ROKHLIN-TYPE PROPERTIES FOR GROUP ACTIONS ON C*-ALGEBRAS.

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Abstract. Since the work of Connes in the classification of von Neumann algebras and their automorphisms, group actions have received a great deal of attention. Amenable group actions on the hyperfinite II_1-factor were completely classified by Ocneanu, extending earlier results of Connes and Jones. In their work, showing that outer actions have the so-called Rokhlin property was fundamental, as this property allows one to prove classification. For C^*-algebras, the picture is more complicated. For once, it is no longer true that (strong) outerness implies the Rokhlin property, and there is little hope to classify general group actions unless they have the Rokhlin property. On the other hand, the Rokhlin property is very restrictive, and there are many C*-algebras that do not admit any action with this property. Several weakenings of the Rokhlin property have been introduced to address this problem. Among them, the weak tracial Rokhlin property and Rokhlin dimension (for which Rokhlin dimension zero is equivalent to the Rokhlin property) have been successfully used to prove structure results for crossed products. Furthermore, actions with these properties seem to be very common.

In this lecture series, we will focus on actions of groups that are either compact or discrete and amenable. We will introduce the Rokhlin property, provide many examples, and show that Rokhlin actions can be classified. We will also see that there are natural obstructions to the Rokhlin property, and will present some weaker variants of it: the (weak) tracial Rokhlin property and Rokhlin dimension (with and without commuting towers). These properties are flexible enough to cover many relevant examples, and are strong enough to yield interesting structural properties for their crossed products. Finally, and inspired by the work of Liao, we will prove a recent analog of Ocneanu's theorem for amenable group actions on C^*-algebras, namely, that for actions on classifiable algebras (which are, in particular, Jiang-Su stable), strong outerness is equivalent to the weak tracial Rokhlin property, and also equivalent to finite Rokhlin dimension (in fact, dimension at most one).

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Contents

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1. Introduction

By the groundbreaking work of Murray and von Neumann, separably acting von Neumann factors can be divided into three types: type I factors have nonzero minimal projections, type II are those that have no minimal projections but contain a finite projection, and type III factors have only infinite projections. Type II factors are further divided into type II_1, when there is a (normalized) finite trace, and type II_∞ if there is a semifinite trace. (The other types also have subdivisions, but we will not go into that here.) Since factors of type II_∞ are all tensor products of type II_1-factors with $\mathcal{B}(\ell^2)$, the study of II_1-factors is in some sense equivalent to the study of type II factors. A remarkable result of Connes asserts that for a II_1-factor, hyperfiniteness is equivalent to injectivity, and moreover there exists a unique such II_1-factor, usually denoted by $\mathcal{R}$. This factor has been extensively studied by a number of authors. A common "regularity" property that a factor $\mathcal{M}$ may satisfy is absorbing $\mathcal{R}$ tensorially (usually known as being *McDuff*). McDuff II_1-factors are much better understood than general II_1-factors. Moreover, if $\mathcal{M}$ is any factor, then $\mathcal{M}\overline{\otimes}\mathcal{R}$ is a McDuff factor (of type II_1 if so is $\mathcal{M}$).

Once the classification of von Neumann factors was completed, the attention quickly shifted to the study of their automorphisms, and, more generally, the study of group actions on them. Automorphisms of the hyperfinite II_1-factor $\mathcal{R}$ which have finite order (that is, actions of a finite cyclic group) were studied by Connes [Con77]. His work was considerably extended by Jones [Jon80], who studied and classified finite group actions on $\mathcal{R}$. These advances culminated in the remarkable work of Ocneanu [Ocn85], who classified general amenable group actions on McDuff factors. In particular, it follows from his work that there exists a unique, up to cocycle equivalence, outer action of any given amenable group on $\mathcal{R}$. We will say more about these results in Section 5.

The study of the structure and classification of C^*-algebras developed, for quite some time, rather independently from the advances on the side of von Neumann algebras. Matui and Sato [MS14] were the first ones to import techniques from von Neumann algebras in a systematic way, obtaining groundbreaking results. These methods were further developed by a number of authors, and these contributions are particularly relevant in the verifications of (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) in the Toms-Winter conjecture:

Conjecture 1.1. (Toms-Winter; see, for example, [ET08]). Let A be a unital, separable, simple, nuclear, infinite dimensional C^*-algebra. Then the following are equivalent:

(1) A has finite nuclear dimension.
(2) A is $\mathcal{Z}$-stable.
(3) A has strict comparison of positive elements.

The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) were shown to hold by Winter [Win12] and Rørdam [Rør04], respectively. As of (3) $\Rightarrow$ (2), the result is known in the case that $T(A)$ is a Bauer simplex and its extreme boundary is finite dimensional, thanks to the independent works of Matui-Sato [MS12a], Kirchberg-Rørdam [KR12], and

Toms-White-Winter [TWW15]. Finally, the implication (2) $\Rightarrow$ (1) is true whenever $T(A)$ is a Bauer simplex, and this was recently shown by Bosa-Brown-Sato-Tikuisis-White-Winter [BBS$^+$15].

Now that the Elliott programme to classify simple, nuclear C^*-algebras is almost completed [EGLN15], it is natural to shift our attention to the study of their automorphisms, and, more generally, group actions on them. By comparison, this area is considerably underdeveloped, and there were, until recently, no systematic efforts to study their structure and make attempts at their classification. Until around 10 years ago, only rather restricted classes of group actions have been studied at a time. Izumi's study and classification of finite group actions with the *Rokhlin property* [Izu04] can be described as the first instance of a systematic study, where the actions under consideration are not described by the way in which they are constructed (namely, as direct limit actions of very special form), but rather characterized by an abstract property. Roughly speaking, for a finite group action, the Rokhlin property says that there exists a partition of unity, indexed by the elements of the group, consisting of approximately central projections which are cyclically translated by the group action (more details are given in Section 2). Izumi's work was extended by the author and Santiago [GS15] to the non-unital case, and also to actions of compact groups [GS17]. The structure of crossed products by actions with the Rokhlin property has also been the object of a number of works by Osaka-Phillips [OP12], Hirshberg-Winter [HW07], Pasnicu-Phillips [PP14], Santiago [San12], the author [Gar16], and Forough [For16].

Actions with the Rokhlin property are rare, and many algebras o not have any. One obstruction is that the Rokhlin property, at least for finite groups, implies certain divisibility properties on K-theory. Attempts to circumvent obstructions of this sort led Phillips to introduce the *tracial Rokhlin property* [Phi11], where the projections are now assumed to have a left over which is small in the tracial sense (more details are given in Section 3). Among other applications, the tracial Rokhlin property has been used by Echterhoff-Lück-Phillips-Walters [ELPW10] to study fixed point algebras of the irrational rotation algebra A_θ under certain finite group actions, and it was also used by Phillips to show that any simple higher-dimensional noncommutative tori is an A$\mathbb{T}$-algebra [Phi06]. The main result used in these works is a theorem of Phillips, asserting that the crossed product of a C^*-algebra with tracial rank zero by a finite group action with the tracial Rokhlin property again has tracial rank zero.

Even the tracial Rokhlin property does not solve what is arguably the strongest restriction that a C^*-algebra can have in order to admit Rokhlin actions: the existence of projections. For example, the Jiang-Su algebra does not admit any action with the tracial Rokhlin property. The need to study weaker versions of these properties was quickly recognized, leading to two further notions. The *weak tracial Rokhlin property*, in which one replaces the projections in the definition of the tracial Rokhlin property with positive elements, has been considered (sometimes under different names) by Archey [Arc08], Hirshberg-Orovitz [HO13], Sato [Sat], Matui-Sato [MS12b], and Wang [Wan13], among others. The main application of this notion has been showing that Jiang-Su absorption is preserved by taking crossed products by actions with the weak tracial Rokhlin property. We say more about this property in Sections 3 and 5.

A different approach was taken by Hirshberg-Winter-Zacharias [HWZ], who introduced the notion of *Rokhlin dimension* for automorphisms and actions of finite groups. In this formulation, the partition of unity appearing in the Rokhlin property is replaced by a multi-tower partition of unity consisting of positive elements, each of which is indexed by the group elements and permuted by the group action (see Section 4 for more details). It is built into the definition that the lowest value of the Rokhlin dimension (which is zero), is equivalent to the Rokhlin property discussed above. Not requiring the existence of projections, actions with finite Rokhlin dimension are more abundant: for actions on the Jiang-Su algebra, Rokhlin dimension equal to one is in fact generic. Despite it being so seemingly commong, finite Rokhlin dimension is a powerful tool to prove bounds of the nuclear dimension of crossed products. An advantage of this approach is that the definition of Rokhlin dimension does not require the C^*-algebra to be simple; in particular, the theory can be applied to actions on compact Hausdorff spaces. The works of Hirshberg-Winter-Zacharias for $\mathbb{Z}$-actions, and of Szabo [Sza13] for $\mathbb{Z}^d$-actions, illustrate this fact nicely. Rokhlin dimension has been defined for actions of much more general groups: for residually finite groups by Szabo-Wu-Zacharias [SWZ14], for compact groups by the author [Gar14c] and [Gar15b], and further by the author, Hirshberg and Santiago [GHS14], and for the reals by Hirshberg-Szabo-Winter-Wu [HSWW16].

With all these seemingly different Rokhlin-type properties, a natural question arises: when does one of these properties imply another one? Except for the obvious implications, it is not clear what the relationship between them is. This is explored in Section 5, where we show that for a large class of simple C^*-algebras, the weak tracial Rokhlin property and having Rokhlin dimension at most one are equivalent. The goal of this series of lectures is to familiarize the audience with all these Rokhlin-type properties, as well as giving a sample of the techniques that are used to work with each of them.

Throughout, we will work mostly with separable, unital C^*-algebras and finite groups. Removing the unitality and separability assumptions assumption is, for the most part, not difficult, and we omit this issue completely here. (The results in Section 5 have really only been proved for separable, unital algebras.) Moving away from finite groups involves more complications. Some results hold in general for compact groups, while others hold for discrete amenable groups, and those concerning Rokhlin dimension require the group to be moreover residually finite. While definitions and proofs will be given for finite groups mostly, we will mention, when appropriate, what generalizations have been obtained in the literature.

2. The Rokhlin property

The Rokhlin property has its origins in Ergodic Theory. In this area, the Rokhlin Lemma asserts, roughly speaking, that an aperiodic measure preserving transformation of a probability space has an approximate decomposition as cyclic shifts. In operator algebras, the Rokhlin property appears as a technical device in the classification of amenable group actions on $\mathcal{R}$. Indeed, in the works of Connes, Jones and Ocneanu, it is shown that outer actions automatically have the Rokhlin property, and this is a key ingredient in showing that any two of them are cocycle conjugate. In C^*-algebras, Herman and Jones [HJ83] studied specific instances of the Rokhlin property, and it was Izumi who introduced the modern definition, which we reproduce below.

Definition 2.1. (Definition 3.2 in [Izu04]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A. We say that α has the *Rokhlin property* if for every $\varepsilon > 0$ and for every finite subset $F \subseteq A$, there exist projections $e_g \in A$, for $g \in G$, satisfying

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$;
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$;
(3) $\sum\limits_{g\in G} e_g = 1$.

The definition was extended to actions of compact groups by Hirshberg and Winter in [HW07]. With the purpose of introducing notation that will be used later, we also present their definition.

Write $\ell^\infty(A)$ for the set of norm-bounded sequences on A, and $c_0(A)$ for the ideal of sequences whose elements converge to zero. We set $A_\infty = \ell^\infty(A)/c_0(A)$, and write $\kappa_A\colon \ell^\infty(A) \to A_\infty$ for the quotient map. There is a canonical map $A \to A_\infty$ given by sending $a \in A$ to the class of the constant sequence with value a. In this way, we regard A naturally as a subalgebra of A_∞, and we write $A_\infty \cap A'$ for its relative commutant. If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a locally compact group G, then the induced action of G on A_∞ may fail to be continuous. We define $A_{\alpha,\infty}$ to be the subalgebra of A_∞ of those elements where G acts continuously, and write $\alpha_\infty\colon G \to \mathrm{Aut}(A_{\alpha,\infty})$ for the induced action. One easily checks that $A \subseteq A_{\alpha,\infty}$.

Definition 2.2. (Definition 3.2 in [HW07]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a unital C^*-algebra A. We say that α has the *Rokhlin property* if there exists a unital, equivariant homomorphism $\varphi\colon (G, \mathtt{Lt}) \to (A_{\alpha,\infty} \cap A', \alpha_\infty)$.

Example 2.3. For trivial reasons, $\mathtt{Lt}\colon G \to \mathrm{Aut}(C(G))$ has the Rokhlin property.

Example 2.4. Let G be a finite group, and let $\lambda\colon G \to \mathcal{U}(\ell^2(G))$ be the left regular representation. Define an action $\alpha\colon G \to \mathrm{Aut}(M_{|G|^\infty})$ by $\alpha_g = \bigotimes\limits_{n\in\mathbb{N}} \mathrm{Ad}(\lambda_g)$ for $g \in G$. Then α has the Rokhlin property.

Example 2.5. By taking tensor products with the action in Example 2.4, we obtain actions with the Rokhlin property on any $M_{|G|^\infty}$-absorbing unital C^*-algebra.

To get a feeling of how strong the Rokhlin property is, we look at actions on commutative C^*-algebras. In the following result, the action on $X/G \times G$ is a diagonal action: trivial on X/G and left translation on G.

Theorem 2.6. Let X be a compact Hausdorff space and let $\alpha\colon G \to \mathrm{Aut}(C(X))$ be an action of a compact group G. Then α has the Rokhlin property if and only if X is equivariantly homeomorphic to $X/G \times G$.

Proof. We take $\varepsilon = 1$ and $F = \emptyset$. Let $e_g \in C(X)$, for $g \in G$, be projections adding up to 1 and satisfying $\|\alpha_g(e_h) - e_{gh}\| < 1$ for all $g, h \in G$. Then $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$, since in commutative C^*-algebras, two projections that are less than one unit apart are actually equal. First observe that $G \curvearrowright X$ is free. Indeed, suppose that $g \in G$ and $x \in X$ satisfy $g \cdot x = x$. Let $h \in G$ be the unique group elements such that $e_h(x) = 1$. Then $e_{gh}(x) = \alpha_{g^{-1}}(e_h(x)) = e_h(x)$, and thus $gh = h$, so $g = 1$ and the action is free.

The support Y of e_1, is a clopen subset of X which satisfies $g \cdot Y \cap Y = \emptyset$ for all $g \in G$, and $\bigcup\limits_{g\in G} g \cdot Y = X$. Define a continuous map $f\colon Y \times G \to X$ by $f(y,g) = g \cdot y$ for $y \in Y$ and $g \in G$. Observe that f is surjective, and it is injective because $G \curvearrowright X$ is free. Endow Y with the trivial action and G with its translation action. Then f is equivariant, and is hence an equivariant homeomorphism. It follows that Y is homeomorphic to X/G, and the proof is finished. □

In particular, the action $\mathbb{Z}_2 \curvearrowright S^1$ given by $z \mapsto -z$, does not have the Rokhlin property (also, $C(S^1)$ does not have any nontrivial projections). We present other non-examples.

Example 2.7. There is no action of $\mathbb{Z}_2$ on M_{3^∞} with the Rokhlin property. Indeed, suppose that $\alpha\colon \mathbb{Z}_3 \to \mathrm{Aut}(M_{3^\infty})$ were one. Then there exist two projections e_0 and e_1 satisfying $\|\alpha_1(e_0) - e_1\| < 1$ and $e_0 + e_1 = 1$. The first condition implies that $\alpha_1(e_0)$ is unitarily equivalent to e_1. Since α_1 is approximately inner (as any automorphism of a UHF-algebra), we deduce that e_0 and e_1 are unitarily equivalent. It follows that the class of the unit of M_{3^∞} is divisible by two in K-theory, which is a contradiction.

Exercise 2.8. Let G be a finite group. Determine for which $n \in \mathbb{N} \cup \{\infty\}$ there exists an action of G on $\mathcal{O}_n$ with the Rokhlin property.

Example 2.9. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then there is no action of any finite group G on the rotation algebra A_θ with the Rokhlin property. Indeed, suppose that $\alpha\colon G \to \mathrm{Aut}(A_\theta)$ were one. Then there exist projections $e_g \in A_\theta$, for $g \in G$, satisfying $\|\alpha_g(e_1) - e_g\| < 1$ and $\sum\limits_{g\in G} e_g = 1$. The first condition implies that $\alpha_g(e_1)$ is unitarily equivalent to e_g, and in particular have the same trace. Since A_θ has a unique trace τ, we must have $\tau \circ \alpha_g = \tau$, and thus $\tau(e_g) = \tau(e_1)$ for all $g \in G$. Thus $\tau(1) = |G|\tau(e_1)$. However, $\tau(1) = 1$ is not divisible by $|G|$ within the image under τ of the projections of A_θ, and this is a contradiction.

2.1. **Crossed products.** The Rokhlin property has had two main uses in the literature: the study of crossed products, and classification. We review both this aspects in this section. Below, we summarize the results that can be found in the literature, and then give proofs of some of them in the case of finite groups. The following statement is essentially Theorem 1.1 in [Gar16] (although many were already known for finite groups), except item (15), which was recently proved by Forough (see [For16]).

Theorem 2.10. The following classes of unital C^*-algebras are closed under formation of crossed products and passage to fixed point algebras by actions of compact groups with the Rokhlin property:

(1) Simple C^*-algebras. More generally, the ideal structure can be completely determined;
(2) C^*-algebras that are direct limits of certain weakly semiprojective C^*-algebras. This includes UHF-algebras, AF-algebras, AI-algebras, A$\mathbb{T}$-algebras, countable inductive limits of one-dimensional NCCW-complexes, and several other classes;
(3) Kirchberg algebras;
(4) Simple C^*-algebras with tracial rank at most one;

(5) Simple, separable, nuclear C^*-algebras satisfying the Universal Coefficient Theorem;
(6) C^*-algebras with nuclear dimension at most n, for $n \in \mathbb{N}$;
(7) C^*-algebras with decomposition rank at most n, for $n \in \mathbb{N}$;
(8) C^*-algebras with real rank zero or stable rank one;
(9) C^*-algebras with strict comparison of positive elements;
(10) C^*-algebras whose order on projections is determined by traces;
(11) (Not necessarily simple) purely infinite C^*-algebras;
(12) Separable $\mathcal{D}$-absorbing C^*-algebras, for a strongly self-absorbing C^*-algebra $\mathcal{D}$;
(13) C^*-algebras whose K-groups are either: trivial, free, torsion-free, torsion, or finitely generated;
(14) Weakly semiprojective C^*-algebras.
(15) C^*-algebras all of whose traces are quasidiagonal.

Many of the above results can be deduced from the following technical proposition.

Proposition 2.11. (see Theorem 3.2 in [OP12]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A. Assume that α has the Rokhlin property. Let $\varepsilon > 0$ and let $S \subseteq A \rtimes_\alpha G$ be a finite subset. Then there exist a projection $p \in A$ and a unital, injective homomorphism $\varphi\colon M_{|G|}(pAp) \to A \rtimes_\alpha G$ such that $\mathrm{dist}(\mathrm{Im}(\varphi), S) < \varepsilon$.

Proof. Without loss of generality, we can assume that S has there exists a finite subset $F \subseteq A$ such that $S = \{u_g\colon g \in G\} \cup F$. Choose ε_0 such that whenever $\{t_{g,h}\colon g, h \in G\}$ are elements in a $A \rtimes_\alpha G$ satisfying

- $\|t_{g,h}t_{k,\ell} - \delta_{h,k}t_{g,\ell}\| < 3\varepsilon_0$;
- $\|t_{g,h}^* - t_{h,g}\| < 3\varepsilon_0$; and
- $t_{g,g}$ is a projection for all $g \in G$ and $\sum\limits_{g\in G} t_{g,g} = 1$

for all $g, h, k, \ell \in G$, then there exists a unital homomorphism $\psi\colon M_{|G|} \to A \rtimes_\alpha G$ satisfying $\|\psi(f_{g,h}) - t_{g,h}\| < \varepsilon$ and $\psi(f_{g,g}) = t_{g,g}$ for all $g, h \in G$.

Let $e_g \in A$, for $g \in G$, be projections satisfying the conditions in Definition 2.1 for ε_0 and F. For $g, h \in G$, define $s_{g,h} = u_{gh^{-1}}e_h$. One checks that these elements form an $3\varepsilon_0$-approximate system of matrix units. By the choice of ε_0, there exists a unital homomorphism $\psi\colon M_{|G|} \to A \rtimes_\alpha G$ satisfying $\|\psi(f_{g,h}) - t_{g,h}\| < \varepsilon$ for all $g, h \in G$. Define a map $\varphi\colon M_{|G|}(e_1Ae_1) \to A \rtimes_\alpha G$ by

$$\varphi(f_{g,h} \otimes a) = \psi(f_{g,1})a\psi(f_{h,1})$$

for $g, h \in G$ and $a \in e_1Ae_1$. One checks that φ is a injective unital homomorphism. Moreover, one easily checks that every element of S is within ε (or a constant times ε) of an element in the image of φ. We omit the details. □

The proposition above is very useful, and it is the main tool to study crossed products by finite group actions with the Rokhlin property. (Nothing like this works for compact groups, and the tools in this context are rather different.) The general principle is that any property that passes to corners and matrices over the algebras, and is "stable" under small perturbations, will be preserved by actions with the Rokhlin property. (The results on compact groups allows one to get rid of the condition on corners; this is crucial in proving part (14) of Theorem 2.10.)

We give proofs for two of the properties in Theorem 2.10. For finite groups, the first one is due to Phillips, and the second one is due to Osaka-Phillips. Recall that a separable C^*-algebra B is an AF-algebra if and only if for every finite subset $S \subseteq B$ and every $\varepsilon > 0$, there exist a finite dimensional C^*-algebra D and a homomorphism $\psi\colon D \to B$ such that $\mathrm{dist}(S, \mathrm{Im}(\psi)) < \varepsilon$.

Proposition 2.12. (Theorem 2.2 in [Phi11]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital AF-algebra A. If α has the Rokhlin property, then $A \rtimes_\alpha G$ is also AF.

Proof. Let $S \subseteq A \rtimes_\alpha G$ be a finite subset and let $\varepsilon > 0$. Use Proposition 2.11 to find a projection $p \in A$ and a unital embedding $\varphi\colon M_{|G|}(pAp) \to A \rtimes_\alpha G$ such that $\mathrm{dist}(S, \mathrm{Im}(\varphi)) < \varepsilon/2$. Let $T \subseteq M_{|G|}(pAp)$ be a finite subset such that $\mathrm{dist}(S, \varphi(T)) < \varepsilon/2$. Since $M_{|G|}(pAp)$ is also an AF-algebra, there exist a finite dimensional C^*-algebra D and a homomorphism $\psi_0\colon D \to M_{|G|}(pAp)$ such that $\mathrm{dist}(T, \mathrm{Im}(\psi_0)) < \varepsilon/2$. The result now follows by taking $\psi = \varphi \circ \psi_0$. $\square$

Recall that a unital C^*-algebra is said to have *stable rank one* if the set of invertible elements is dense.

Proposition 2.13. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A with stable rank one. If α has the Rokhlin property, then $A \rtimes_\alpha G$ also has stable rank one.

Proof. It is known that if A has stable rank one, then so does $M_n(pAp)$ for any projection $p \in A$ and any $n \in \mathbb{N}$. Let $x \in A \rtimes_\alpha G$ and let $\varepsilon > 0$. We want to find an invertible element $y \in A \rtimes_\alpha G$ such that $\|x - y\| < \varepsilon$. Use Proposition 2.11 to find a projection $p \in A$, a unital embedding $\varphi\colon M_{|G|}(pAp) \to A \rtimes_\alpha G$, and an element $y_0 \in M_{|G|}(pAp)$ such that $\|\varphi(y_0) - x\| < \varepsilon/2$. By stable rank one, there exists an invertible element $y \in M_{|G|}(pAp)$ such that $\|y - y_0\| < \varepsilon/2$. Then $\varphi(y)$ is invertible and is within ε of x. $\square$

2.2. **Classification and model actions.** We now turn to classification of Rokhlin actions. For finite groups on unital C^*-algebras, this was done by Izumi. The nonunital case was obtained by the author and Santiago, and the general case of compact group actions on separable C^*-algebras was obtained by the author and Santiago in subsequent work. The following formulation is accurate only for finite groups. The proof is not particularly difficult, but it is lengthy, and we omit it.

Theorem 2.14. (Theorem 3.5 in [Izu04] and Theorem 3.11 in [GS15]). Let α and β be actions of a finite group G on a unital C^*-algebra A with the Rokhlin property. Then there exists an approximately inner automorphism $\theta \in \mathrm{Aut}(A)$ satisfying $\alpha_g \circ \theta = \theta \circ \beta_g$ for all $g \in G$ if and only if α_g is approximately unitarily equivalent to β_g for all $g \in G$.

Here is an immediate consequence.

Corollary 2.15. Let A be a unital C^*-algebra with the property that any automorphism is approximately inner. (For instance, a UHF-algebra, a Cuntz algebra, etc.) Then any two Rokhlin actions on A are conjugate. In particular, there exists a “unique” Rokhlin action of G on $M_{|G|^\infty}$ and on $\mathcal{O}_2$.

In some sense, a converse to Example 2.5 holds. The proof of (1) $\Rightarrow$ (2) is Example 2.5, while the proof of (2) $\Rightarrow$ (3), which we omit, combines elementary computations with standard facts about UHF-algebras and UHF-absorption.

Theorem 2.16. Let G be a finite group and let A be a separable, unital C^*-algebra. Then the following are equivalent:

(1) $A \otimes M_{|G|} \cong A$;
(2) There exists an action of G on A with the Rokhlin property which is pointwise approximately inner.

If the conditions above hold, then every action of G on A with the Rokhlin property absorbs the model action constructed in Example 2.4. In particular, all Rokhlin actions on A are as in Example 2.5.

Model actions satisfying the conclusion of Theorem 2.16 only exist for totally disconnected compact groups, but their construction is rather involved and will not be presented here.

3. The tracial Rokhlin property

The tracial Rokhlin property was defined by Phillips in order to study certain finite group actions which failed to have the Rokhlin property. Here is the definition.

Definition 3.1. (Definition 1.2 in [Phi11]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action on a simple, unital C^*-algebra A. We say that α has the *tracial Rokhlin property* if for every $\varepsilon > 0$, for every finite subset $F \subseteq A$, and every positive contraction $x \in A$ with $\|x\| = 1$, there exist orthogonal projections $e_g \in A$, for $g \in G$, satisfying

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$;
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$;
(3) With $e = \sum\limits_{g\in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$;
(4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

The definition above makes sense even if the algebra is not simple. However, in general, it may be equivalent to the Rokhlin property. For instance, suppose that $A = B \oplus C$, and that there are actions β and γ of G on B and C, respectively, such that $\alpha = (\beta, \gamma)$. The projections $e_g \in A$ from Definition 3.1 will have the form (p_g, q_g) for projections $p_g \in B$ and $q_g \in C$. By choosing $x = (0, 1_C)$, we see that $\sum\limits_{g\in G} p_g = 1_B$, so the action β has the Rokhlin property. Similarly, γ has the Rokhlin property, and hence so does α.

Quite possibly, a 'good' definition of the tracial Rokhlin property must require the element x in Definition 3.1 to be *full* in A.

Remark 3.2. When A has *strict comparison (of positive elements by traces)*, then condition (3) can be replaced by

(3') $\tau(1 - e) < \varepsilon$ for every (extreme) trace $\tau \in T(A)$.

Remark 3.3. Condition (4) is automatic if A is finite; see Lemma 1.16 in [Phi11]. The proof, which we omit, is not difficult, but uses a fair amount of Cuntz comparison, which we will not review here.

Remark 3.4. In condition (3), we may moreover require that $\alpha_g(e) = e$ for all $g \in G$. The reason is that e is almost fixed, so it is unitarily equivalent to a projection $\widetilde{e}$ in A^α. Use this unitary to perturb the projections e_g, to obtain projections satisfying all conditions in the definition, and such that their sum is $\widetilde{e}$.

Of course every action with the Rokhlin property has the tracial Rokhlin property. The converse does not hold, as the following example shows.

Example 3.5. For $n \in \mathbb{N}$, set

$$u_n = \mathrm{diag}(\underbrace{1, \ldots, 1}_{\frac{3^n-1}{2} \text{ times}}, \underbrace{-1, \ldots, -1}_{\frac{3^n-1}{2} \text{ times}}, 1),$$

which is a unitary in M_{3^n} of order two. Let the nontrivial element in $\mathbb{Z}_2$ act on M_{3^∞} as $\bigotimes\limits_{n\in\mathbb{N}} \mathrm{Ad}(u_n)$. We claim that the resulting action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(M_{3^\infty})$ has the tracial Rokhlin property but not the Rokhlin property.

Let $n \in \mathbb{N}$, and denote by τ_n the unique trace on M_{3^n}. We first observe that there exist projections $p_0^{(n)}, p_1^{(n)} \in M_3$ such that

$$\mathrm{Ad}(u_n)(p_0^{(n)}) = p_1^{(n)} \quad \text{and} \quad \tau_n(1 - p_0^{(n)} - p_1^{(n)}) = \frac{1}{3^n}.$$

The easiest way to see this is probably the following. With $m = \frac{3^n-1}{2}$, observe that u_n is unitarily equivalent to

$$v_n = \begin{pmatrix} 0 & 1_{M_m} & 0 \\ 1_{M_n} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that it is enough to find projections $q_0^{(n)}$ and $q_1^{(n)}$ for v_n. We may take $q_0^{(n)} = \mathrm{diag}(1_{M_m}, 0, 0)$ and $q_1^{(n)} = \mathrm{diag}(0, 1_{M_m}, 0)$.

Now let $F \subseteq M_{3^\infty}$ be a finite subset and let $\varepsilon > 0$. We think of M_{3^∞} as the infinite tensor product $\bigotimes_{n\in\mathbb{N}} M_{3^n}$. Find $n \in \mathbb{N}$ and a finite subset $F' \subseteq M_3 \otimes \cdots \otimes M_{3^{n-1}}$ such that every element of F is within ε of an element of F', and such that $\frac{1}{3^n} < \varepsilon$. Set

$$e_0 = 1_{M_3} \otimes \cdots \otimes 1_{M_{3^{n-1}}} \otimes p_0^{(n)} \quad \text{and} \quad e_1 = 1_{M_3} \otimes \cdots \otimes 1_{M_{3^{n-1}}} \otimes p_1^{(n)}.$$

It is an exercise to check that these projections satisfy conditions (1) and (2) in Definition 3.1 as well as condition (3') in Remark 3.2. Since condition (4) is automatic by Remark 3.3, the claim follows.

Finally, α does not have the Rokhlin property by Example 2.7.

It is easy to see that an action with the tracial Rokhlin property is (pointwise) outer, and the converse is false in general. However, the converse does hold for actions on Kirchberg algebras:

Theorem 3.6. (Theorem 2.10 in [GHS14]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action on a Kirchberg algebra A. Then α has the tracial Rokhlin property if and only if α_g is outer for every $g \in G$.

The proof is similar in spirit to Example 3.5, using a structure result for pointwise outer actions on Kirchberg algebras of Goldstein-Izumi [GI11]. (The result basically says that it is some kind of product type action.) The projections are constructed using a result of Kishimoto, in combination with the fact that a Kirchberg algebra has real rank zero. Condition (4) does have to be checked in this case, since Kirchberg algebras are not finite.

The tracial Rokhlin property is a far weaker assumption than the Rokhlin property, and therefore it is to be expected that one cannot prove a result as strong as

Theorem 2.10. Indeed, at least in a general form, most of the items there fail: most importantly, (2), (6), (7), (12) (except for $\mathcal{D} = \mathcal{Z}$ in some special cases), and (13) all fail. (1) is guaranteed by a result of Kishimoto, as well as (3). (15) holds under the additional assumption that A be exact, by a result of Forough [For16].

Algebras with tracial rank zero, also known as tracially AF-algebras, are preserved by actions with the tracial Rokhlin property, by a result of Phillips. This is indeed the main application that motivated the study of the tracial Rokhlin property.

Definition 3.7. Let A be a separable, unital, simple C^*-algebra. We say that α has *tracial rank zero* if for every $\varepsilon > 0$, for every finite set $F \subseteq A$ and every positive contraction $x \in A$ with $\|x\| = 1$, there exist a projection $p \in A$, a finite dimensional C^*-algebra D and a unital homomorphism $\varphi\colon D \to pAp$ such that

(1) $\|pa - ap\| < \varepsilon$ for every $a \in F$;
(2) $\mathrm{dist}(\mathrm{Im}(\varphi), F) < \varepsilon$;
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$.

One of the main technical steps in proving preservation of tracial rank zero is the following tracial approximation of the crossed product, which should be compared with Proposition 2.11.

Proposition 3.8. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A. Assume that α has the tracial Rokhlin property. Let $\varepsilon > 0$ and let $S \subseteq A \rtimes_\alpha G$ be a finite subset, and let $x \in A$ be a positive contraction with $\|x\| = 1$. Then there exist projections $p, e \in A$ and a unital, injective homomorphism $\varphi\colon M_{|G|}(pAp) \to e(A \rtimes_\alpha G)e$ such that

(1) $e^\perp = 1 - e$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$;
(2) for every $s \in S$ there is $b \in M_{|G|}(pAp)$ such that $\|\varphi(b) + e^\perp s e^\perp - s\| < \varepsilon$.

In other words, S is approximated by elements in the image of φ up to elements living in a very small corner of A.

Proof. Without loss of generality, we can assume that S has there exists a finite subset $F \subseteq A$ such that $S = \{u_g \colon g \in G\} \cup F$. Choose ε_0 sufficiently small. Let $e_g \in A$, for $g \in G$, be projections satisfying the conditions in Definition 2.1 for ε_0, F, and x. By Remark 3.4, we may assume that $a \in A^\alpha$. Denote by $\beta\colon G \to \mathrm{Aut}(eAe)$ the induced action. Then $(eAe) \rtimes_\beta G = e(A \rtimes_\alpha G)e$. The argument in the proof of Proposition 2.11 gives us a unital homomorphism

$$\varphi\colon M_{|G|} \otimes e_1 A e_1 \to (eAe) \rtimes_\beta G = e(A \rtimes_\alpha G)e$$

satisfying $\mathrm{dist}(eSe, \Im(\varphi)) < \varepsilon/2$.

One checks that

$$\|s - \left(ese + e^\perp s e^\perp\right)\| < \frac{\varepsilon}{2}$$

for all $s \in S$. Let $s \in S$, and find $b \in M_{|G|}(pAp)$ such that $\|\varphi(b) - ese\| < \varepsilon/2$. Using this and the above estimate at the second step, we conclude that

$$\|\varphi(b) + e^\perp s e^\perp - s\| \leq \|\varphi(b) - ese\| + \|ese + e^\perp s e^\perp - s\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. $\square$

If one tries to use an argument similar to the one in Proposition 2.12 to prove, for example, that crossed products of tracially AF-algebras by actions with the tracial Rokhlin property are again tracially AF, one runs into the following problem. The error projection in the definition of tracially AF that one gets by using Proposition 3.8, which should be Murray-von Neumann equivalent to a projection in a prescribed hereditary subalgebra of the crossed product, comes out to be Murray-von Neumann equivalent to a projection in a prescribed hereditary subalgebra of A. To bridge this discrepancy, the following of Kishimoto is crucial. His result holds much more generally, but we state it here exactly the way we need it.

Theorem 3.9. (Kishimoto). Let A be a tracially AF-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action with the Rokhlin property. Let $B \subseteq A \rtimes_\alpha G$ be a hereditary subalgebra. Then there exists a nonzero projection $p \in B$ which is Murray-von Neumann equivalent to a projection in A.

These results can be combined to prove, in a way similar to Proposition 2.12, the following:

Theorem 3.10. (Theorem 2.6 in [Phi11]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple, separable, unital tracially AF C^*-algebra. If α has the tracial Rokhlin property, then $A \rtimes_\alpha G$ is also tracially AF.

3.1. **The weak tracial Rokhlin property.** The weak tracial Rokhlin property is the weakening of the tracial Rokhlin property in which projections are replaced by positive contractions, as follows.

Definition 3.11. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action on a simple, unital C^*-algebra A. We say that α has the *weak tracial Rokhlin property* if for every $\varepsilon > 0$, for every finite subset $F \subseteq A$, and every positive contraction $x \in A$, there exist orthogonal positive contractions $f_g \in A$, for $g \in G$, satisfying

(1) $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;
(2) $\|f_g a - a f_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$;
(3) With $f = \sum\limits_{g\in G} f_g$, we have $1 - f \precsim x$;
(4) With f as in (3), we have $\|fxf\| > 1 - \varepsilon$.

As in Remark 3.2 and Remark 3.3, in special cases, conditions (3) and (4) above can be replaced by simpler ones that allow one to forget about x.

Remark 3.12. Definition 3.11 has been extended to all amenable groups by Wang [Wan13], and independently by the author and Hirshberg [GH17]. Wang's definition is formally stronger, but it is equivalent to ours in a fairly general setting; see Section 5.

As one could expect, in the presence of sufficiently many projections, the weak tracial Rokhlin property is in fact equivalent to the tracial Rokhlin property.

Proposition 3.13. (Phillips). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action on a tracially AF-algebra A. Then α has the tracial Rokhlin property if and only if it has the weak tracial Rokhlin property.

Question 3.14. Can we relax the condition on A in the proposition above? How about tracial rank at most one? And how about real rank zero?

Here we give an application of Proposition 3.13.

Example 3.15. For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A_\theta)$ be the action determined by $u \mapsto u$ and $v \mapsto -v$. We claim that α has the tracial Rokhlin property. By Proposition 3.13, it suffices to verify the weak tracial Rokhlin property.

Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of integers satisfying $\mathrm{dist}(n_k\theta, \mathbb{Z}) \to 0$ as $k \to \infty$. Then $(u^{n_k})_{k\in\mathbb{N}}$ is asymptotically central. Moreover, $\alpha_1(u^{n_k}) = -u^{n_k}$. Thus, we get an asymptotically central sequence of equivariant unital homomorphisms $\varphi\colon C(S^1) \to A_\theta$, where the action on $C(S^1)$ is $z \mapsto -z$. Hence, it suffices to find the positive contractions in $C(S^1)$. The restriction of the (unique) trace of A_θ to the image of φ induces a Borel probability measure μ on S^1 which is invariant under $z \mapsto -z$. One can check that there exist disjoint open sets U_0 and U_1 which satisfy $-U_0 = U_1$ and $\mu\left(S^1 \setminus (U_0 \cup U_1)\right) < \varepsilon$. By taking appropriate positive contractions supported on these sets, we conclude that α has the weak tracial Rokhlin property, and hence the tracial Rokhlin property.

The main use of the weak tracial Rokhlin property so far has been the following result of Hirshberg-Orovitz [HO13] (for finite groups and $\mathbb{Z}$), and of the author and Hirshberg [GH17] (in the general amenable case):

Theorem 3.16. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital, separable, nuclear, $\mathcal{Z}$-stable C^*-algebra. If A has the weak tracial Rokhlin property, then $A \rtimes_\alpha G$ is $\mathcal{Z}$-stable.

Before closing this section, we wish to rephrase the tracial Rokhlin property and weak tracial Rokhlin property in terms of central sequence algebras. We need to introduce some notation.

Definition 3.17. Let A be a unital C^*-algebra. Denote by $\ell^\infty(A)$ the unital C^*-algebra of bounded sequences on A, and let $c_0(A)$ denote the ideal of $\ell^\infty(A)$ of those sequences that converge to zero in norm. Write $A_\infty = \ell^\infty(A)/c_0(A)$ for the quotient. There is a natural inclusion $A \to A_\infty$ by constant sequences, so we may take the relative commutant of A in A_∞; concretely:
(1)
$$A_\infty \cap A' = \left\{ [(x_n)_{n\in\mathbb{N}}]\colon \sup_{n\in\mathbb{N}} \|x_n\| < \infty \text{ and } \lim_{n\to\infty} \|x_n a - a x_n\| = 0 \text{ for all } a \in A \right\}.$$

If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a discrete group G on A, then there are induced actions $\alpha_\infty\colon G \to \mathrm{Aut}(A_\infty)$ and $\alpha_\infty\colon G \to \mathrm{Aut}(A_\infty \cap A')$.

The construction can be carried out for a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, with the objects being denoted A_ω, $A_\omega \cap A'$, and α_ω. The advantage of using free ultrafilters is that a trace τ on A naturally induces a trace τ_ω on A_ω given by

$$\tau_\omega\left([(x_n)_{n\in\mathbb{N}}]\right) = \lim_{n\to\omega} \tau(x_n).$$

We denote by J_A the ideal in A_ω given by

$$J_A = \{x \in A_\omega \colon \tau_\omega(x^*x) = 0 \text{ for all } \tau \in T(A)\}.$$

Recall that a completely positive map $\varphi\colon A \to B$ between C^*-algebras A and B is said to be of *order zero* if $\varphi(a)\varphi(b) = 0$ whenever $a, b \in A_+$ satisfy $ab = 0$. We can rephrase the tracial Rokhlin property and the weak tracial Rokhlin property as follows (without strict comparison, a similar characterization holds, but it is not as neat to write down):

Proposition 3.18. Let $\alpha\colon G\to \mathrm{Aut}(A)$ be an action of a finite group G on a separable, unital, simple C^*-algebra A, and assume that A has strict comparison. Let $\omega\in\beta\mathbb{N}\setminus\mathbb{N}$.

(1) α has the tracial Rokhlin property if and only if there exists an equivariant homomorphism $\varphi\colon C(G)\to A_\omega\cap A'$ satisfying $\varphi(1)\in J_A$.

(2) α has the weak tracial Rokhlin property if and only if there exists an equivariant completely positive contractive order zero map $\varphi\colon C(G)\to A_\omega\cap A'$ satisfying $\varphi(1)\in J_A$.

The difference between the two statements is that in (2), we have the *weak* tracial Rokhlin property, and φ is assumed to be *order zero*, not a homomorphism.

4. Rokhlin dimension

In this section, we introduce a different weakening of the Rokhlin property, in a spirit similar to how the weak tracial Rokhlin property was obtained as a weakening of the tracial Rokhlin property; see Proposition 3.18. Simply replacing the projections in Definition 2.1 with orthogonal positive elements and still requiring that they add up to one would not yield a different notion. Indeed, if $a_1,\ldots,a_n$ are orthogonal positive contractions in a unital C^*-algebra A satisfying $\sum\limits_{j=1}^{n} a_j=1$, then each a_j is necessarily a projection. The alternative then is to introduce more "towers", leading to a dimensional notion:

Definition 4.1. Let $\alpha\colon G\to\mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A. Given $d\in\mathbb{N}$, we say that α has *Rokhlin dimension at most* d, and write $\dim_{\mathrm{Rok}}(\alpha)\leq d$, if the following holds: for every $\varepsilon>0$ and for every finite subset $F\subseteq A$, there exist positive contractions $f_g^{(j)}\in A$, for $g\in G$ and $j=0,\ldots,d$, satisfying

(1) $\left\|\alpha_g(f_h^{(j)})-f_{gh}^{(j)}\right\|<\varepsilon$ for all $g,h\in G$ and all $j=0,\ldots,d$;

(2) $f_g^{(j)}f_h^{(j)}=0$ whenever $g\neq h$, for all $j=0,\ldots,d$;

(3) $\left\|f_g^{(j)}a-af_g^{(j)}\right\|<\varepsilon$ for all $g\in G$, all $j=0,\ldots,d$ and all $a\in F$;

(4) $\sum\limits_{j=0}^{d}\sum\limits_{g\in G} f_g^{(j)}=1$.

We write $\dim_{\mathrm{Rok}}(\alpha)$ for the smallest $d\in\mathbb{N}$ such that $\dim_{\mathrm{Rok}}(\alpha)\leq d$.

Similarly, we say that α has *Rokhlin dimension with commuting towers at most* d, and write $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha)\leq d$, if the following holds: for every $\varepsilon>0$ and for every finite subset $F\subseteq A$, there exist positive contractions $f_g^{(j)}\in A$, for $g\in G$ and $j=0,\ldots,d$, satisfying conditions (1) through (4) above, in addition to

(5) $\left\|f_g^{(j)}f_h^{(k)}-f_h^{(k)}f_g^{(j)}\right\|<\varepsilon$ for all $g,h\in G$ and all $j,k=0,\ldots,d$.

We write $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha)$ for the smallest $d\in\mathbb{N}$ such that $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha)\leq d$.

One way of thinking of the above definition is as a "colored" version of the Rokhlin property. "Coloring" is really just a meta-mathematical term, but in this context it usually means replacing homomorphisms or projections with a finite number (usually bounded, the bound essentially being the value of the relevant "dimension") of order zero maps or positive elements. Here is an equivalent formulation of Rokhlin dimension:

Proposition 4.2. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on aunital C^*-algebra A, and let $d \in \mathbb{N}$.

(1) We have $\dim_{\mathrm{Rok}}(\alpha) \leq d$ if and only if there exist an equivariant completely positive conractive order zero maps $\varphi_0, \ldots, \varphi_d\colon C(G) \to A_\infty \cap A'$ satisfying $\sum\limits_{j=0}^{d} \varphi_j(1) = 1$.

(2) We have $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) \leq d$ if and only if there exist an equivariant completely positive conractive order zero maps $\varphi_0, \ldots, \varphi_d\colon C(G) \to A_\infty \cap A'$ with commuting ranges satisfying $\sum\limits_{j=0}^{d} \varphi_j(1) = 1$

Remark 4.3. The inequality $\dim_{\mathrm{Rok}}(\alpha) \leq \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha)$ always holds and can be strict. In fact, there are actions with $\dim_{\mathrm{Rok}}(\alpha) = 1$ and $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = \infty$. Even worse, there are actions with $\dim_{\mathrm{Rok}}(\alpha) = 1$ and $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = 2$, even on classifiable C^*-algebras.

Remark 4.4. We have $\dim_{\mathrm{Rok}}(\alpha) = 0$ if and only if $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = 0$, and if and only if α has the Rokhlin property.

To see how much more general the notion of finite Rokhlin dimension is with respect to the Rokhlin property, we look at the case of commutative C^*-algebras.

Theorem 4.5. (Theorem 4.4 of [Gar14c]; see also Lemma 1.7 of [HP15]). Let $\alpha\colon G \to \mathrm{Aut}(C(X))$ be an action of a finite group G on a unital commutative C^*-algebra $C(X)$. Then $\dim_{\mathrm{Rok}}(\alpha) < \infty$ if and only if the induced action of G on X is free.

While the Rokhlin property corresponds to (global) triviality of the fiber bundle $X \to X/G$ (Theorem 2.6), finite Rokhlin dimension corresponds to local triviality, which for compact Lie groups is equivalent to freeness.

On purely infinite C^*-algebras, finite Rokhlin dimension is also very common:

Theorem 4.6. (Theorem 4.20 in [Gar14c]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital Kirchberg algebra A. Then α is pointwise outer if and only if $\dim_{\mathrm{Rok}}(\alpha) \leq 1$.

The definition of Rokhlin dimension was extended to compact groups [Gar14c], residually finite groups [SWZ14], and reals [HSWW16]. The main application in all of these works is showing that finiteness of the nuclear dimension is preserved (by taking crossed products). We recall the definition of the nuclear dimension of a C^*-algebra, which can also be thought as a "colored" version of being an AF-algebra (also, as a colored version of hyperfiniteness for von Neumann algebras).

Definition 4.7. Let A be a C^*-algebra, and let $r \in \mathbb{N}$. We say that A has *nuclear dimension at most* r, and write $\dim_{\mathrm{nuc}}(A) \leq r$, if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exist finite dimensional C^*-algebras $E_0, \ldots, E_r$, a completely positive contractive map $\varphi\colon A \to \bigoplus\limits_{k=0}^{r} E_k$, and completely positive contractive order zero maps $\psi_k\colon E_k \to A$, for $k = 0, \dots, r$, such that the diagram

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\theta"'] & & A \\ & \bigoplus_{k=0}^{r} E_k. \arrow[ur, "\sum\limits_{k=0}^{r} \psi_k"'] & \end{tikzcd}$$

In other words, $\left\| \left(\sum\limits_{k=0}^{d} \psi_k \circ \varphi \right)(a) - a \right\| < \varepsilon$ for every $a \in F$.

Here is the main application of finite Rokhlin dimension:

Theorem 4.8. (Theorem 1.4 in [HWZ] and Theorem 3.4 in [Gar15b]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A. Then

$$\dim_{\mathrm{nuc}}(A^\alpha) + 1 = \dim_{\mathrm{nuc}}(A \rtimes_\alpha G) + 1 \leq (\dim_{\mathrm{nuc}}(A) + 1)(\dim_{\mathrm{Rok}}(\alpha) + 1).$$

Proof. We sketch an idea of the proof. Set $\dim_{\mathrm{Rok}}(\alpha) = d$ and $\dim_{\mathrm{nuc}}(A) = r$. Let $S \subseteq A \rtimes_\alpha G$ be a finite subset. Without loss of generality, we assume that S has the form $F \cup \{u_g \colon g \in G\}$ for some finite subset $F \subseteq A$. Let $\varphi_0, \dots, \varphi_d \colon C(G) \to A_\infty \cap A'$ be equivariant completely positive contractive maps as in Proposition 4.2. Denote by $\iota\colon A \to C(G) \otimes A$ the map $\iota(a) = 1 \otimes a$ for all $a \in A$. By tensoring with A and the identity on it, we obtain the following commutative diagram of equivariant maps:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\iota"'] & & A_\infty \\ & C(G) \otimes A. \arrow[ur, "\sum\limits_{j=0}^{d} \widetilde{\varphi}_j"'] & \end{tikzcd}$$

By taking crossed products, and using that $C(G, A) \rtimes_{\mathtt{Lt} \otimes \alpha} G \cong A \otimes \mathcal{K}(L^2(G))$, we obtain the following commutative diagram:

$$\begin{tikzcd} A \rtimes_\alpha G \arrow[rr] \arrow[dr, "\iota"'] & & (A \rtimes_\alpha G)_\infty \\ & A \otimes \mathcal{K}(L^2(G)). \arrow[ur, "\sum\limits_{j=0}^{d} \widehat{\varphi}_j"'] & \end{tikzcd}$$

Let $E_0, \dots, E_r$, φ and $\psi_0, \dots, \psi_r$ determine an r-colored approximation of F up to ε as in Definition 4.7. For $k = 0, \dots, r$ and $j = 0, \dots, d$, set

$$E_k^{(j)} = E_k \quad \text{and} \quad \psi_k^{(j)} = \widehat{\varphi_j} \circ \psi_k \colon E_k^{(j)} \to (A \rtimes_\alpha G)_\infty.$$

Then the following diagram, which commutes on S up to ε, shows that $\dim_{\mathrm{nuc}}(A \rtimes_\alpha) \leq (d+1)(r+1)-1$, as desired (details omitted):

$$\begin{tikzcd}
A \rtimes_\alpha G \arrow[rrr] \arrow[dr, "\iota"'] & & & (A \rtimes_\alpha G)_\infty \\
& A \otimes \mathcal{K}(L^2(G)) \arrow[dr, "\theta"'] & & \\
& & \bigoplus_{j=0}^{d} (E_0 \oplus \cdots \oplus E_r)\,. \arrow[uur, "\sum_{k=0}^{r}\sum_{j=0}^{d}\psi_k^{(j)}"'] &
\end{tikzcd}$$

□

For more general groups, one sometimes has to include a multiplicative constant (depending only in G) on the right-hand side of the inequality in Theorem 4.8; see [SWZ14] and [HSWW16]. A similar statement holds for the decomposition rank in the case of compact groups, but fails for infinite discrete groups (in fact, already for $\mathbb{Z}$).

4.1. **Rokhlin dimension with commuting towers.** Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A with $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq d$. We will derive some general facts first, and then explain how they can be used to prove results about crossed products.

By Proposition 4.2, there exist completely positive contractive order zero maps $\varphi_0, \ldots, \varphi_d\colon C(G) \to A_\infty \cap A'$ with commuting images and such that $\sum\limits_{j=0}^{d} \varphi_j(1) = 1$. Denote by C the unital C^*-algebra generated by the image of all these maps. Then this algebra is commutative, so it has the form $C(X)$ for some compact Hausdorff space X. One can check that X must have covering dimension at most d. On the other hand, since G acts on the image of each of the maps φ_j, it also acts on C, and this induces an action on X. This action is also easily checked to be free. The result is then a unital equivariant homomorphism $C(X) \to A_\infty \cap A'$ from the free G-space X into the central sequence algebra of A. Since the fiber bundle $X \to X/G$ has local cross-sections (that is, it is locally trivial), the dimension of the orbit space X/G is also at most d.

Arguing similar to the proof of Theorem 4.8, there is a commutative diagram of equivariant unital homomorphisms as follows:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\iota"'] & & A_\infty \\
& C(X) \otimes A. \arrow[ur] &
\end{tikzcd}$$

By taking crossed products we arrive at a local approximation of $A \rtimes_\alpha G$ by $C(X,A) \rtimes G$ as follows:

$$\begin{tikzcd} A \rtimes_\alpha G \arrow[rr] \arrow[dr, "\iota"'] & & (A \rtimes_\alpha G)_\infty \\ & C(X,A) \rtimes G. \arrow[ur] & \end{tikzcd}$$

We would like to transfer properties from A to $A \rtimes_\alpha G$. Using the approach above, an intermediate step would be to first transfer properties from A to $C(X,A) \rtimes G$, and then use the above diagram to transfer them to $A \rtimes_\alpha G$. Unfortunately, $C(X,A) \rtimes G$ is not even Morita equivalent to $(C(X) \rtimes G) \otimes A$. However, it is a continuous $C(X/G)$-alebra with fibers Morita equivalent to A:

Theorem 4.9. (Corollary 3.6 in [GHS14]). Let G be a finite group, and let X be a free G-space with finite covering dimension. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be any action of G on a unital C^*-algebra A. Then $C(X,A) \rtimes G$ is a locally trivial continuous $C(X/G)$-algebra with fibers canonically isomorphic to $A \otimes \mathcal{K}(L^2(G))$.

The next ingredient is a family of results in the literature, that assert that if Y is a compact Hausdorff space of finite covering dimension, and B is a (locally trivial) continuous $C(Y)$-algebra, then certain properties pass from the fibers of B to all of B. Examples of such properties are $\mathcal{D}$-stability ([HRW07]), the UCT in the nuclear case ([Dad03]), and many others.

In the following theorem, we summarize some of the properties that are preserved by actions with finite Rokhlin dimension with commuting towers (the list is not exhaustive).

Theorem 4.10. (Theorem 3.14 in [GHS14]). The following classes of unital C^*-algebras are closed under formation of crossed products and passage to fixed point algebras by actions of compact groups with the Rokhlin property:

(1) Simple C^*-algebras. More generally, the ideal structure can be completely determined;
(2) Kirchberg algebras;
(3) Simple C^*-algebras with tracial rank zero;
(4) Separable, nuclear C^*-algebras satisfying the Universal Coefficient Theorem;
(5) C^*-algebras with finite nuclear dimension or decomposition rank;
(6) C^*-algebras with finite real rank zero or stable rank;
(7) Separable $\mathcal{D}$-absorbing C^*-algebras, for a strongly self-absorbing C^*-algebra $\mathcal{D}$;
(8) C^*-algebras whose K-groups are either: trivial, free, torsion-free, torsion, or finitely generated;

In the next result, we relate finite Rokhlin dimension with commuting towers with the weak tracial Rokhlin property, which gives a proof of item (3) in the theorem above. The proof uses the free G-space X constructed before, and is in fact very similar to the argument given in Example 3.15 to show that the gauge action of $\mathbb{Z}_2$ on A_θ has the tracial Rokhlin property.

Theorem 4.11. (Theorem 2.3 in [GHS14]). Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a separable, unital C^*-algebra A with strict comparison and countably many extreme traces. If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$, then α has the weak tracial Rokhlin property.

Finally, we state another result from [GHS14], which allows us to conclude that an action has the Rokhlin property, just by knowing that it has finite Rokhlin dimension *with commuting towers*.

Theorem 4.12. (Theorem 3.34 in [GHS14]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C^*-algebra A, and suppose that $A \otimes M_{|G|^\infty} \cong A$. If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$, then α has the Rokhlin property.

5. Equivalence between the weak tracial Rokhlin property and finite Rokhlin dimension

We begin by summarizing the so-far discussed relations between the Rokhlin-type properties mentioned in these lectures. A full arrow represents a (trivial) implication, while a dotted arrow means that in some circumstances, an additional implication exists, and a reference is given:

$$
\begin{tikzcd}
 & \mathrm{tRp} \arrow[r, Rightarrow] & \mathrm{wtRp} \arrow[l, Rightarrow, dashed, bend right, "\text{Theorem 3.6}"'] \arrow[l, Rightarrow, dashed, bend left, "\text{Proposition 3.13}"] \arrow[dd, Leftrightarrow, dashed, "\text{Theorem 3.6 and}"', "\text{Theorem 4.6}"] \\
\mathrm{Rp} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty \arrow[r, Rightarrow] \arrow[ul, Rightarrow, dashed, bend left, "\text{Theorem 4.12}"] & \dim_{\mathrm{Rok}}(\alpha) < \infty
\end{tikzcd}
$$

There are two other easy implications: the tracial Rokhlin property implies the Rokhlin property for actions on algebras without property (SP), and (obviously) $\dim_{\mathrm{Rok}}(\alpha) < \infty$ implies $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$ for actions on commutative algebras.

The goal of this section is to show that the weak tracial Rokhlin property and finite Rokhlin dimension are equivalent for finite group actions on classifiable C^*-algebras. (The result holds more generally for amenable groups.) To motivate the result, we revisit a celebrated result of Ocneanu [Ocn85] (what he showed is much more general). The case of $G = \mathbb{Z}_n$ for $\mathcal{M} = \mathcal{R}$ was first proved by Connes [Con77], while the case of finite G and $\mathcal{M} = \mathcal{R}$ was obtained by Jones [Jon80].

We need a definition first (we give it for finite groups, but it can be given for amenable groups). We denote the unique trace of a II_1-factor $\mathcal{M}$ by τ, and write $\|\cdot\|_{2,\tau}$ for the associated 2-norm: $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$ for $x \in \mathcal{M}$.

Definition 5.1. Let $\gamma\colon G \to \mathrm{Aut}(\mathcal{M})$ be a finite group action on a II_1-factor $\mathcal{M}$. We say that γ has the *(von Neumann) Rokhlin property* if for every $\varepsilon > 0$ and every finite subset $F \subseteq \mathcal{M}$, there exist projections $e_g \in \mathcal{M}$, for $g \in G$, satisfying

(1) $\|\gamma_g(e_h) - e_{gh}\|_{2,\tau} < \varepsilon$ for all $g, h \in G$;
(2) $\|e_g a - a e_g\|_{2,\tau} < \varepsilon$ for all $g \in G$ and all $a \in F$;
(3) $\sum\limits_{g\in G} e_g = 1$.

For a free ultrafilter ω, we denote by $\mathcal{M}^\omega \cap \mathcal{M}'$ the tracial central sequence of a II$_1$-factor $\mathcal{M}$, and if $\gamma\colon G \to \mathrm{Aut}(\mathcal{M})$ is a group action, we denote by $\gamma^\omega\colon G \to \mathrm{Aut}(\mathcal{M}^\omega \cap \mathcal{M}')$ the induced action.

Theorem 5.2. (See [Ocn85]). Let G be a countable amenable group, let $\mathcal{M}$ be a separably acting McDuff II$_1$-factor, and let $\gamma\colon G \to \mathrm{Aut}(\mathcal{M})$ be an action. Then the following are equivalent:
(1) γ^ω is (pointwise) outer;
(2) γ has the Rokhlin property.
Moreover, if the above conditions hold, then $\gamma\overline{\otimes}\mathrm{id}_{\mathcal{R}}$ is cocycle conjugate to γ.

We wish to obtain a C*-analog of the above result, and for this we need to find substitutes for $\mathcal{M}$, ponintwise outerness, the von Neumann Rokhlin property, and for $\mathrm{id}_{\mathcal{R}}$. The following are natural choices:

- Instead of a separably acting McDuff II$_1$-factor $\mathcal{M}$, we consider a separable, stably finite, exact, simple unital C^*-algebra A, which is $\mathcal{Z}$-stable.
- Instead of pointwise outerness in the central sequence algebra, we ask for outerness on every factor representation.
- Instead of the von Neumann Rokhlin property we ask for the weak tracial Rokhlin property, or, equivalently, finite Rokhlin dimension.
- Instead of absorbing $\mathrm{id}_{\mathcal{R}}$, we ask for absorption of $\mathrm{id}_{\mathcal{Z}}$.

At the moment, we need to adopt an additional assumption on A, which we hope to remove in the future. Namely, we only work with algebras A as above for which $T(A)$ is a Bauer simplex and $\partial_e T(A)$ is finite dimensional. Our main result is then:

Theorem 5.3. (See [GH17]). Let G be a finite group, let A be a C^*-algebra as above, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then the following are equivalent:
(1) α is strongly outer;
(2) α has the weak tracial Rokhlin property;
(3) $\dim_{\mathrm{Rok}}(\alpha) \leq 1$.
Moreover, if the above conditions hold, then $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ is (cocycle) conjugate to α.

There's also a version for arbitrary amenable groups, which we omit. Observe that any A as in the theorem is in particular finite and has strict comparison (so that the definition of the weak tracial Rokhlin property does not require to take the positive contraction $x \in A$.) A version of this theorem for $G = \mathbb{Z}$ was proved first by Liao; see [Lia16]. Our methods are, however, different.

Proof. The proof that (3) implies (1) is not difficult, and we omit it. We will show that (1) implies (2), and that (2) implies (3). We will sketch the proof in the case in which A has a unique trace τ. (The result is new even in this case.)

We show that (1) implies (2). The assumptions imply that the weak closure $\mathcal{M}$ of A in the GNS representation associated to τ is a McDuff factor, and that the action $\gamma\colon G \to \mathrm{Aut}(\mathcal{M})$ induced by α satisfies the assumptions of Theorem 5.2. Let $\varepsilon > 0$ and $F \subseteq A$ be a finite set. Find projections $e_g \in \mathcal{M}$, for $g \in G$, satisfying the conditions in Definition 5.1. Since the unit ball of $\mathcal{M}$ is the completion of the unit ball of A with respect to the norm $\|\cdot\|_{2,\tau}$, and by working in the central sequence algebra $A_\omega \cap A'$, we can find orthogonal positive contractions $f_g \in A_\omega \cap A'$, with $g \in G$, satisfying

- $(\alpha_\omega)_g(f_h) - f_{gh} \in J_A$, for all $g, h \in G$;
- $f_g a - a f_g \in J_A$ for all $g \in G$ and all $a \in F$;
- $1 - \sum\limits_{g \in G} f_g \in J_A$.

Let C denote the (separable) C^*-algebra generated by A and $\{(\alpha_\omega)_g(f_h) \colon g, h \in G\}$. One can show that there exists $x \in (J_A \cap C')^{\alpha_\omega}$ such that $xc = c$ for all $J_A \cap C$. It is then easy to check that the elements $\widetilde{f_g} = (1-x)f_g(1-x)$ satisfy the conditions in the definition of the weak tracial Rokhlin property.

Before proving that (2) implies (3), we need to establish equivariant $\mathcal{Z}$-stability. By Theorem 5.2, the weak extension γ of α absorbs $\mathrm{id}_{\mathcal{R}}$, so there exists a unital embedding of $\mathcal{R}$ into $(\mathcal{M}^\omega \cap \mathcal{M}')^{\gamma^\omega}$. Since M_2 is a unital subalgebra of $\mathcal{R}$, and $(\mathcal{M}^\omega \cap \mathcal{M}')^{\gamma^\omega}$ is the quotient of $(A_\omega \cap A')^{\alpha_\omega}$ (by J_A), we get the following diagram:

$$\begin{tikzcd}
 & & (A_\omega \cap A')^{\alpha_\omega} \arrow[d] \\
M_2 \arrow[r] \arrow[urr, dashed, "\rho"] & \mathcal{R} \arrow[r] & (\mathcal{M}^\omega \cap \mathcal{M}')^{\gamma^\omega}.
\end{tikzcd}$$

Since M_2 is finite dimensional, there exists a completely positive contractive order zero map ρ making the diagram commute. Since A is $\mathcal{Z}$-stable and α has the weak tracial Rokhlin property, one can show that (A, α) satisfies an equivariant version of property (SI) which allows one to extend ρ to a unital homomorphism $\psi \colon I_{2,3} \to (A_\omega \cap A')^{\alpha_\omega}$ from the dimension drop algebra $I_{2,3}$. This is known to imply that $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ is (cocycle) conjugate to α.

Finally, we show that (2) implies (3). The idea is to "break" the tower coming from the weak tracial Rokhlin property, as well as the left over, into two Rokhlin dimension towers. The extra copy of $\mathcal{Z}$ is crucial in this step. Indeed, earlier methods to obtain results like ours, not relying on equivariant $\mathcal{Z}$-stability, had the disadvantage of breaking down for finite groups or infinitely generated groups like $\mathbb{Z}^\infty$.

Concretely, we do the following. Let $f_g \in A$, for $g \in G$, be positive contractions as in the definition of the weak tracial Rokhlin property, and set $f = \sum\limits_{g \in G} f_g$. Then $\tau(f) = \tau(1) = 1$. Find a positive contraction $h_0 \in \mathcal{Z}$ with $\mathrm{sp}(h_0) = [0,1]$, and set $h_1 = 1 - h_0$. Classifications results for positive contractions of full trace imply that there exist unitaries $u_0, u_1 \in \mathcal{U}((A^\alpha \otimes \mathcal{Z})_\omega)$ such that $u_j(f \otimes h_j)u_j^* = 1 \otimes h_j$ for $j = 0, 1$. Finally, for $j = 0, 1$ and $g \in G$, we define $f_g^{(j)} = u_j f_g u_j^*$. One checks that these elements witness the fact that $\dim_{\mathrm{Rok}}(\alpha) \leq 1$, finishing the proof. $\square$

6. Problems

Here, we list some open problems and questions whose solution should be of interest. They are listed roughly in increasing order of (expected!) difficulty.

(1) We have seen that the weak tracial Rokhlin property implies the tracial Rokhlin property in some cases (for actions on Kirchberg algebras, and for actions on TAF algebras). Does this result hold in a more general context?

How about tracial rank at most one? How about real rank zero and $\mathcal{Z}$-stability?

(2) Develop a theory of the Rokhlin property (and maybe Rokhlin dimension) for actions of more general objects, such as groupoids, semigroups, or partial actions.

(3) There are by now some projectionless versions of being TAF.Are these preserved by the weak tracial Rokhlin property?

(4) Prove that for simple unital AF-algebras, the weak tracial Rokhlin property and Rokhlin dimension at most one are equivalent, and equivalent to strong outerness. (The point of this problem is not to assume that $T(A)$ is a Bauer simplex and that its boundary has finite covering dimension. One could assume that the action of G on $T(A)$ is trivial to begin with.)

(5) Alternatively, for (non-simple) unital AF-algebras, study whether strong outerness implies finite Rokhlin dimension. (There may be things to figure out here, for example what traces one should use: does one have to allow for unbounded traces?)

(6) Does finite Rokhlin dimension preserve tracial $\mathcal{Z}$-stability? In the case of commuting towers, this seems to be true (S. Jamali).

(7) Classify finite group actions with finite Rokhlin dimension with commuting towers on Kirchberg algebras. Is it enough to have α_g approximately unitarily equivalent to β_g for all $g \in G$? I don't know of any counterexamples, but they may exist.

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