DIMENSION THEORIES IN TOPOLOGY, COARSE GEOMETRY, AND C^*-ALGEBRAS

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Abstract. These notes are written as supplementary material for a series of talks given at IPM (Institute for Research in Fundamental Sciences) in Tehran from February 5th to February 8th, 2018. The intent is to introduce interesting links between topological covering dimension, asymptotic dimension, and nuclear dimension. We also insert a quick introduction to basic notions in coarse geometry.

Please note that only little proofreading has been done for this version of the notes. If you find any mistakes or typos, please let me know.

1. Topological covering dimension

Definition 1.1. Let X be a topological space, and let $\alpha = \{U_i\}_{i\in I}$ be an open cover of X. The *order* (or *multiplicity*) of α, written as $\operatorname{ord}(\alpha)$, is the largest integer n such that there exist distinct $i_0, i_1, ..., i_n \in I$ satisfying $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_n} \neq \emptyset$.

Here are various ways of saying the same thing.

Lemma 1.2. *Let X be a topological space, and let $\alpha = \{U_i\}_{i\in I}$ be an open cover of X. Then the following are equivalent:*

(1) $\operatorname{ord}(\alpha) \leq n$;
(2) for any $n+2$ distinct indices $i_0, ..., i_{n+1} \in I$ we have $U_{i_0} \cap \cdots \cap U_{i_{n+1}} = \emptyset$;
(3) every $x \in X$ is contained in at most $n+1$ distinct members of α.

Proof. This is immediate from the definition. □

In particular, the order of α can be defined as the smallest integer n such that every point $x \in X$ is contained in at most n members of α.

Definition 1.3. Let X be a topological space, and let $\alpha = \{U_i\}_{i\in I}$ and $\beta = \{V_j\}_{j\in J}$ be open covers of X. We say β *refines* α if for each $j \in J$, there exists $i \in I$ such that $V_j \subseteq U_i$. In this case we say β is an *(open) refinement* of α.

Definition 1.4. Let X be a topological space. The *topological covering dimension* (or *Čech-Lebesgue covering dimension*) of X, written as $\dim(X)$, is the smallest integer $n \in \mathbb{N} \cup \{0\}$ such that every finite open cover of X has an open refinement of order at most n.

If such an integer does not exist, then we write $\dim(X) = \infty$.

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Remark 1.5. The usual convention is that $\dim(\emptyset) = -1$.

Remark 1.6. Since the definition only involves open subsets, the covering dimension is invariant under homeomorphisms.

Suppose X is a topological space with $\dim(X) \leq n$. Let $\alpha = \{U_i\}_{i\in I}$ be a finite open cover of X. Then by definition there exists an open refinement $\beta := \{V_j\}_{j\in J}$ with $\mathrm{ord}(\beta) \leq n$. By collecting sets which belong to the same U_i, we may assume that β has the form $\beta = \{W_i\}_{i\in I}$ such that $W_i \subseteq U_i$ for every $i \in I$. In particular, we may assume that β is a finite cover.

The next two proposition are straightforward. The proofs are left as exercises.

Proposition 1.7. *Let X be a topological space and let $F \subseteq X$ be a closed subspace. Then* $\dim(F) \leq \dim(X)$.

Proposition 1.8. *Let X be a topological space and let $F_1, ..., F_n$ be closed subsets of X such that $\bigcup_{k=1}^n F_k = X$. Then we have* $\dim(X) = \max_{1\leq k\leq n} \dim(F_k)$.

Recall that a topological space X is *normal* if given any pair of closed subsets A and B of X, there exist disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. Examples of normal spaces include all metrizable spaces (in particular, all second countable locally compact Hausdorff spaces) and all compact Hausdorff spaces. For our purposes, the greatest advantage of working with normal spaces is that for every finite open cover we have a *partition of unity.*

Definition 1.9. Let X be a topological space, and let $\alpha = \{U_1, ..., U_n\}$ be a finite open cover of X. A family of continuous functions $h_i : X \to [0, 1]$ for $i = 1, 2, ..., n$ is called a *partition of unity subordinate to* α if

1. The support of h_i is contained in U_i for each $i = 1, 2, ..., n$.
2. $\sum_{i=1}^n h_i(x) = 1$ for each $x \in X$.

Theorem 1.10. [Mun00, Theorem 36.1] *Let X be a normal space and let α be a finite open cover of X. Then there exists a partition of unity subordinate to α.*

Using partition of unity we can give an alternative characterization of covering dimension for normal spaces. We will see that this motivates the definition of *nuclear dimension* for C^*-algebras.

Definition 1.11. Let X be a topological space and $\beta = \{V_j\}_{i\in J}$ be an open cover of X. We say β is *n-decomposable* if there is a decomposition $J = \bigsqcup_{i=0}^n J_i$ such that $V_j \cap V_{j'} = \emptyset$ whenever j and j' belong to the same J_i.

Proposition 1.12. [KW04, Proposition 1.5] *Let X be a normal space. Then the following are equivalent:*

(i) $\dim(X) \leq n$,
(ii) *every finite open cover of X has an n-decomposable finite open refinement.*

Proof. (Sketch) It is clear that (ii) implies (i). The proof of the converse makes use of a standard simplicial complex technique, which we briefly outline here.

Given a finite open cover $\alpha = \{U_i\}_{i\in I}$ of X, we can associate to α a finite simplicial complex C (called the *nerve* of α) in the following way. The set of vertices of C is the index set I, and the simplices are the subsets $J \subseteq I$ such that $\bigcap_{i\in J} U_i \neq \emptyset$. Observe that the dimension of C is equal to the order of α. As $\dim(X) \leq n$, by passing to open refinement we may assume that the order of α is at most n.

Let $\{h_i\}_{i\in I}$ be a partition of unity subordinate to α, and let $|C| \subseteq \mathbb{R}^I$ be the geometric realization of C. Define the map $f : X \to |C|$ by

$$f(x) := \sum_{i\in I} h_i(x)e_i,$$

where $\{e_i\}_{i\in I}$ is the standard basis for $\mathbb{R}^I$. Let e_i be a vertex of $|C|$ and let $\mathrm{St}(e_i)$ be the open star of C at e_i. Now the crucial observation is that the preimage $f^{-1}(\mathrm{St}(e_i))$ is contained in U_i for all $i \in I$. It is not hard to convince ourselves that $\beta := \{St(e_i)\}_{i\in I}$ is an open cover of $|C|$ and that β has an n-decomposable open refinement γ (one can obtain γ systematically using barycentric division). Now the pullback of γ along f would be a desired n-decomposable open refinement of α. □

Meanwhile, normality also gives us a countable version of Proposition 1.8.

Theorem 1.13. [Pea75, Theorem 3.2.5] *Let X be a normal space and let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X such that $X = \bigcup_{k\in\mathbb{N}} F_k$. Then we have* $\dim(X) = \sup_{k\in\mathbb{N}} \dim(F_k)$.

Theorem 1.14 (Lebesgue)**.** *For any $n \in \mathbb{N}$ we have* $\dim([0,1]^n) = n$.

Proof. It is not hard to show that $\dim([0,1]^n) \leq n$. In fact, for compact metrizable spaces one has the product formula: $\dim(X \times Y) \leq \dim(X) + \dim(Y)$ (see, for example, [BD08, Theorem 5]). The reverse inequality can be deduced from the Lebesgue covering theorem (see [HW41, Theorem IV.2]). □

Corollary 1.15. *For any $n \in N$ we have* $\dim(\mathbb{R}^n) = n$. *In particular, $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.*

Proof. This follows from the previous result and the countable sum theorem. □

Remark 1.16. The previous corollary can also be obtained using methods from algebraic topology. For example see the online notes [Sch12].

Definition 1.17 (Winter)**.** Let A and B be C^*-algebras, and let $\varphi : A \to B$ be a completely positive (c.p.) map. We say φ has *order zero* if for all $a, b \in A_+$ we have that $ab = 0$ implies $\varphi(a)\varphi(b) = 0$.

The following structure theorem for order zero maps should be compared to Stinespring's theorem for c.p. maps. Roughly, it says that order zero maps are nothing but compression of $*$-homomorphisms by a positive element in the commutant.

Theorem 1.18. [WZ09, Theorem 3.3] *Let A, B be C^*-algebras and $\varphi : A \to B$ a c.p. order zero map. Set $C := C^*(\varphi(A)) \subseteq B$.*

Then there exists a positive element $h \in \mathcal{M}(C) \cap C'$ with $\|h\| = \|\varphi\|$ and a $*$-homomorphism $\pi_\varphi : A \to \mathcal{M}(C) \cap \{h\}'$ such that

$$\pi_\varphi(a)h = h\pi_\varphi(A) = \varphi(a)$$

for all $a \in A$.

If A is unital, then $h = \varphi(1_A) \in C$.

Definition 1.19 (Winter-Zacharias)**.** Let A be a C^*-algebra. The *nuclear dimension* of A, written as $\dim_{\mathrm{nuc}}(A)$, is the smallest integer $n \in \mathbb{N} \cup \{0\}$ such that the following holds.

For every finite subset $F \subseteq A$ and $\varepsilon > 0$, there exist a finite-dimensional C^*-algebra F, a c.p.c. map $\psi : A \to F$, and a c.p. map $\varphi : F \to A$ such that

(1) $\|\varphi\psi(a) - a\| < \varepsilon$ for every $a \in F$.
(2) F decomposes into $F = F^{(0)} \oplus \cdots \oplus F^{(n)}$ such that $\varphi^{(i)} := \varphi|_{F^{(i)}}$ is c.p.c. order zero for each $i = 0, 1, ..., d$.

The following proposition, at least in the case that the space is second countable, is essentially contained in [Win03, Proposition 2.5]. A proof is included here only for the reader's convenience.

Proposition 1.20. *Let X be a locally compact Hausdorff normal space. Then we have $\dim_{\mathrm{nuc}}(C_0(X)) \leq \dim(X)$.*

Proof. Let $n = \dim(X) < \infty$. Fix a finite subset $F \subseteq C_0(X)$ and $\varepsilon > 0$. Let $K \subseteq X$ be a compact subset such that $|f(x)| < \frac{\varepsilon}{2}$ for all $f \in F$ and $x \in X \setminus K$. Using compactness of K, we can find a finite open cover $\alpha = \{U_1, ..., U_r\}$ of X such that $|f(x) - f(y)| < \varepsilon$ for all $f \in F$ and all x, y in the same U_i. Since $\dim(X) = n$ and X is normal, by Proposition 1.12 there exists an n-decomposable open refinement $\beta = \{V_1, ..., V_s\}$ of α. In other words, we have a decomposition $\beta = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(n)}$ such that each $\mathcal{V}^{(i)}$ consists of disjoint members.

Since X is normal, there exists a partition of unity $\{h_j\}_{j=1}^s$ subordinate to the open cover β. For each $j = 1, ..., s$ choose a point x_j in V_j. Define

$$\begin{aligned} &F := \mathbb{C}^s; \\ &\psi : C_0(X) \to F, \qquad \psi(f) := (f(x_1), ..., f(x_s)); \\ &\varphi : F \to C_0(X), \qquad \varphi(\lambda_1, ..., \lambda_s) := \sum_{j=1}^s \lambda_j h_j. \end{aligned}$$

It is clear that ψ is a $*$-homomorphism (hence a c.p.c. map). One can directly check that φ is c.p.c. as well. For every $f \in F$ and $x \in X$ we have

$$|\varphi\psi(f)(x) - f(x)| = \left|\sum_{j=1}^{s} f(x_j)h_j(x) - \sum_{j=1}^{s} f(x)h_j(x)\right|$$
$$\leq \sum_{j=1}^{s} |f(x_j) - f(x)|h_j(x) < \varepsilon.$$

Therefore $\|\varphi\psi(f) - f\| < \varepsilon$ for all $f \in F$. Finally, observe that the map $(\lambda_1, ..., \lambda_m) \mapsto \sum_{j=1}^{m} \lambda_j h_j$ has order zero if the functions $h_1, ..., h_m$ have disjoint supports. It follows that φ can be decomposed into $(n+1)$-order zero maps. This completes the proof. ☐

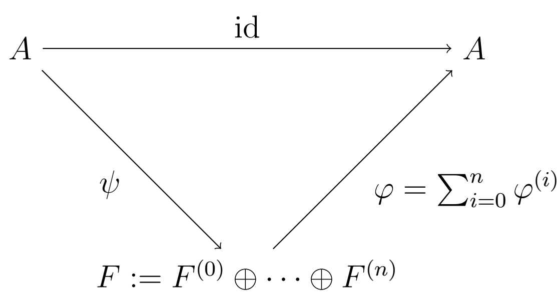
The next theorem is also due to Winter, though it is not easy to find a direct proof in the literature. Therefore we supply a complete proof here.

Theorem 1.21 (Winter)**.** *Let X be a compact Hausdorff space. Then we have* $\dim_{\mathrm{nuc}}(C(X)) \geq \dim(X)$.

Proof. For notational convenience let us write $A = C(X)$. Let $n = \dim_{nuc}(A) < \infty$. Given a finite open cover $\alpha = \{U_1, ..., U_r\}$ of X, we need to find an open refinement $\beta = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(n)}$ of α such that each $\mathcal{V}^{(i)}$ consists of disjoint members. Since X is normal, there exists a partition of unity $\{h_k\}_{k=1}^{r}$ subordinate to the open cover α. We define the constants

$$C := \frac{1}{4(n+1)}, \qquad \varepsilon := \frac{C}{16r(n+1)} = \frac{1}{64r(n+1)^2}.$$

By definition of nuclear dimension, we can find a c.p. approximation



for $\{h_1, ..., h_r, 1_A\}$ within ε in the way that ψ is c.p.c. and each $\varphi^{(i)} = \varphi|_{F^{(i)}}$ is c.p.c. order zero. Note that we have

$$\frac{1}{2} \leq \|1_A\| - \varepsilon \leq \|\varphi\psi(1_A)\| \leq \|\varphi\|\|\psi(1_A)\| \leq (n+1)\|\psi(1_A)\|.$$

Hence $\|\psi(1_A)\| \geq \frac{1}{2(n+1)}$.

Since A is abelian, each matrix direct summand of F is necessarily one-dimensional For each $i = 0, 1, ..., n$, write $F^{(i)} = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (k_i copies) and let $\{e_j^{(i)}\}_{j=1}^{k_i}$ be the minimal projections in $F^{(i)}$. For each element T in F we write $T(i,j)$ for the cut down of T by $e_j^{(i)}$ (note that $T(i,j)$ is a complex number). Define the index set

$$I := \left\{ (i,j) : i \in \{0,...,n\}, j \in \{1,...,k_i\}, \psi(1_A)(i,j) \geq \frac{1}{4(n+1)} \right\}.$$

Roughly speaking I consists of the places where $\psi(1_A)$ is "large". Note that I is nonempty by our lower bound estimate of $\|\psi(1_A)\|$ above. For each pair $(i,j) \in I$, define

$$V_j^{(i)} := \varphi(e_j^{(i)})^{-1}\left((C, +\infty)\right)$$

and for each $i = 0, 1, ..., n$ let

$$\mathcal{V}^{(i)} := \bigcup_{j:(i,j)\in I} V_j^{(i)}.$$

We claim that $\beta = \{\mathcal{V}^{(0)}, ..., \mathcal{V}^{(n)}\}$ is a desired open refinement of α.

First of all, it is clear that each $\mathcal{V}^{(i)}$ consists of disjoint members since each $\varphi^{(i)}$ is order zero and $\{e_j^{(i)}\}_j$ are mutually orthogonal. We show that β is indeed a cover of X. Write

$$q := \sum_{(i,j)\in I} e_j^{(i)}.$$

Then by construction we have

$$\|\psi(1_A) - q\psi(1_A)\| \leq \frac{1}{4(n+1)}.$$

Therefore we have

$$\begin{aligned} \|\varphi\psi(1_A) - \varphi(q\psi(1_A))\| &\leq \|\varphi\|\|\psi(1_A) - q\psi(1_A)\| \\ &\leq (n+1) \cdot \frac{1}{4(n+1)} \\ &= \frac{1}{4}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{(i,j)\in I} \varphi(e_j^{(i)}) = \varphi(q) &\geq \varphi(q\psi(1_A)) \\ &\geq \varphi\psi(1_A) - \frac{1}{4}1_A \\ &\geq 1_A - \left(\varepsilon + \frac{1}{4}\right) 1_A \\ &\geq \frac{1}{2}1_A. \end{aligned}$$

Since for each $i = 0, 1, ..., n$ the functions $\{\varphi(e_j^{(i)})\}_j$ have disjoint supports, for every $x \in X$ there exists a pair $(i, j) \in I$ such that

$$\varphi(e_j^{(i)})(x) \geq \frac{1}{2(n+1)} > C.$$

This shows that $\beta = \{\mathcal{V}^{(0)}, ..., \mathcal{V}^{(n)}\}$ is a cover of X.

It remains to show that β refines α. By construction we have

$$\frac{1}{4(n+1)} \sum_{(i,j)\in I} e_j^{(i)} \leq \psi(1_A) \sum_{(i,j)\in I} e_j^{(i)} = \sum_{(i,j)\in I} e_j^{(i)} \left(\sum_{k=1}^{r} \psi(h_k) \right).$$

Therefore for each pair $(i, j) \in I$ there is some $k \in \{1, ..., r\}$ such that $\frac{1}{4r(n+1)} \leq \psi(h_k)(i, j)$. Now fix $(i, j) \in I$ and choose k as above. For every $x \in V_j^{(i)}$ we have

$$C < \varphi(e_j^{(i)})(x) \leq 4r(n+1)\varphi\psi(h_k)(x) \approx_{4r(n+1)\varepsilon} 4r(n+1)h_k(x).$$

By our choices of C and ε we have

$$4r(n+1)\varepsilon = 4r(n+1) \cdot \frac{C}{16r(n+1)} = \frac{C}{2} < C.$$

Therefore $h_k(x) > 0$. This shows that $V_j^{(i)}$ is completely contained in U_k, and it follows that β is a refinement of α. □

Corollary 1.22. *Let X be a locally compact Hausdorff and σ-compact space. Then we have $\dim_{\text{nuc}}(C_0(X)) = \dim(X)$.*

Proof. Since every locally compact Hausdorff and σ-compact space is normal (see Munkres Theorem 41.1 and Theorem 41.5), we have $\dim_{\text{nuc}}(C_0(X)) \leq \dim(X)$. For the reverse inequality, write $X = \bigcup_{n\in\mathbb{N}} K_n$ as a countable union of compact subsets. By the countable union theorem we have $\dim(X) = \sup_{n\in\mathbb{N}} \dim(K_n)$. Now each K_n is compact Hausdorff, so by the previous theorem we have $\dim(K_n) = \dim_{\text{nuc}}(C(K_n))$ for all $n \in \mathbb{N}$. As K_n is closed in X, we can identify $C(K_n)$ as a quotient of $C_0(X)$. By [Winter-Zacharias Proposition 2.3] we have $\dim_{\text{nuc}}(C(K_n)) \leq \dim_{\text{nuc}}(C_0(X))$. Combining, we obtain

$$\dim(X) = \sup_{n\in\mathbb{N}} K_n \leq \sup_{n\in\mathbb{N}} \dim_{\text{nuc}}(C(K_n)) \leq \dim_{\text{nuc}}(C_0(X)).$$

This completes the proof. □

Remark 1.23. Let X be a locally compact Hausdorff space. Note that if X is second countable, then X is necessarily σ-compact. The converse is false since a compact Hausdorff space does not have to be second countable.

Exercises

Exercise 1.1. Prove Proposition 1.7.

Exercise 1.2. Prove Proposition 1.8 (note that by induction, it suffices to prove the case $X = F_1 \cup F_2$).

Exercise 1.3. Show that $\dim([0,1]) = 1$ using the definition. (Hint: one way to show that $\dim([0,1]) \leq 1$ is to make use of Lebesgue numbers)

Exercise 1.4. Let X be a Hausdorff space. Show that if $\dim(X) = 0$ then X is totally disconnected. (Remark: the converse is also true if we further assume that X is compact)

Exercise 1.5. Let A be an abelian C^*-algebra and $\varphi : M_n(\mathbb{C}) \to A$ be a nonzero c.p. order zero map. Show that we must have $n = 1$. (Hint: Use the structure theorem for order zero maps)

2. Asymptotic dimension and coarse geometry

Asymptotic dimension was first introduced by Gromov [Gro93]. Soon after Yu showed that finitely generated groups with finite asymptotic dimension satisfies the Novikov conjecture [Yu98]. Since then asymptotic dimension, and more generally large-scale geometric methods, has found many applications in geometry and topology of manifolds.

Asymptotic dimension is a large-scale analogue of the covering dimension. The intuition is the following: when we look at the set $\mathbb{Z}^n$ from far away, the distances between points become very small and the entire $\mathbb{Z}^n$ looks like $\mathbb{R}^n$. Therefore $\mathbb{Z}^n$ should have large-scale covering dimension n.

Definition 2.1. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a (not necessarily open) cover of a metric space, and let $R > 0$. The *R-multiplicity* of $\mathcal{U}$ is the smallest integer $n \in \mathbb{N} \cup \{0\}$ such that for every $x \in X$, the ball $B(x, R)$ intersects at most $n+1$ members of $\mathcal{U}$.

We say that a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of a metric space is *uniformly bounded* if we have $\sup_{i\in I} \operatorname{diam}(U_i) < \infty$.

Definition 2.2 (Gromov)**.** Let X be a metric space. The *asymptotic dimension* of X, written as $\operatorname{asdim}(X)$, is the smallest integer $n \in \mathbb{N} \cup \{0\}$ such that for every $R > 0$ there exists a uniformly bounded cover $\mathcal{U} = \{U_i\}_{i\in I}$ with R-multiplicity at most n.

If such an integer does not exist, then we write $\operatorname{asdim}(X) = \infty$.

Example 2.3. Let $X = \{\pm n^2 : n \in \mathbb{N}\}$ equipped with the usual metric. Then $\operatorname{asdim}(X) = 0$.

Example 2.4. We have $\operatorname{asdim}(\mathbb{Z}) \leq 1$.

Proposition 2.5. *Let X be a metric space and let $Y \subseteq X$ be a subspace. Then* $\operatorname{asdim}(Y) \leq \operatorname{asdim}(X)$*.*

Proof. When $\operatorname{asdim}(X) = \infty$ there is nothing to prove. Therefore assume $\operatorname{asdim}(X) = n < \infty$. Given $R > 0$, there exists a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that $\mathcal{U}$ has R-multiplicity n. For each $i \in I$ define $V_i := U_i \cap Y$ and consider the cover $\mathcal{V} = \{V_i\}_{i\in I}$ of Y. Since a ball $B_Y(x, R)$ in Y is precisely the intersection $B_X(x, R) \cap Y$, the R-multiplicity of $\mathcal{V}$ is smaller than the R-multiplicity of X. □

We mentioned that since $\mathbb{Z}^n$ looks like $\mathbb{R}^n$ when viewed from far away, the asymptotic dimension of $\mathbb{Z}^n$ should be equal to n. As in the topological case, the difficult part is to find the lower bound. However, a short proof can be given if we assume the knowledge $\dim([0, 1]^n) = n$. Here we only sketch the idea. Full details can be found in [NY12, Example 2.2.6].

Theorem 2.6. *For any $n \in \mathbb{N}$ we have* $\operatorname{asdim}(\mathbb{Z}^n) = n$.

Proof. (Sketch) We only show that $\operatorname{asdim}(\mathbb{Z}^n) \geq n$. For the sake of contradiction, assume that $\operatorname{asdim}(\mathbb{Z}^n) \leq n - 1$. Set $R = 5$ and find a uniformly bounded cover $\mathcal{U}$ of $\mathbb{Z}^n$ with R-multiplicity $n - 1$. Consider the floor map $\lfloor\cdot\rfloor^n : \mathbb{R}^n \to \mathbb{Z}^n$ defined by

$$\lfloor(x_1, ..., x_n)\rfloor^n := (\lfloor x_1\rfloor, ..., \lfloor x_n\rfloor).$$

The pullback of $\mathcal{U}$ along $\lfloor\cdot\rfloor^n$ forms a uniformly bounded cover $\mathcal{V}$ of $\mathbb{R}^n$ with order $n - 1$. By taking a small open neighborhood for each member of $\mathcal{V}$, we obtain a uniformly bounded open cover β of $\mathbb{R}^n$ with order $n - 1$.

Now let α be a finite open cover of $[0, 1]^n$. Since $[0, 1]^n$ is compact, there exists a Lebesgue number $\delta > 0$ for α. Since the open cover β is uniformly bounded, we can re-scale β so that all members of β have diameters less than δ. In this way the restriction of β to $[0, 1]^n$ is an open refinement of α with order $n-1$. Since α is arbitrary, we conclude that $\dim([0, 1]^n) \leq n-1$, a contradiction. □

Remark 2.7. Like in the topological case, this result can also be obtained using homological methods. See, for example, [Roe03, Section 9.2].

Recall that for normal spaces, the covering dimension can be characterized using n-decomposable covers (see Proposition 1.12). We have a similar characterization for asymptotic dimension.

Definition 2.8. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a family of subsets of a metric space X, and let $R > 0$. We say $\mathcal{U}$ is *R-disjoint* if $d(U_i, U_{i'}) \geq R$ whenever i and i' are distinct indices in I.

Proposition 2.9. [BD08, Theorem 19] *Let X be a metric space. Then the following are equivalent:*

(1) $\operatorname{asdim}(X) \leq n$;
(2) *for every $R > 0$ there exist families $\mathcal{U}^{(0)}, ..., \mathcal{U}^{(n)}$ of subsets of X which together form a uniformly bounded covering of X, and each $\mathcal{U}^{(i)}$ is R-disjoint.*

Proof. (sketch) We only look at the implication (1) $\Longrightarrow$ (2), as the converse is obvious. The proof is very similar to the one we saw in Proposition 1.12.

Let $R > 0$ be given, and let $R' > 0$ be a constant (depending on R) to be determined later. By definition of asymptotic dimension there exists a uniformly bounded cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X with $(5R')$-multiplicity at most n. For each $i \in I$, let V_i be the $(2R')$-neighborhood of U_i. Then $\mathcal{V} = \{V\}_{i\in I}$ is a uniformly bounded cover of X with order at most n. Observe that $\mathcal{V}$ has a Lebesgue number δ greater than R'.

For each $i \in I$, define a function $\varphi_i : X \to \mathbb{R}$ by

$$\varphi_i(x) := \frac{d(x, X \setminus V_i)}{\sum_{j\in I} d(x, X \setminus V_j)}.$$

Note that the sum in the denominator is finite since each x belongs to at most $n+1$ distinct members. It is straightforward to check that $\{\varphi_i\}_{i\in I}$ is a partition of unity for X subordinate to the cover $\mathcal{V}$. A more involved computation shows that each φ_i is $\left(\frac{2n+3}{R'}\right)$-Lipschitz (the Lebesgue number plays a role here; see [NY12, Lemma 4.3.5]).

Let C be the nerve of the cover $\mathcal{V}$. Then C is a simplicial complex with dimension at most n. Let $|C|$ be the geometric realization of C (the complex C may have infinitely many vertices, but we can still realize it in $\ell^2(I)$). Define $\varphi : X \to |C|$ by

$$\varphi(x) := \sum_{i\in I} \varphi_i(x)\delta_i,$$

where $\{\delta_i\}_{i\in I}$ is the standard orthonormal basis for $\ell^2(I)$. Since $\mathcal{V}$ is uniformly bounded, the map φ is *uniformly cobounded*, meaning that $\sup_{y\in|C|} \operatorname{diam}(\varphi^{-1}(B(y, r))) < \infty$ for all $r > 0$. Now there exists a constant $c > 0$ (which only depends n) such that $|C|$ admits a uniformly bounded cover $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(n)}$ such that each $\mathcal{W}^{(k)}$ consists of c-disjoint members. As φ is uniformly cobounded, the pullback of $\mathcal{W}$ along φ is again uniformly bounded. Finally, if R' is sufficiently large, then the Lipschitz constant for φ is very small. This guarantees that the pullback of each $\mathcal{W}^{(k)}$ consists of R-disjoint members. □

The partition of unity constructed in the proof is very useful. For example, it allows us to show that finite asymptotic dimension implies *Property (A)*, which is an important notion in large-scale geometry. A slight variation of this partition of unity is also going to be very useful later when we consider uniform Roe algebras and other C^*-algebras. Therefore we single it out in the following lemma.

Lemma 2.10. *Let X be a metric space with asymptotic dimension at most n. Then given any $R > 0$ we can find an (infinite) partition of unity $\{h_j^{(i)}\}_{i=0,1,...,n}^{j\in J(i)}$ on X such that*

1. *for each $i = 0, 1, ..., n$, the functions $\{h_j^{(i)}\}_{j\in J(i)}$ have pairwise R-disjoint supports,*
2. *each $h_j^{(i)}$ is $\frac{1}{R}$-Lipschitz, and*
3. *there is a uniform bound D on the diameters of the supports of $h_j^{(i)}$.*

Proof. Let $R' > 0$ be a suitable constant (depending on R). Find a uniformly bounded cover $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(n)}$ be a uniformly bounded cover such that each $\mathcal{U}^{(i)}$ consists of $(5R')$-disjoint members. Enlarge each set in the cover by $2R'$ and obtain a cover with a Lebesgue number greater than R'. Then carry out the same construction of partition of unity. □

Now we make precise what it means by saying $\mathbb{Z}^n$ and $\mathbb{R}^n$ are "the same in large scale". Although these are both metric spaces, clearly the notion of isometry would be too restrictive for our purposes. Therefore we consider instead the so-called *quasi-isometries.*

Definition 2.11. Let X be a metric space. We say a subset N of X is a *net* in X if there exists $C > 0$ such that for every $x \in X$ there is $y \in N$ satisfying $d(x, y) \leq C$.

Definition 2.12. Let X, Y be metric spaces. A map $f : X \to Y$ is called a *quasi-isometric embedding* if there exists constants $L, C > 0$ such that

$$\frac{1}{L} d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + C$$

for all $x_1, x_2 \in X$.

A quasi-isometric embedding $f : X \to Y$ is called a *quasi-isometry* if the image $f(X)$ is a net in Y. In this case we say X is quasi-isometric to Y

Of course we expect being quasi-isometric to be an equivalence relation. To see this let us reformulate the definition. Let $f, g : X \to Y$ be two maps between metric spaces. We say f and g are *close* if there exists $K > 0$ such that $d(f(x), g(x)) \leq K$ for all $x \in X$.

Proposition 2.13. *Let $f : X \to Y$ be a quasi-isometric embedding. Then the following are equivalent:*

(1) *f is a quasi-isometry;*
(2) *there exists a quasi-isometric embedding $g : Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y.*

Proof. (1) $\Longrightarrow$ (2): Suppose $f : X \to Y$ is a quasi-isometry with constants $L, C > 0$. Since $f(X)$ is a net in Y, there exists $C' > 0$ such that for every $y \in Y$ there exists $x \in X$ satisfying $d(f(x), y) \leq C'$. Given $y \in Y$, choose any point $x \in X$ such that $d(f(x), y)$, and define $g(y) := x$. It is straightforward to check that g is a quasi-isometric embedding (using the fact that f is a quasi-isometric embedding). For any $y \in Y$ we have $d(fg(y), y) < C'$ by construction. On the other hand, given any $x \in X$ we have

$$d(gf(x), x) \leq Ld(fgf(x), f(x)) + C \leq LC' + C.$$

Therefore g satisfies all the requirements.

(2) $\Longrightarrow$ (1): Since $f \circ g$ is close to id_Y, we see that $f(X)$ is a net in Y. □

From this proposition it is easy to see that being quasi-isometric is an equivalence relation.

Example 2.14. Let N be a net in a metric space X. Then the inclusion $\iota : N \to X$ is a quasi-isometry. In particular, the inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$ is a quasi-isometry.

Example 2.15. Any bounded metric space is quasi-isometric to a point.

Proposition 2.16. *Let X and Y be quasi-isometric spaces. Then we have* $\operatorname{asdim}(X) = \operatorname{asdim}(Y)$. *In particular, $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are quasi-isometric if and only if $n = m$.*

Proof. See [NY12, Theorem 2.2.5]. □

We mentioned in the beginning of the section that the first applications of asymptotic dimension were to finitely generated groups. What we haven't discussed is how to view groups as metric spaces. This is one of the central themes in geometric group theory.

Definition 2.17. Let G be a finitely generated group and let S be a finite symmetric generating set (symmetric means that if $s \in S$ then $s^{-1} \in S$). The *word metric d_S* on G with respect to the generating set S is defined by

$$d_S(g, h) = \min\{n \in \mathbb{N} : gh^{-1} = s_1 s_2 \cdots s_n, s_i \in S \text{ for all } i = 1, 2, ..., n\}.$$

In other words, $d_S(g, h)$ is the minimal number of generators needed to connect g and h. If we view G as a subset of its Cayley graph $\operatorname{Cay}(G; S)$, then d_S is precisely the restriction of the path metric on $\operatorname{Cay}(G; S)$.

A natural question is: how much does the metric space structure depend on the choice of generating set S? The following fundamental observation says that even though the metric itself depends on S, the large-scale structure does not.

Proposition 2.18. *Let G be a finitely generated group, and let S, S' be two finite symmetric generating sets of G. Then the identity map*

$$\operatorname{id} : (G, d_S) \to (G, d_{S'})$$

is a (bijective) quasi-isometry.

Proof. Let $L := \max\{d_S(s', e) : s' \in S'\}$. Given $g, h \in G$, by definition of the word metric we have

$$d_{S'}(g, h) = \min\{n \in \mathbb{N} : gh^{-1} = s'_1 s'_2 \cdots s'_n, s'_i \in S'\}.$$

As each s'_i above can be written as a product of at most L generators in S, we see that

$$d_S(g, h) \leq L d_{S'}(g, h).$$

Now the proof is completed by reversing the roles of S' and S. □

Remark 2.19. It follows from the proof that the identity map $\operatorname{id} : (G, d_S) \to (G, S')$ is actually *bi-Lipschitz*, i.e, a bijective quasi-isometry without the translation constant C.

It follows that the quasi-isometry type of G does not depend on the choice of generating set. In particular, the asymptotic dimension of G is well-defined.

We have seen that $\operatorname{asdim}(\mathbb{Z}^n) = n$. Below we give some other examples of groups with finite asymptotic dimension. Proofs and more examples can be found in [NY12].

Example 2.20. For any $n \in \mathbb{N}$, the free group $\mathbb{F}_n$ has asymptotic dimension 1.

Example 2.21. If G is finitely generated group and H is a finitely generated subgroup of G with finite index, then G and H are quasi-isometric. Since $SL_2(\mathbb{Z})$ contains a copy of $\mathbb{F}_2$ with index 12, we conclude that $\operatorname{asdim}(SL_2(\mathbb{Z})) = 1$.

Remark 2.22. In fact, if G is finitely generated and H is a subgroup of G with finite index, then H is automatically finitely generated. This can be seen, for example, using the Milnor-Švarc lemma [NY12, Proposition 1.3.13].

Before closing the section, let us mention one interesting connection between topological covering dimension and asymptotic dimension. Recall that a metric space is *proper* if every closed ball is compact. For example, every finitely generated group is proper with respect to a word metric.

Definition 2.23. [Roe03, Definition 2.35] Let X be a proper metric space. A continuous bounded function $g : X \to \mathbb{C}$ is called a *Higson function* if for every $R > 0$ and $\varepsilon > 0$, there exists a compact subset $A \subseteq X$ such that $|g(x) - g(y)| < \varepsilon$ for all $x, y \in X \setminus A$ with $d(x, y) < R$.

Let $C_h(X)$ denote the set of all Higson functions on X.

Proposition 2.24. *$C_h(X)$ is a unital C^*-subalgebra of $C_b(X)$.*

Proof. This is rather straightforward. For example, to see that $C_h(X)$ is closed under (pointwise) multiplication, use the identity

$$fg(x) - fg(y) = f(x)[g(x) - g(y)] + [f(x) - f(y)]g(y).$$

□

Since $C_h(X)$ is a unital C^*-subalgebra of $C_b(X)$, by the Gelfand-Naimark theorem the spectrum of $C_h(X)$ can be identified as a compactification of X. We write hX for this compactification.

Definition 2.25. The boundary $\nu X := hX \setminus X$ is called the *Higson corona* of X.

Historically the Higson corona is related to index theoretic approaches to the Novikov conjecture. Here we only focus on its dimensional property. The following theorem is proved in [DKU98].

Theorem 2.26 (Dranishnikov-Keesling-Uspenskij)**.** *Let X be a proper metric space. Then* $\dim(\nu X) \leq \operatorname{asdim}(X)$.

It was further shown by Dranishnikov in [Dra00] that the equality holds whenever X has finite asymptotic dimension.

Exercises

Exercise 2.1. Show that being quasi-isometric is an equivalence relation.

Exercise 2.2. Let X be a metric space and $R > 0$. Two elements x and y in X are called *R-connected* if there exists a finite sequence $x_0, x_1, ..., x_n$ in X such that $x = x_0$, $y = x_n$, and $d(x_i, x_{i+1}) \leq R$ for all $i = 0, 1, ..., n-1$. Being R-connected is an equivalence relation, and the equivalence classes are called the *R-connected components* of X.

Show that $\text{asdim}(X) = 0$ if and only if for every $R > 0$, there is a uniform bound on the diameters of the R-connected components of X.

Exercise 2.3. Let G be a finitely generated group. Show that $\text{asdim}(G) = 0$ if and only if G is finite. (Hint: use the previous exercise)

Exercise 2.4. Prove Proposition 2.18.

3. Uniform Roe algebras

In this section we look at C^*-algebras which encode large-scale geometric properties of metric spaces, and we will focus on their dimensional properties.

Let X be a metric space. We say X has *bounded geometry* if X is uniformly discrete and for every $r > 0$, the map $x \mapsto \text{Card}(B(x, r))$ is a bounded function on X. In plain English, this says that given any $r > 0$, the balls of radius r have uniformly bounded cardinalities. Note that every metric space with bounded geometry is necessarily countable and discrete.

Example 3.1. Let G be a finitely generated group equipped with a word metric. Then G has bounded geometry.

Now suppose X is a metric space with bounded geometry. Then $\ell^2(X)$ is a (separable) Hilbert space with orthonormal basis $\{\delta_x\}_{x \in X}$. An operator T in $B(\ell^2(X))$ is said to have *finite propagation* (or *finite width*) if there exists $R > 0$ such that $\langle T\delta_x, \delta_y \rangle = 0$ whenever $d(x, y) > R$.

Let $\mathbb{C}_u[X]$ be the collection of all operators in $B(\ell^2(X))$ with finite propagation. Then $\mathbb{C}_u[X]$ is actually a $*$-subalgebra of $B(\ell^2(X))$.

Definition 3.2. The norm closure of $\mathbb{C}_u[X]$ is called the *uniform Roe algebra* of X, which is denoted $C^*_u(X)$.

In practice we often view elements in $B(\ell^2(X))$ as X-by-X indexed matrices, and we visualize an operator with finite propagation as a matrix supported in a "band" around the main diagonal.

From this picture it is clear that $C^*_u(X)$ contains $\ell^\infty(X)$ as diagonal matrices. Therefore $C^*_u(X)$ is non-separable whenever X is infinite. Also it is not hard to see that $C^*_u(X)$ contains all compact operators on $\ell^2(X)$.

Remark 3.3. The name suggests that there are "non-uniform" versions of Roe algebras. This is indeed the case, and those have important applications in index theory and geometry.

Example 3.4. Let X be a finite space. Then every operator in $B(\ell^2(X))$ has finite propagation. Therefore $C^*_u(X) = B(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$.

It is already clear from this example that $C_u^*(X)$ is not invariant under quasi-isometry (as all finite spaces are quasi-isometric). However, in some sense this discrepancy of cardinalities is the only thing that can go wrong.

Theorem 3.5. [STY02, Corollary 3.6], [BNW07, Theorem 4] *Let X, Y be metric spaces with bounded geometry. If X and Y are quasi-isometric, then $C_u^*(X)$ and $C_u^*(Y)$ are stably-isomorphic.*

It is also well-known that if there exists a quasi-isometry $f : X \to Y$ which is also a bijection, then $C_u^*(X) \cong C_u^*(Y)$. For an explicit proof, see for example [LL17, Proposition 2.3].

Suppose G is a finitely generated group equipped with a word metric. We have seen that up to bijective quasi-isometry the choice of the word metric is irrelevant. Therefore the uniform Roe algebra $C_u^*(G)$ is also independent of the choice of generating set. In fact, in this case $C_u^*(G)$ is a crossed product in disguise.

Proposition 3.6. [BO08, Proposition 5.1.3] *Let G be a finitely generated group. Then $C_u^*(G) \cong \ell^\infty(G) \rtimes G$, where the action is given by left translation.*

Uniform Roe algebras reflect many large-scale properties of the group (or more generally, the metric space). The following theorem is due to Ozawa.

Theorem 3.7. [Oza00, Theorem 3] *$C_u^*(G)$ is nuclear if and only if G is exact.*

The goal now is to show that the nuclear dimension of $C_u^*(X)$ is bounded by $\operatorname{asdim}(X)$. Given an operator $T \in B(\ell^2(X))$, we write $T_{x,y} = \langle T\delta_y, \delta_x\rangle$ (this resembles the usual notation for matrices).

Lemma 3.8. [WZ10, Lemma 8.1] *Let X be a metric space with bounded geometry. Let $a = [a_{x,y}]$ be an operator in $\mathbb{C}_u[X]$ with width S and uniform bound M on the entries. Then $\|a\| \leq b(a)M$, where*

$$b(a) = \sup\{|B(x, S)| : x \in X\}$$

(note that $b(a) < \infty$ since X has bounded geometry).

Proof. Let $v = \sum_{x\in X} v_y\delta_y$ be a vector in $\ell^2(X)$ (so $\sum |v_y|^2 < \infty$). Then

$$(av)_x := \langle av, \delta_x\rangle = \sum_{y\in X} v_y\langle a\delta_y, \delta_x\rangle = \sum_{y\in X} a_{x,y}v_y = \sum_{y\in B(x,S)} a_{x,y}v_y.$$

Now the desired norm estimate follows from a computation:

$$\begin{aligned}
\|av\|^2 &= \sum_{x\in X} |(av)_x|^2 \\
&= \sum_{x\in X} \left| \sum_{y\in B(x,S)} a_{x,y}v_y \right|^2 \\
&\leq \sum_{x\in X} \left(\sum_{y\in B(x,S)} |a_{x,y}|^2 \right) \left(\sum_{y\in B(x,S)} |v_y|^2 \right) \qquad \text{(Cauchy} - \text{Schwarz)} \\
&\leq b(a)M^2 \cdot \sum_{x\in X} \sum_{y\in B(x,S)} |v_y|^2 \\
&= b(a)M^2 \cdot \sum_{(x,y):d(x,y)<S} |v_y|^2 \\
&= b(a)M^2 \cdot \sum_{y\in X} \sum_{x\in B(y,S)} |v_y|^2 \\
&\leq b(a)M^2 \cdot b(a) \sum_{y\in X} |v_y|^2 \\
&\leq b(a)^2 M^2 \|v\|^2.
\end{aligned}$$

□

Lemma 3.9. [WZ10, Lemma 8.4] *Let K be any index set and $(n_k)_{k\in K}$ be a bounded family of positive integers. Then $\prod_{k\in K} M_{n_k}$ is an AF algebra.*

Theorem 3.10. [WZ10, Theorem 8.5] *Let X be a metric space with bounded geometry. Then $\dim_{\mathrm{nuc}}(C_u^*(X)) \leq \mathrm{asdim}(X)$.*

Proof. (Sketch) Write $n := \mathrm{asdim}(X)$. Let $\mathcal{F}$ be a finite subset in $C_u^*(X)$ consisting of positive contractions and let $\varepsilon > 0$. By perturbation we may assume each a_k belongs to $\mathbb{C}_u[X]$. Let S be an upper bound for the propagations of elements in $\mathcal{F}$, and let M be an upper bound for the (absolute values of) all matrix entries appearing in $\mathcal{F}$.

Let $R > 0$ be a large number to be determined later. Find an $\left(\frac{1}{R}\right)$-Lipschitz partition of unity $\{h_j^{(i)}\}_{i=0,1,\ldots,n}^{j\in J(i)}$ as in Lemma 2.10. Write $h^{(i)} := \sum_{j\in J(i)} h_j^{(i)}$. Note that each $h^{(i)}$ is again $\left(\frac{1}{R}\right)$-Lipschitz and $\sum_{i=0}^n h^{(i)} = 1$. For any $a \in \mathbb{C}_u[X]$ we have

$$\begin{aligned}
[h^{(i)}, a]_{x,y} &:= \langle (h^{(i)}a - ah^{(i)})\delta_y, \delta_x\rangle \\
&= (h^{(i)}(x) - h^{(i)}(y))\langle a\delta_y, \delta_x\rangle \\
&= (h^{(i)}(x) - h^{(i)}(y))a_{x,y}.
\end{aligned}$$

In particular the commutator $[h^{(i)}, a]$ has the same propagation as a. It follows from Lemma 3.8 and the assumption of $h^{(i)}$ being $\left(\frac{1}{R}\right)$-Lipschitz that

$$\begin{aligned}\|[h^{(i)}, a]\| &\leq b(a)M \sup\{|h^{(i)}(x) - h^{(i)}(y)| : d(x, y) < S\}\\ &\leq b(a)M \cdot \frac{1}{R} \cdot S\end{aligned}$$

for all $a \in \mathcal{F}$. Since we can choose R to be arbitrarily large, we may assume the commutator $[{h^{(i)}}^{1/2}, a]$ $(a \in \mathcal{F})$ also has small norm (compared to ε). For each $i = 0, 1, ..., n$, define a c.p.c map

$$\psi^{(i)} : C^*_u(X) \to C^*_u(X)$$

by

$$\psi^{(i)}(a) = \sum_{j \in J(i)} {h^{(i)}_j}^{1/2} a {h^{(i)}_j}^{1/2}.$$

By the assumptions of the partition of unity, it is not hard to see that the image of $\psi^{(i)}$ is contained in a subalgebra $A^{(i)}$ of the form $\prod_{j \in J(i)} M_{n_j}(\mathbb{C})$, where (n_j) is a bounded sequence (they correspond to the diameters of the supports of $h^{(i)}_j$). Define

$$\psi : C^*_u(X) \to A^{(0)} \oplus \cdots \oplus A^{(n)}, \qquad \psi(a) := (\psi^{(0)}(a), ..., \psi^{(n)}(a))$$

and

$$\varphi : A^{(0)} \oplus \cdots \oplus A^{(n)} \to C^*_u(X), \qquad \varphi(x_0, x_1, ..., x_n) = x_0 + x_1 + \cdots + x_n.$$

If R is sufficiently large, then $\varphi\psi(a)$ approximates a within ε for all $a \in \mathcal{F}$. Moreover the restriction of φ to each $A^{(i)}$ is a $*$-homomorphism (in particular has order zero). Finally, each $A^{(i)}$ is AF by Lemma 3.9. Therefore upon combining our approximation and a standard approximation for AF algebras, we have shown that $\dim_{\mathrm{nuc}}(C^*_u(X)) \leq n$. □

Together with Kang Li and Wilhelm Winter, we are developing a dimension theory for diagonal pairs of C^*-algebras. In the case of uniform Roe algebras, there is a canonical diagonal subalgebra $\ell^\infty(X)$. We have been able to show that the nuclear dimension of $C^*_u(X)$ relative to $\ell^\infty(X)$ is precisely the asymptotic dimension of X.

Exercises

Exercise 3.1. Let X be a metric space with bounded geometry. Show that the set $\mathbb{C}_u[X]$ of all finite propagation operators is a $*$-subalgebra of $B(\ell^2(X))$.

Exercise 3.2. Let X be a metric space with bounded geometry. Show that there exists a faithful conditional expectation $E : C^*_u(X) \to \ell^\infty(X)$.

4. QUASI-LOCAL OPERATORS AND VERY LIPSCHITZ SEQUENCES

In this final section we study a recent article [ŠT17] by Špakula and Tikuisis. More precisely, we discuss how asymptotic dimension enters the study of uniform Roe algebra, and we will see further connection between asymptotic dimension and C^*-algebras obtained from a metric space.

To motivate the main question, we begin with a simple lemma about propagation of an operator.

Lemma 4.1. *Let X be a metric space with bounded geometry, and let $R > 0$. Then an operator $a \in B(\ell^2(X))$ has propagation at most R if and only if for any f, f' in $\ell^\infty(X)$ with R-disjoint supports, we have $faf' = 0$.*

Proof. The "if" direction is obvious. Conversely, assume that a has propagation at most R. Then given $f, f' \in \ell^\infty(X)$ with R-disjoint supports, we have

$$\langle faf'\delta_y, \delta_x\rangle = f(x)f'(y)\langle a\delta_y, \delta_x\rangle.$$

for all $x, y \in X$. By assumption at least one of the terms in the product is zero. Therefore $faf' = 0$. □

Definition 4.2. Let X be a metric space of bounded geometry, and let $R, \varepsilon > 0$.

- We say an operator $a \in B(\ell^2(X))$ has *ε-propagation at most R* if for any f, f' in $\ell^\infty(X)_1$ with R-disjoint supports we have $\|faf'\| < \varepsilon$.
- We say an operator $a \in B(\ell^2(X))$ is called *quasi-local* if a has finite ε-propagation for any $\varepsilon > 0$.

Although limits of finite propagation operators need not have finite propagation, it is a simple exercise to show that limits of quasi-local operators is quasi-local. In particular, the set of quasi-local operators is a C^*-subalgebra of $B(\ell^2(X))$, and all operators in the uniform Roe algebra are quasi-local.

Is the converse true? Roe predicted an affirmative answer under the assumption of finite asymptotic dimension, and this was confirmed in the paper of Špakula and Tikuisis.

Theorem 4.3 (Špakula-Tikuisis)**.** *Let X be a metric space of bounded geometry. Suppose X has finite asymptotic dimension. Then every quasi-local operator is in the uniform Roe algebra $C^*_u(X)$.*

Though we won't be able to prove this result here, we shall discuss some ingredients in the proof. A crucial observation made in the paper is a characterization of quasi-local operators in terms of relative commutant in the sequence algebra of $\ell^\infty(X)$.

Definition 4.4. Let X be a metric space. A bounded sequence (f_n) from $\ell^\infty(X)$ is called *very Lipschitz* if for every $L > 0$ there exists n_0 such that f_n is L-Lipschitz for all $n \geq n_0$.

We write $VL(X)$ for the set of all very Lipschitz bounded sequences from $\ell^\infty(X)$.

In the definition above, one should think of L as a small number. Therefore in a very Lipschitz sequence the functions get flatter and flatter as n grows.

Lemma 4.5. *Let X be a metric space of bounded geometry. Then $VL(X)$ is a C^*-subalgebra of $\ell^\infty(\mathbb{N}, \ell^\infty(X))$.*

Proof. We only show that $VL(X)$ is closed in $\ell^\infty(\mathbb{N}, \ell^\infty(X))$. Recall that a metric space of bounded geometry is by definition also uniformly discrete, i.e., there exists $\delta > 0$ such that $d(x, y) \geq \delta$ for all x, y in X. Let (f_n) be in the closure of $VL(X)$. Given $L > 0$, we can find $L' > 0$ and $\varepsilon > 0$ such that $L' + \frac{2\varepsilon}{\delta} = L$. Choose an element (g_n) in $VL(X)$ such that $\|(g_n) - (f_n)\|_{\ell^\infty(\mathbb{N},\ell^\infty(X))} < \varepsilon$. Since (g_n) is very Lipschitz, there exists $n_0 \in \mathbb{N}$ such that g_n is L'-Lipschitz whenever $n \geq n_0$. Now we compute, for any $n \geq n_0$ and any $x, y \in X$,

$$\begin{aligned}
|f_n(x) - f_n(y)| &\leq |f_n(x) - g_n(x)| + |g_n(x) - g_n(y)| + |g_n(y) - f_n(x)| \\
&< 2\varepsilon + L'd(x, y) \\
&= \frac{2\varepsilon}{\delta} \cdot \delta + L'd(x, y) \\
&\leq \left(\frac{2\varepsilon}{\delta} + L'\right) d(x, y) \\
&= L \cdot d(x, y).
\end{aligned}$$

This shows that (f_n) is very Lipschitz. □

Recall that given a C^*-algebra we can build the sequence algebra

$$A_\infty := \ell^\infty(\mathbb{N}, A)/\{(a_n) \in \ell^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\}.$$

Define

$$VL_\infty(X) := VL(X)/\{(f_n) \in VL(X) : \lim_{n\to\infty} \|f_n\| = 0\}$$

(note that this is different from $VL(X)_\infty$). Then $VL_\infty(X)$ is a C^*-subalgebra of the sequence algebra $\ell^\infty(X)_\infty$. The following theorem characterizes quasi-local operators in terms of commutation with very Lipschitz sequences.

Theorem 4.6 (Špakula-Tikuisis)**.** *Let X be a metric space with bounded geometry, and b be an operator in $B(\ell^2(X))$. Then the following are equivalent:*

(1) *b is quasi-local, i.e., b has finite ε-propagation for every $\varepsilon > 0$;*
(2) *b belongs to the relative commutant $VL_\infty(X)' \cap B(\ell^2(X))_\infty$.*

Remark 4.7. In the paper the authors showed that being quasi-local is also equivalent to essential commutation with Higson functions. More precisely, an operator $b \in B(\ell^2(X))$ is quasi-local if and only if the commutator $[b, g]$ belongs to $\mathcal{K}(\ell^2(X))$ for all $g \in C_h(X)$. See [ŠT17, Section 5].

Finally, let us discuss a relationship between the asymptotic dimension of X and the nuclear dimension of $VL(X)$.

Theorem 4.8 (Špakula-Tikuisis)**.** *Let X be a metric space with bounded geometry. Then $\dim_{\mathrm{nuc}}(VL(X)) \leq \mathrm{asdim}(X)$.*

Proof. Let $n := \operatorname{asdim}(X)$. Fix $m \in \mathbb{N}$ for a moment. By Lemma 2.10 there exists a partition of unity $\{h_j^{(i)}\}_{i=0,1,\ldots,n}^{j\in J(i)}$ on X such that

(1) for each $i = 0, 1, ..., n$, the functions $\{h_j^{(i)}\}_{j\in J(i)}$ are pairwise orthogonal,
(2) each $h_j^{(i)}$ is $(\frac{1}{m})$-Lipschitz, and
(3) there is a uniform bound D on the diameters of the supports of $h_j^{(i)}$.

For each $i = 0, 1, ..., d$ and $j \in J(i)$ pick a point $x_j^{(i)}$ in the support of $h_j^{(i)}$. Define a $*$-homomorphism

$$\psi : \ell^\infty(X) \to \ell^\infty(J(0)) \oplus \cdots \oplus \ell^\infty(J(n)),$$
$$f \mapsto (f_0, ..., f_n),$$

where

$$f_i(j) := f(x_j^{(i)})$$

for all $i = 0, 1, ..., n$ and $j \in J(i)$. In plain words, the map ψ is nothing but point evaluations (and grouping according to the colors). For each $i = 0, 1, ..., n$, define a $*$-homomorphism

$$\varphi^{(i)} : \ell^\infty(J(i)) \to \ell^\infty(X)$$

by

$$\varphi^{(i)}(g) = \sum_{j\in J(i)} g(j)h_j^{(i)}.$$

and set

$$\varphi = \sum_{i=0}^{n} \varphi^{(i)} : \ell^\infty(J(0)) \oplus \cdots \oplus \ell^\infty(J(n)) \to \ell^\infty(X).$$

Note that by the assumptions on $h_j^{(i)}$, each function in the image of φ is $\left(\frac{2(n+1)}{m}\right)$-Lipschitz. Moreover, for each $f \in \ell^\infty(X)$ and $x \in X$ we have

$$|\varphi\psi(f)(x) - f(x)| = \left|\sum_{i=0}^{n}\sum_{j\in J(i)} [f(x_j^{(i)}) - f(x)]h_j^{(i)}(x)\right|.$$

There is at most one nonzero term for the inner sum, namely the index j such that x is contained in the support of $h_j^{(i)}$. In this case the distance between $x_j^{(i)}$ and x is at most D. Hence $\|\varphi\psi(f) - f\|_\infty < \varepsilon$ whenever f is $\left(\frac{\varepsilon}{(n+1)D}\right)$-Lipschitz.

Now we allow m to vary, so from now on everything carries a label m. Let $\mathcal{F} = \{(f_{1,m})_{m\in\mathbb{N}}, ..., (f_{p,m})_{m\in\mathbb{N}}\}$ be a finite subset of $VL(X)$ and fix $\varepsilon > 0$. By passing to subsequences we may assume that each $f_{r,m}$ is $\left(\frac{\varepsilon}{(n+1)D(m)}\right)$-Lipschitz. Define a $*$-homomorphism

$$\Psi : VL(X) \to \prod_{m\in\mathbb{N}}\left(\bigoplus_{i=0}^{n} \ell^\infty(J(i,m))\right) \cong \bigoplus_{i=0}^{n}\left(\prod_{m\in\mathbb{N}} \ell^\infty(J(i,m)\right)$$

by setting

$$\Psi((f_m)_{m\in\mathbb{N}}) := (\psi_m(f_m))_{m\in\mathbb{N}}\,.$$

For each $i = 0, 1, ..., n$, define a $*$-homomorphism

$$\Phi^{(i)} : \prod_{m\in\mathbb{N}} \ell^\infty(J(i,m)) \to \ell^\infty(\mathbb{N}, \ell^\infty(X))$$

by

$$\Phi^{(i)}((g_m)_{m\in\mathbb{N}}) := \left(\varphi_m^{(i)}(g_m)\right)_{m\in\mathbb{N}},$$

and set

$$\Phi := \sum_{i=0}^{n} \Phi^{(i)} : \bigoplus_{i=0}^{n} \left(\prod_{m\in\mathbb{N}} \ell^\infty(J(i,m))\right) \to \ell^\infty(\mathbb{N}, \ell^\infty(X)).$$

Then for each $r = 1, 2, ..., p$ we have

$$\begin{aligned}
\|\Phi\Psi((f_{r,m})_{m\in\mathbb{N}}) - (f_{r,m})_{m\in\mathbb{N}}\|_{\ell^\infty(\mathbb{N},\ell^\infty(X))} &= \|(\varphi_m\psi_m(f_{r,m}))_{m\in\mathbb{N}} - (f_{r,m})_{m\in\mathbb{N}}\|_{\ell^\infty(\mathbb{N},\ell^\infty(X))} \\
&= \sup_{m\in\mathbb{N}} \|\varphi_m\psi_m(f_{r,m}) - f_{r,m}\|_{\ell^\infty(X)} \\
&\leq \varepsilon.
\end{aligned}$$

Since the algebra $\prod_{m\in\mathbb{N}}(\bigoplus_{i=0}^{n} \ell^\infty(J(i,n)))$ is AF (by Lemma 3.9), we have shown that $\dim_{\text{nuc}}(VL(X)) \leq n$. □

In the paper the authors further proved that $\dim_{\text{nuc}}(VL(X)) = \dim_{\text{nuc}}(VL_\infty(X)) = \text{asdim}(X)$ under the assumption that X is known to have finite asymptotic dimension.

5. Some open problems

Question 5.1. Find the precise value of $\dim_{\text{nuc}}(C_u^*(X))$. Is it true that $\dim_{\text{nuc}}(C_u^*(X)) = \text{asdim}(X)$? What happens if we already know that X has finite asymptotic dimension? At the moment we don't even know the precise value of $\dim_{\text{nuc}}(C_u^*(\mathbb{Z}^2))$.

Question 5.2. Does $C_u^*(\mathbb{Z})$ have real rank zero? If the answer is "no", then real rank zero for $C_u^*(X)$ actually characterizes $\text{asdim}(X) = 0$.

Question 5.3. Is there an example of X such that not all quasi-local operators belong to $C_u^*(X)$?

Question 5.4. If $VL_\infty(X) \cong VL_\infty(Y)$, do we know that X and Y are coarsely equivalent?

Question 5.5. Do we have $\dim_{\text{nuc}}(VL(X)) = \text{asdim}(X)$ always? What is the relationship between $\dim_{\text{nuc}}(VL(X))$ and $\dim(\nu X)$?

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