Prime ideals in rings of power series & polynomials

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(work of W. Heinzer, C. Rotthaus, SW &
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Question 2 What is Spec(R) for a two-dimensional Noetherian domain R? What is Spec(R) for a particular ring R ?

Question 3 What is Spec(R) for a two-dimensional Noetherian polynomial ring R? Or a ring of power series? Or homomorphic image of a ring of polynomials and power series?

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Setting: $R =$ 1-dim Noetherian domain, $k =$ field; eg. $R = \mathbb{Z}$, $k = \mathbb{Q}$.
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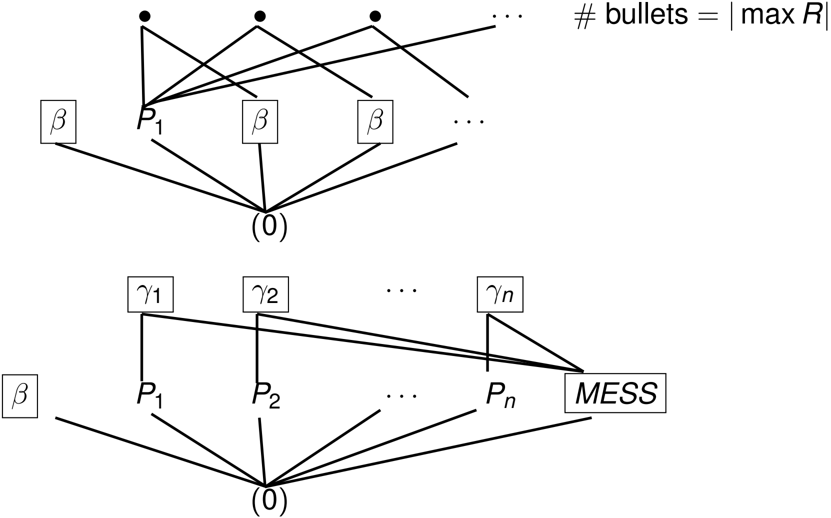
- Spec($R[y]$) for $R = D[g/f]$ and $D =$ order in algebraic number field:

Spec($R[y]$) $\cong$ Spec($\mathbb{Z}[y]$) [rW; Li, sW; Saydam, sW].
For many other R, Spec($R[y]$) is unknown.

Spec($R[y]$) for R semilocal; Henselian, non-Hens.



2-dim. polynomial-power series rings, cont.

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So we turn to the rest—rings with power series in them. MUCH easier!!

What is $Spec(\mathbb{Z}[[x]])$??

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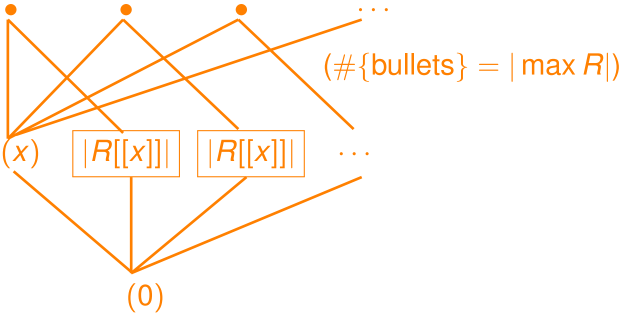

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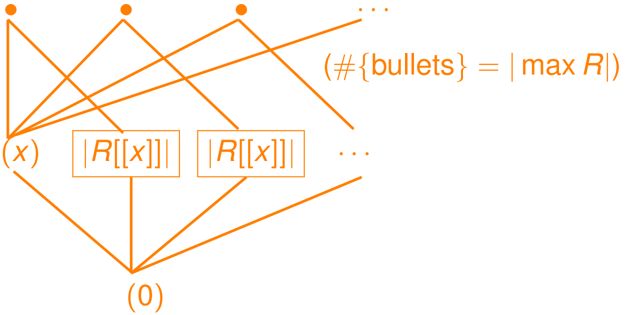

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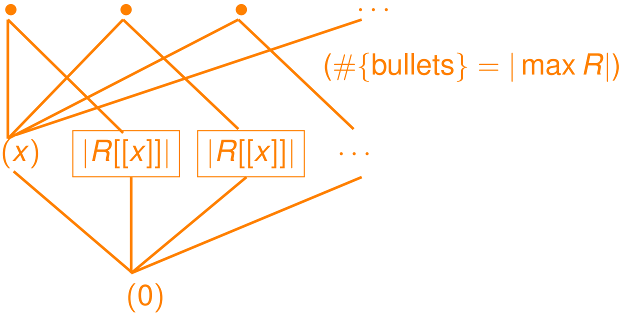

$R = \mathbb{Z} \Longrightarrow |\max \mathbb{Z}| = |\mathbb{Z}|, \quad |\mathbb{Z}[[x]]| = |\mathbb{R}|.$

Corollary: $\text{Spec}(\mathbb{Q}[y][[x]]) \cong \text{Spec}(\mathbb{Z}[[x]])$.

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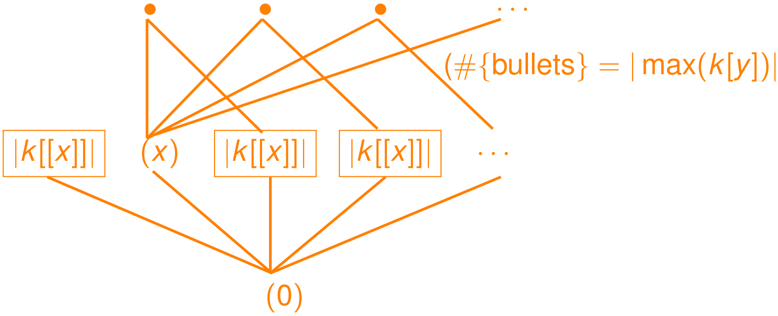

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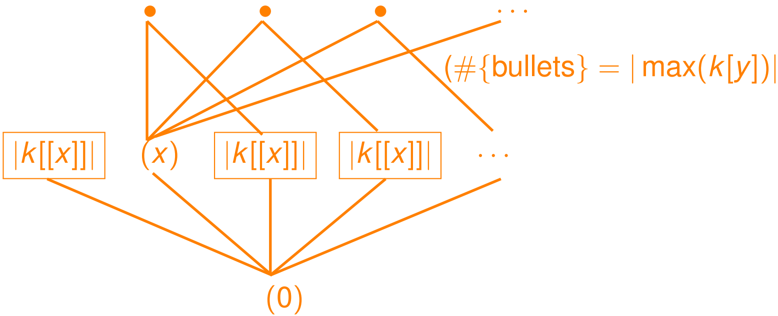

Spec($k[[x]][y]$), for k a field.

Thus Spec($\mathbb{Q}[[x]][y]$) is just like Spec($\mathbb{Q}[y][[x]]$), except that it has an “arm" sticking out on the left.

Part II: Spec(E/Q), for E poly-power series

Let $E = k[[x]][y, z]$, $R[[x]][y]$, $R[y][[x]]$, or $R[[x, y]]$,
Here $k =$ a field or $R =$ a 1-dim Noetherian integral domain, and
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Notice that these rings are catenary, and Noetherian.

A ring A is catenary provided for every pair $P \subsetneq Q$ in Spec(A), the number of prime ideals in every maximal chain of form

$$P = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_n = Q \text{ is the same.}$$

Question: What is Spec($R[\![x,y]\!]/Q$)? ($Q = (x)$ is okay.)

Spec($R[[x, y]]/Q$) $Q = (x)$ is okay in (case i

Theorem [CEW, Theorem 4.1] $R =$ 1-dim Noetherian domain, $Q \in \text{Spec}(R[\![x, y]\!])$, ht $Q = 1$. Set $B = R[\![x, y]\!]/Q$. Then:

Case i: $Q \nsubseteq (x, y)R[\![x, y]\!] \implies \exists n \in \mathcal{N}, \mathfrak{m}_1, \ldots, \mathfrak{m}_n \in$ max(R) and Spec(B) is:

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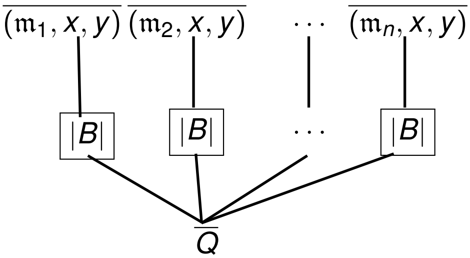
Case i: $Q \not\subseteq (x,y)R[\![x,y]\!] \implies \exists n \in \mathcal{N}, \mathfrak{m}_1, \ldots, \mathfrak{m}_n \in \max(R)$ and Spec(B) is:



OR

Spec($R[[x, y]]/Q$) case ii

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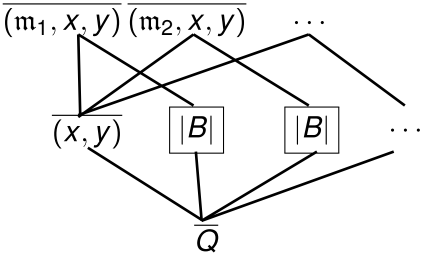

where the $\mathfrak{m}_i$ range over all the elements of max(R).

The diagrams show Spec($R[[x,y]]/Q$) is characterized for each case.

Spec(E/Q), in the dim 1 case.

Here $E = k[[x]][y,z]$, $R[[x]][y]$, or $R[y][[x]]$ a mixed poly-power series, where $k =$ a field or $R =$ a 1-dim Noetherian integral domain, and $Q \in \operatorname{Spec} E$, $\operatorname{ht} Q = 1$, $Q \neq xE$.

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Case i: $\dim(E/Q) = 1$. Then $\dim(\operatorname{Spec}(E/Q)) = 1$, $\implies$ a “fan".

(This case occurs if $Q \mid (\mathfrak{m}, Q)E \in \max(E)$ or $= E, \forall \mathfrak{m} \in \max R$— Not for $E = R[y][[x]]$.)

eg. $Q = (2xy + 1) \subseteq \mathbb{Z}[[x]][y] \implies \operatorname{Spec}(E/Q)$ is:

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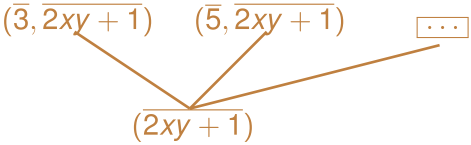


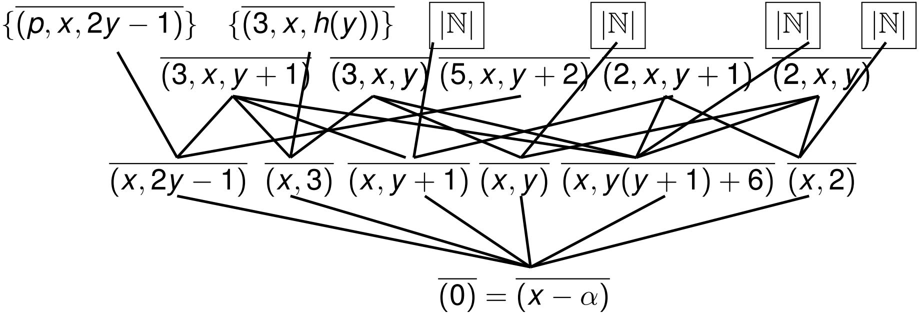
$\operatorname{Spec}(\, \mathbb{Z}[[x]][y]/(2xy+1)\,)$

Note: The *hgt*-3 max ideals of $\mathbb{Z}[[x]][y]$ are $\{(p, x, h_i(y))\}$, where p

Spec($R[y][[x]]/Q$) with R countable, max(R) infinite.

(dim 2 always for this case)

e.g. $E = \mathbb{Z}[y][[x]]$, $Q = (x - \alpha)$,
$\alpha = 2 \cdot 3 \cdot y \cdot (2y - 1) \cdot (y + 1) \cdot (y(y + 1) + 6)$.



Note; Every height-two element has a set of $|\mathbb{R}|$ elements below it and below no other height-two element (not shown).

Features of $U = \text{Spec}(R[y][[x]]/Q)$ if R, Q as above.

Theorem: If $U = \text{Spec}(R[y][[x]]/Q)$, where R a countable 1-dim Noetherian domain, $Q \in \text{Spec}(R[y][[x]])$, ht $Q = 1$, $Q \neq (x)$, then

1. U has a unique minimal element, $|U| = |\mathbb{R}|$, dim $U = 2$.
2. $\forall t \in U, \text{ht}\, t = 2 \implies |t^{\downarrow,e}| = |\mathbb{R}|$.
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   (c) $\forall f \neq g \in F_0, |f^{\uparrow} \cap g^{\uparrow}| < \infty$.
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Define $F := (\bigcup_{f \neq g \in F_0} f^{\uparrow} \cap g^{\uparrow}) \cup F_0$, a finite set by item c.
Then F determines U.

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Theorem: For every finite poset F of dim 1, $\exists Q \in \mathrm{Spec}(Z[y][[x]])$ such that F “determines" $\mathrm{Spec}(\mathbb{Z}[y][[x]]/Q)$.
(Want every ht-1 element of F above 2 ht-0 elements of F.)

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3. For $E = k[[x]][y][z]$, $\#\{$ height-one maximal ideals of $E/Q\} = |k[[x]]|$. So again, k countable $\implies$ Spec(E/Q) is determined

Part III. (from [HRW]) Noetherian and Non-Noetherian Examples. Let $R = k[x,y]_{(x,y)}$ or $R = k[x,y,z]_{(x,y,z)}$. $\therefore \mathcal{Q}(R) = k(x,y)$ or $k(x,y,z)$. Take $R^* = k[y]_{(y)}[[x]]$ or $k[y,z]_{(y,z)}[[x]]$.

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a) "Intersection Domain" $\boxed{A = K \cap R^*}$, $K =$ a field $\subseteq \mathcal{Q}(R^*)$. Take $K = \mathcal{Q}(R)(f) \subseteq \mathcal{Q}(R^*)$, where $f \in xR^*$ are algebraically independent over $\mathcal{Q}(R)$.

Spectra for $A = K \cap R^*$, K a field, $R^* =$ power series

Part III. (from [HRW]) Noetherian and Non-Noetherian Examples. Let $R = k[x,y]_{(x,y)}$ or $R = k[x,y,z]_{(x,y,z)}$. $\therefore \mathcal{Q}(R) = k(x,y)$ or $k(x,y,z)$. Take $R^* = k[y]_{(y)}[[x]]$ or $k[y,z]_{(y,z)}[[x]]$.

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b). "Approximation domain" B (approximating A) formed by adding "endpieces" of f to R. (Described below.)

Notes 1. B is Noetherian $\iff$ ($B = A$ and A is Noetherian.)

2. Sometimes A is Noetherian, but B is not.

3. If B is Noetherian then B is catenary, in fact universally catenary. (For catenary, non-universally catenary Noetherian examples, we use another version of the construction given above.)

$yz\tau$ Example:

Let $R = k[x,y,z]_{(x,y,z)} \implies R^* = k[y,z]_{(y,z)}[[x]]$. Choose $\tau \in xk[[x]]$, algebraically independent over $k(x)$.

The $yz\tau$ Example: Let $f = yz\tau \in xR^*$. Then $f = \sum_{i=1}^{\infty} a_i x^i$, $a_i \in yzk$
For every $n \in \mathbb{N}$, the n^{th} *endpiece* f_n of f is:

$$\boxed{f_n := \sum_{i=n+1}^{\infty} a_i x^{i-n}.}$$

Note: $f_n := a_{n+1}x + x\sum_{i=n+2}^{\infty} a_i x^{i-n-1} = a_{n+1}x + xf_{n+1}$.
Set $B_n := k[x,y,z,f_n]_{(x,y,z,f_n)}$. By Note, $B_n \subseteq B_{n+1}$.

Define the *Approximation Domain B*

$$\boxed{B := \bigcup B_n.}$$

In this example, B is not Noetherian, by our methods.

The 1-coefficient example

Let $R = k[x, y]_{(x,y)} \implies R^* = k[y]_{(y)}[[x]]$. Choose $\sigma \in xk[[x]]$, algebraically independent over $k(x)$.

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The 1-coefficient example

Let $f = y\sigma \in xR^*$. Let B be the approximation domain,
$B = \bigcup k[x, y, z, f_n]_{(x,y,z,f_n)}$.
B is not Noetherian.

The 2-coefficient example

Let $R = k[x, y, z]_{(x,y,z)} \implies R^* = k[y, z]_{(y,z)}[[x]]$. Choose $\tau, \sigma \in xk[[x]]$, algebraically independent over $k(x)$.

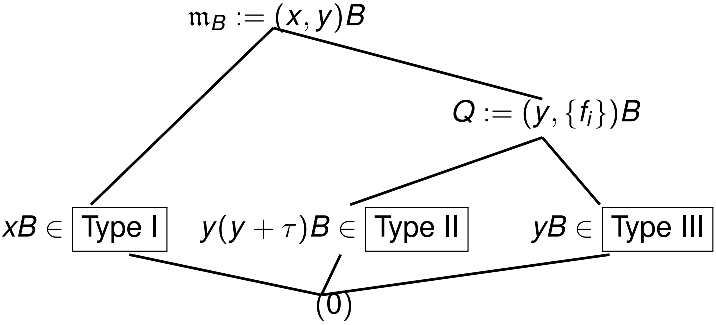
The 2-coefficient example

Let $R = k[x, y, z]_{(x,y,z)} \implies R^* = k[y, z]_{(y,z)}[[x]]$. Choose $\tau, \sigma \in xk[[x]]$, algebraically independent over $k(x)$.

The 2-coefficient example

Let $f = y\sigma + z\tau \in xR^*$. Let B be the approximation domain. So $B = \bigcup k[x, y, z, f_n]_{(x,y,z,f_n)}$.
B is not Noetherian.

Spec B, for the 1-coeff example, $B \subset k[[x, y]]$



Spec B

"Type I" ="B/P is Noetherian"; "Type III"= "P not contracted."

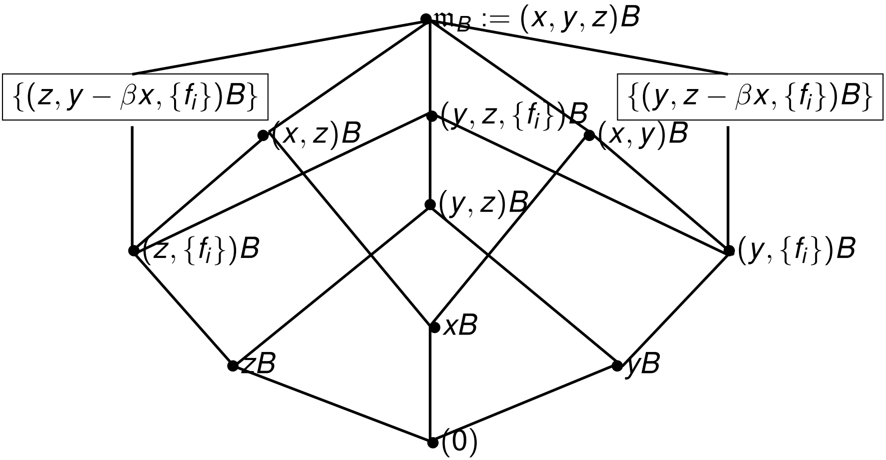
"Type II" = "$P = P^* \cap B$, $\exists P^* \in \text{Spec}(k[[x, y]]$."

Properties of the 1-coefficient example

B is a non-Noetherian local integral domain $(B, \mathfrak{m}_B)$ such that:

1. $\dim B = 3$.
2. The ring B is a UFD that is not catenary.
3. The maximal ideal $\mathfrak{m}_B$ of B is $(x, y)B$.
4. The $\mathfrak{m}_B$-adic completion of B is a two-dimensional regular local domain.
5. For every non-maximal prime ideal P of B, the ring B_P is Noetherian.
6. The ring B has precisely 1 prime ideal of height two.
7. Every prime ideal of B of height two is not finitely generated; all other prime ideals of B are finitely generated.

Part of Spec B, for the $yz\tau$ example, $B \subset k[[x, y, z]]$



$yz\tau$ Theorem:

Theorem Let B be the $yz\tau$ example. Then:

1. B = 4-dim local UFD, max ideal $\mathfrak{m}_B = (x, y, z)B$, $\widehat{B} = k[[x, y, z]]$.
2. $B[1/x]$ = Noetherian regular UFD, $\dim(B/xB) = 2$. If $P \in \operatorname{Spec} B$, B_P an RLR $\iff$ B_P is Noetherian $\iff$ $(yz, x)R^* \cap B \not\subseteq P$. $\therefore \operatorname{ht} P \leq 2 \implies B_P$ is an RLR.
3. What ideals of B are finitely generated? Partial answer:
   1. Every height-one prime ideal is principal.
   2. $Q_1 := (y, \{f_n\})B = yR^* \cap B$, $Q_2 := (z, \{f_n\})B = zR^* \cap B$, $Q_3 := (y, z, \{f_n\})B = (y, z)R^* \cap B$ are prime ideals, not finitely generated; $\operatorname{ht} Q_1 = \operatorname{ht} Q_2 = 2$, $\operatorname{ht} Q_3 = 3$.
   3. The prime ideals $(x, y)B$ and $(x, z)B$ have height three.
   4. If P is a height-two prime ideal of B that contains an element of the form $y + g(z, x)$ or $z + h(x, y)$, where $0 \neq g(z, x) \in (x, z)k[x, z]$ and $0 \neq h(x, y) \in (x, y)k[x, y]$, then P is generated by two elements.
   5. If $\mathfrak{a}$ is an ideal of B that contains $x + yzg(y, z)$, for some polynomial $g(y, z) \in k[y, z]$, then $\mathfrak{a}$ is finitely generated.
   6. $\exists\infty$ many ht-3 non-finitely generated prime ideals, e.g. $Q_{i,\alpha} = (y - \alpha x^i, z, \{f_n\})B$, where $i \in \mathbb{N}$ and $\alpha \in k$.

d-coefficient Example

Let $R = k[x, y_1, \ldots, y_d]_{(x,y_1,\ldots,y_d)}$, $R^* = k[y_1, \ldots, y_d]_{(-)}[[x]]$, $\tau_1, \ldots, \tau_d \in xk[[x]]$ algebraically independent over $k(x)$, and $f = y_1\tau_1 + \ldots y_d\tau_d \in xR^*$ Define $B = \bigcup k[x, y_1, \ldots, y_d, f_n]_{(x,y_1,\ldots,y_d,f_n)}$. Then B is a non-catenary, non-Noetherian local UFD of dimension $d + 2$ such that:

(i) B has exactly 1 prime ideal of height $d + 1$;

(ii) The height-$(d + 1)$ prime ideal is not finitely generated;

(iii) The localization of B at every nonmaximal prime ideal of B is Noetherian.

THANKS!