}^n(M) = \inf\{0 \leq i \in \mathbb{Z}|\ \dim H_{\mathfrak{a}}^i(M)/N \geq n$ *for any finite submodule N of* $H_{\mathfrak{a}}^i(M)\}$. *As a consequence, it follows that the set*

$$\operatorname{Ass}_R(\oplus_{i=0}^{f_{\mathfrak{a}}^n(M)} H_{\mathfrak{a}}^i(M)) \cap \{\mathfrak{p} \in \operatorname{Spec}(R)|\ \dim R/\mathfrak{p} \geq n\}$$

is finite.

(ii) *The Bass numbers $\mu^j(\mathfrak{p}, H_{\mathfrak{a}}^i(M))$ are finite for all $0 \leq i \leq f_{\mathfrak{a}}^n(M)-1$ and all integers j, where $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} \geq n$.*

(iii) *The Bass numbers $\mu^j(\mathfrak{p}, H_{\mathfrak{a}}^{f_{\mathfrak{a}}^n(M)}(M))$ are finite for $j = 0, 1$, where $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} \geq n$.*

(iii) $f_{\mathfrak{a}}^1(M) = \inf\{i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(M)$ is not minimax$\}$.

(iv) $f_{\mathfrak{a}}^2(M) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M)$ is not weakly Laskerian$\}$, *whenever R is semi-local.*

(v) *The R-modules $H_{\mathfrak{a}}^i(M)$ are $\mathfrak{a}$-cofinite for all $i < f_{\mathfrak{a}}^2(M)$ and for all minimax submodules N of $H_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M)}(M)$, the R-modules*

$$\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M)}(M)/N) \text{ and } \operatorname{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^{f_I^2(M)}(M)/N)$$

are finitely generated.

(vi) *If $\mathfrak{a}$ has dimension one, then the R-modules $H_{\mathfrak{a}}^i(M)$ are $\mathfrak{a}$-cofinite for all i.*

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