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Lyubeznik Tables of Ideals of Cycle Graphs

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Let A be a commutative Noetherian local ring which admits a surjective ring homomorphism $\pi : S \to A$, where (S, η, k) is a regular local ring of dimension n containing a field. Set $I = Ker(\pi)$. In 1993, Huneke and sharp in characteristic $p > 0$ using the Frobenius map, and Lyubeznik in characteristic zero using D-modules, showed that the local cohomology module $H^i_\eta(H^{n-j}_I(S))$ is injective and supported at η. In the cases $H^i_\eta(H^{n-j}_I(S))$ is a finite direct some of many ($= \lambda_{i,j}(A)$ known as Lyubeznik number A) copies of the injective hull $E(S/\eta)$ of the residue field of S. In fact the Lyubeznik numbers $\lambda_{i,j}(A)$, $i, j \geq 0$ are some of the invariants of A defined as i-th Bass number of the local cohomology module $H^{n-j}_I(S)$ is as follows:

$$\lambda_{i,j}(A) := \mu_i(\eta, H^{n-j}_I(S)) = \mu_0(\eta, H^i_\eta(H^{n-j}_I(S))) = \dim_k \mathrm{Ext}^i_S(k, H^{n-j}_I(S))$$

depend only on A, i and j. Lyubeznik numbers carry some geometrical and topological information. Moreover, note that in the case of isolated singularities, Lyubeznik numbers can be described in terms of certain singular cohomology groups in characteristic zero or étale cohomology groups in positive characteristic. These invariants satisfy the following properties:

i) $\lambda_{i,j}(A) = 0$ if $j > d$,

ii) $\lambda_{i,j}(A) = 0$ if $i > j$ and $\lambda_{d,d}(A) \neq 0$,

iii) Euler characteristic: $\sum_{0 \leq i,j \leq d} (-1)^{i-j} \lambda_{i,j}(A) = 1$,

where $d = \dim(A)$. Therefore, we denote all nonzero Lyubeznik numbers as follows:

$$\Lambda(A) = \begin{bmatrix} \lambda_{0,0} & \dots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{bmatrix},$$

that so-called Lyubeznik table. The Lyubeznik table of A is trivial if $\lambda_{d,d}(A) = 1$ and the rest of these invariants vanish. The highest Lyubeznik number $\lambda_{d,d}(A)$ can be described as the number of connected components of the so-called Hochster-Huneke graph G.

However not so much is known about the Lyubeznik tables except in some partial cases. Furthermore in the works of Walther and Kawasaki the Lyubeznik table for rings of dimension less than 3 understood well. In this talk, we are interested in examining the Lyubeznik table of cycle graphs. We show that:

Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k, m denotes it's homogeneous maximal ideal $(x_1, ..., x_n)$. Let C_n, the n-cycle graph, be a graph on the vertex set $\{x_1, ..., x_n\}$ and the edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, ..., \{x_{n-1}, x_n\}, \{x_n, x_1\}\}$. Then the edge ideal of C_n is the ideal

$$I_{C_n} = (x_1x_2, x_2x_3, ..., x_{n-1}x_n, x_nx_1).$$

we compute the last column of the Lyubeznik table of R/I_{C_n}.

Theorem A. *Let n be an even integer. Then the last column of the Lyubeznik table of R/I is as follows:*

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 2 \end{bmatrix}.$$

Theorem B. *Let $k \geq 3$ be a positive integer and $n = 2k + 1$. Then $\lambda_{k,k}(R/I) = \lambda_{k-1,k}(R/I) = 1$ and the last column of Lyubeznik table of R/I is*

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

The talk is based on my joint work with Farhad Rahmati and Majid Eghbali.