Ozawa I

$\Gamma = \langle S \rangle$ $|x| = \min\{n : \exists s_1 \cdots s_n \in S \ \ x = s_1 \cdots s_n\}$ $|e| = 0$

$\gamma_\Gamma(n) := \#\{x \in \Gamma : |x| \leq n\}$ $d(x,y) = |x^{-1}y|$ left inv.

for finite generating sets S, S', $\exists C > 0$ $|x| \leq C|x|'$ $x \in \Gamma$

Two metric spaces X, Y are quasi-isometric if $\exists \theta : X \to Y$ $\downarrow$ not bijective

$\exists C, K > 0$ $\frac{1}{C} d_X(x,y) - K \leq d_Y(\theta x, \theta y) \leq C d_X(x,y) + K$

and $\theta(X)$ is K-dense in Y.

Ex X = bounded metric space $\iff X \underset{Q\,iso}{\simeq} \{\cdot\}$

$\mathbb{R}^d \underset{Q\,iso}{\simeq} \mathbb{Z}^d$ $(\Gamma, d_S) \underset{Q\,iso}{\simeq} (\Gamma, d_{S'})$

$[\Gamma, \Gamma_0] < \infty \Rightarrow \Gamma_0 \underset{Q\,iso}{\simeq} \Gamma$

Exercise If $X \underset{Q}{\simeq} Y$

$$\left.\begin{array}{l} X, Y \text{ have bdd geometry} \\ (\text{i.e. } \sup_{x_0 \in X} \#\{x \in X : d(x, x_0) \leq R\} < \infty \\ \quad \forall R \text{ same for } Y \end{array}\right\} \Rightarrow \gamma_{X, x_0}(n) \leq C\gamma_{Y, y_0}(Cn+K) + K$$

i.e. γ_X', γ_Y' have the same growth rate.

Thm (Milnor - Švarc) X = proper geodesic metric space

$\Gamma \curvearrowright X$ properly^{1} cocompact^{2} then $\Gamma \ni g \mapsto g x_0 \in X$ is QI

[i.e. $\{g : gK \cap K \neq \emptyset\}$ = finite] $[\exists K \underset{cpt.}{\subseteq} X \ \ \Gamma K = X]$
$K \subseteq X$ cpt

Thm (Milnor, Wolf, 1968) $\overset{f.g.}{\Gamma}$ is nilpotent, then Γ has polynomial growth (PG)

The converse is true for solvable gps, i.e. Γ = PG + solvable $\Rightarrow \Gamma \overset{\wedge\,\text{virtually}}{=}$ nilpotent

(i.e. $\exists \Gamma_0 \underset{fi.\,index}{\leq} \Gamma$ s.t. Γ_0 = nilpotent.

These are used [illegible] by Milnor to show that if M = mfld with ≥ 0 curvature then $\pi_1(M)$ has PG

Conversely: [illegible] Tits alternative:

$$\Gamma \leq GL_n(\mathbb{F}) \text{ f.g.} \Rightarrow \begin{cases} \mathbb{F}_2 \leq \Gamma \to \Gamma = EG \\ \text{or } \Gamma = \text{virtually solvable.} \end{cases}$$

Thm (Gromov 1981)

$\overset{f.g.}{\Gamma} = PG \iff \Gamma =$ virtually nilpotent.

Pf/ Induction on $d =$ deg. of growth

Need to find a virtual $\mathbb{Z}$-quotient. i.e. $\Gamma \underset{f.ind.}{\geq} \Gamma_0 \overset{\delta}{\twoheadrightarrow} \mathbb{Z}$

(or equivalently, $\Gamma \twoheadrightarrow \Lambda \underset{f.ind}{\geq} \mathbb{Z}$)

$\because \ker\delta =$ f.g. + PG of deg $\leq d-1$

by inductive hypothesis $\ker\delta =$ v. nilp.

$\therefore \Gamma_0 = \ker\delta \rtimes \mathbb{Z}$ v. nilp. (by Milnor)

You can go a little further to super-poly. growth.

Grigorchuk's conjecture (1990)

$\overset{f.g.}{\Gamma}$; $\gamma_\Gamma(n) \ll e^{\sqrt{n}}$ (e.g. $\gamma_\Gamma(n) < Ce^{n^\delta}$, $0<\delta<\frac{1}{2}$) $\Rightarrow \Gamma =$ virt. nilp.

· Grigorchuk constructed the first example of a gp with subexp. superpol. growth rate.

$$e^{\sqrt{n}} \lneq \underset{\exp(n^\alpha)\quad \alpha\in(\frac{1}{2},1)}{\gamma_G(n)} \leq e^n$$

We don't know if

∃ one virtual $\mathbb{Z}$-quotient?

How to find a v. $\mathbb{Z}$-quotient?

By Tits alternative (TA), need to find infinite finite dim. repn $\pi : \Gamma \to GL(n)$ or $O(n)$
$|\pi(\Gamma)| = \infty$

by TA : $\pi(\Gamma)$ is virt. solvable.

$\to \exists$ v. $\mathbb{Z}$-quotient.

How to find infinite f.d. repn?

Shalom 2004 : Use reduced cohomology.

Reduced Cohomology

Fix prob. measure μ on Γ, finitely supported, symmetric $\langle \text{supp}\,\mu \rangle = \Gamma$.

Consider any $(\pi, \mathcal{H})$ = orth. rep$\underline{n}$ (i.e. real setting)

Cocycle $b : \Gamma \to \mathcal{H}$ i.e. $b(gx) = b(g) + \pi_g b(x)$

Coboundary $b : \Gamma \to \mathcal{H}$ i.e. $(g, x \in \Gamma)$

$\exists \xi \in \mathcal{H}$ $b = b_\xi$ $b_\xi(g) = \xi - \pi_g(\xi) \in \mathcal{H}$

b = cocycle : b = coboundary $\iff \sup_g \|b(g)\| < \infty$

b is μ-harmonic if $\sum_x b(gx) = b(g)$ $\forall g$ (or equivalently for $g = e$) $\sum_x = \int d\mu$

$Z^1(\Gamma, \pi) = \{ b : b = \text{cocycle} \}$

$B^1(\Gamma, \pi) = \{ b : b = \text{coboundary} \}$ (not closed)

$$\overline{H}^1(\Gamma, \pi) := \frac{Z^1(\Gamma, \pi)}{\overline{B}^1(\Gamma, \pi)}$$

Z^1 = Hilbert space wrt $\|b\|^2 = \sum_x \|b(x)\|^2$ $\overline{H}^1 = Z^1 / \overline{B^1} \underset{\text{[illegible]}}{\cong} (B^1)^\perp$ dep on μ

$(B^1)^\perp$ = harmonic cocycles

$$\because \langle b, b_\xi \rangle = \sum_x \langle b(x), \xi - \pi_x \xi \rangle$$
$$= \sum_x \langle b(x) + b(x^{-1}), \xi \rangle$$
$$= 2 \sum$$

<u>Bönicke I</u>

G = groupoid

$\underline{\text{Ex}}$ $G = \bigcup_x \{x\} \times \Gamma_x$

$G = \Gamma \times X$

$s|_U$ = homeo onto open set

U = bi-section

$r|_U$ = homeo onto open set

We say that G is étale if $r : G \to G$ is a local homeomorphism. This is equivalent to $G^{(0)}$ being open in G. In this case, $x \mapsto \sum_{g \in G^x} f(g)$ is continuous

Also G has a basis consisting of open bi-sections

<u>Ex</u> ① $\Gamma \times X$ is étale $\Leftrightarrow$ Γ = discrete

② $\Lambda = (V, E)$ be a $\overset{\text{directed}}{\text{graph}}$ (row finite)

$$\underset{\substack{\downarrow \\ \text{totally disconnected}}}{E^{\infty}} := \{ \text{inf. paths} \} \leq \prod_{1}^{\infty} E \qquad \begin{array}{l} \sigma : E^{\infty} \to E^{\infty} \\ \sigma(x)_i = x_{i+1} \end{array}$$

$$G_{\Lambda} = \{ (x, \ell, y) : x, y \in E^{\infty}, \underset{\text{some } m, n}{\sigma^{m}(x) = \sigma^{n}(y)}, \ \ell = m - n \}$$

$$(x, \ell, y)(y, k, z) = (x, \ell + k, z)$$

$$(x, \ell, y)^{-1} = (y, -\ell, x) \qquad G_{\Lambda}^{(0)} = E^{\infty}.$$

G_{Λ} has basis of bi-sections:

$$Z(\alpha, \beta) = \{ (x, k, y) \in G_{\Lambda} : \exists z \in E^{\infty} \ \begin{array}{l} x = \alpha z \\ y = \beta z \\ k = |\beta| - |\alpha| \end{array} \}$$

with $\alpha, \beta \in E^{<\infty}$

$$r : Z(\alpha, \beta) \twoheadrightarrow Z(\alpha)$$
$$s : Z(\alpha, \beta) \twoheadrightarrow Z(\beta) \qquad Z(\alpha) = \{ \alpha x \mid x \in E^{\infty} \}$$

for $\Lambda = \mathbb{Z}$

$$G_{\Lambda} = \mathbb{Z} \times \mathbb{Z}$$

$\Lambda =$

n-loops $\qquad G_{\Lambda}$ = Cuntz-groupoid

③ (X, d) = discrete metric space with bdd geometry, for $\varphi : D_{\varphi} \to R_{\varphi}$ with $\sup_{x} d(x, \varphi(x)) < \infty$, then φ extends to $\bar{\varphi} : \bar{D}_{\varphi} \to \bar{R}_{\varphi}$ with closure inside βX

$$G_X = \{ (\bar{\varphi}, x) : x \in \bar{D}_{\varphi} \} / \sim \qquad \begin{array}{l} (\bar{\varphi}, x) \sim (\bar{\psi}, y) \Leftrightarrow x = y \ \& \\ \bar{\varphi}|_U = \bar{\psi}|_U \text{ some nhd} \\ \qquad x \in \underset{\text{open}}{U} \subseteq \beta X \end{array}$$

$$[\bar{\varphi}, \psi(x)] [\bar{\psi}, x] = [\varphi \circ \bar{\psi}, x]$$

$$[\bar{\varphi}, x]^{-1} = [\bar{\varphi}^{-1}, \varphi(x)]$$

For G = étale, $x \in G^{(0)}$

$$\pi_x : C_c(G) \to B(\ell^2(G_x))$$

$$\pi_x(f)\xi = f * \xi$$

$\|f\|_r = \sup_{x \in G^{(0)}} \|\pi_x f\|$ $\quad$ $\|f\| = \sup_{\pi} \|\pi(f)\|$

$$C^*(G) \twoheadrightarrow C^*_r(G)$$

Ex

$C^*(\Gamma \times X) = C(X) \rtimes \Gamma$ $\quad$ $C^*_r(\Gamma \times X) = C(X) \rtimes_r \Gamma$

$C^*(\Gamma \times \Gamma) = C^*_r(\Gamma \times \Gamma) = \mathcal{K}(\ell^2(\Gamma))$

$C^*(G_\Lambda) = C^*(\Lambda)$ $\quad$ $C^*_r(E_\Lambda) = C^*_r(\Lambda)$

$C^*(G_X) =$ $\quad\quad$ $C^*_r(G_X) = C^*_u(X)$

Liao (I)

For a Cover $\alpha = \{U_i\}$ of top. space X,

$$\text{ord}\,\alpha \le n \underset{\text{Def}}{\Leftrightarrow} \forall x \in X \quad x \in \text{at most } n+1 \text{ diffirent } U_i\text{'s}$$

for $\beta = \{V_j\}$; $\beta \geq \alpha \underset{\text{Def}}{\Leftrightarrow} \forall j\ \exists i \quad V_j \subseteq U_i$

$$\dim_{top}(X) \le n \underset{\text{Def}}{\Leftrightarrow} \forall \alpha\ \exists \beta \ge \alpha \quad \text{ord}\,\beta \le n.$$

Ex $\quad \dim_{top}(\mathbb{Z}) = 0 \qquad \dim_{top}[0,1] = 1$ connected so not zero-dim.

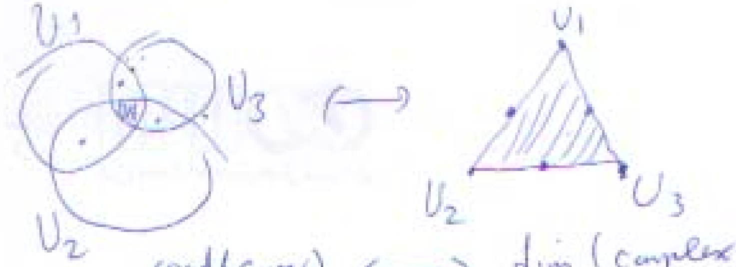
[δ = Leb. no. of α i.e. $\text{diam} A < \delta \Rightarrow \exists U_i \ A \subseteq U_i$]

Thm (Lebesgue) $\dim_{top}[0,1]^n = n$.

We say α is n-decomposable if $\alpha \geq \alpha^{(0)} \sqcup \dots \sqcup \alpha^{(n)}$ with $\alpha^{(i)}$'s consisting of disjoint sets.

Prop (Kirchberg-Winter) $X = T_4$, $\dim_{top} X \le n \Leftrightarrow \forall \alpha\ \exists \beta \ge \alpha$ β = n-deco

Pf/ ($\Leftarrow$) Easy ($\Rightarrow$) We use simplecial Complex for each α



$\dim(\text{Nerve}(\alpha)) = \text{ord}(\alpha)$

ord(cover) $\longleftrightarrow$ dim (complex)

Take a p.o.u. $\{h_i\}$ subordinated to $\alpha = \{U_i\}$

$$f : X \longrightarrow \text{Nerve}(\alpha)$$
$$x \longmapsto \sum h_i(x)[U_i]$$

Any open cover of Nerve(α) is pulled back to a refinement of α.

Def/ A cpc map : $A \xrightarrow{\varphi} B$ has order zero ($\perp$) if $\forall a,b \in A_+$

$a \perp b \Rightarrow \varphi(a) \perp \varphi(b)$ $\qquad (a \perp b \underset{\text{Def}}{\Leftrightarrow} ab = 0)$

<u>Thm</u> (Winter-Zacharias) $\quad \varphi : A \to B$ cpc, $\perp$

$\exists C \geqslant B$ $\exists$ pos. contraction $\underset{\overset{\uparrow}{C'}}{h}$ and $\pi_\varphi : A \underset{*\text{-homo}}{\to} C$ $\qquad \varphi = h\pi_\varphi(\cdot) = \pi_\varphi(\cdot)h$
(e.g. $C = B^{**}$) $\qquad = h^{1/2}\pi_\varphi(\cdot)h^{1/2}$

<u>Def</u> (Winter-Zacharias)

$\dim_{\text{nuc}} A \leqslant n \underset{\text{Def}}{\Leftrightarrow} \forall \mathcal{F} \subset\subset A \ \forall \varepsilon > 0$

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, "\psi"'] & \overset{\partial}{(\mathcal{F},\varepsilon)} & A \\ & F \arrow[ur, "\varphi"'] & \end{tikzcd}$$

more restriction

ψ = cpc $\qquad \varphi\psi \approx id$ on $\mathcal{F}$

$\varphi = \text{cp} + (n+1)$-decomp.

(i.e. $F = F^{(0)} \oplus \cdots \oplus F^{(n)}$ with $\varphi|_{F^{(i)}}$ = cpc $\perp$

<u>Thm</u> (Winter-Zacharias)

$X = \text{cpt} + T_2 \qquad \dim_{\text{nuc}} C(X) = \dim_{\text{top}} X$

Pf/ ($\leqslant$) Easy Take $\mathbb{C}^s$; s = number of sets in a good refinement

($\geqslant$) let $\overset{n}{d} = \dim_{\text{top}} X < \infty$, $\alpha = \{U_1 \cdots, U_r\}$ with $\underbrace{\{h_1 \cdots, h_r\}}_{\mathcal{F}}$ = p.o.u.

then $\exists$ $(\mathcal{F},\varepsilon)$-approx. decomp., and for $x, y \in F_{\perp}^{(i)}$;

$\text{supp}\,\varphi(x) \cap \text{supp}\,\varphi(y) = \phi$; $F = F^{(0)} \oplus \cdots \oplus F^{(n)}$, with $F^{(i)} = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$

(since $\nexists$ cpc $\perp$ map : $M_n \to C(X)$. for $n \neq 1$); take the supports for the $(n+1)$-family of disj. sets. ▨

• Liao (II)

Def/ X = metric space, $\alpha=\{U_i\}$ cover, $R>0$, Then the R-multiplicity of α is at most n, $\text{mult}_R\,\alpha \le n$, if $B(x,R)$ meets at most $n+1$ members of α, for each $x\in X$.

Def/ (Gromov) $\dim_{asy} X \le n \overset{\forall R>0}{\underset{Def}{\iff}} \exists$ unif bdd α, $\text{mult}_R\,\alpha\le n$
$(\sup_{U_i\in\alpha} \text{diam}\, U_i < \infty)$

Thm (Yu) groups with finite asy. dimension satisfy Novikov Conjecture.

Ex 1) $X=\{\pm n^2\}_0^\infty$, then $\dim_{asy}(X)=0$

2) $X=\mathbb{Z}$ $\dim_{asy}(X)\leqslant 1$

3) $\dim_{asy}(\mathbb{Z}) = \dim_{top}(\mathbb{R})$?

4) $\dim_{asy}(\mathbb{Z}^n) = n$

If it is $\leqslant n-1$; then find a unif. bdd. Cover α with $\text{mult}_R\,\alpha\leqslant n-1$, for $R=5$, consider $\lfloor\ \rfloor : \mathbb{R}^n\to\mathbb{Z}^n$ and pullback α, to get a u.b. cover β of $\mathbb{R}^n$, take open nhd's of elements of β, to get a unif. bdd open cover of $\mathbb{R}^n$, with order $\leq n-1$. Since $\dim_{top}[0,1]^n=n$, given any cover γ of $[0,1]^n$, using Lebesgue number, you could rescale β to get a refinement of γ →←.

5) $\dim_{asy}(\text{Tree}) \leq 1$

<u>Prop</u> X = metric, $\dim_{asy} X \leq n \iff \forall R>0 \ \exists$ R-disjoint (i.e. $d(U_i, \overset{i\neq j}{U_j}) \geq R$) families $\alpha^{(0)}, \dots, \alpha^{(n)}$ which together form a unif. bdd. cover of X.

Pf/ ($\Leftarrow$) Easy

($\Rightarrow$) Fix $R>0$, let $R'>0$ and pick a u.b. cover α of X with $\text{mult}_{5R'} \alpha \leq n$. Set $V_i = B(U_i, 2R')$ for $U_i \in \alpha$, then $\beta = \{V_i\}$ is a ub-cover of X with order $\leq n$ and Lebesgue number $\geq R'$. Define $\varphi_i : X \to [0,1]$; $\varphi_i(x) = d(x, V_i^c) / \sum_j \underbrace{d(x, V_j^c)}_{\neq 0 \text{ when } x \in V_j}$, $\sum_i \varphi_i = 1$

supp $\varphi_i \subseteq V_i$; i.e. $\{\varphi_i\}$ = pou Subordinated to β.

Now each φ_i is $\frac{2n+3}{R'}$-Lipschitz; Consider the corresponding simplicial complex Nerve(β). and $\varphi : X \to \text{Nerve}(\beta)$; $\varphi(x) = \sum_i \varphi_i(x)[V_i]$; take nhd's of barrycenters of faces of different dimension s.t. $W^{(0)}, \dots, W^{(n)}$ together cover Nerve(β) s.t. each $W^{(j)}$ is $C = C(n) > 0$ disjoint; Since φ_i is very flat ($R'>0$ very large) the pullback could be R-disjoint, and we are done! ▨

<u>Lemma</u> (POU) Suppose $\dim_{asy} X \leq n$, then $\forall R > 0$

$\exists \{h_j^{(i)}\}_{\substack{0 \leq i \leq n \\ j \in J(i)}}$ = pou s.t.

(1) supp $h_j^{(i)}$, supp $h_{j'}^{(i)}$ R-disjoint ($j \neq j'$)

(2) $h_j^{(i)}$ = $1/R$ - Lipschitz

(3) $\sup \text{diam}(\text{supp } h_j^{(i)}) < \infty$

· Taka (II)

$\Gamma = \langle S \rangle$, μ = f. supported prob. measure with $S = \operatorname{supp} \mu$.

$(\pi, \mathcal{H})$ = orth. rep$\underline{n}$

Thm (Mok 95, Korevaar-Schoen 97)

$\Gamma^{\text{f.g.}}$ amenable $\Rightarrow$ $\exists$ non zero harmonic cocycle
(or not Kazhdan (T)) wrt some $(\pi, \mathcal{H})$

Pf/ $(\pi, \mathcal{H})$ given, Fix $\mathcal{U}$ = free ultrafilter

$\lim_{\mathcal{U}} : \ell_\infty(\mathbb{N}) = C(\beta\mathbb{N}) \to \mathbb{R}$ nontrivial character

$u \in \beta\mathbb{N} \setminus \mathbb{N}$

$\mathcal{H}^u := \ell_\infty(\mathbb{N}; \mathcal{H}) / c_u(\mathbb{N})$ $\quad \langle (\xi_n), (\eta_n) \rangle = \lim_u \langle \xi_n, \eta_n \rangle$

$\pi^u(g)(\xi_n) := (\pi(g)\xi_n)$ $\quad (\pi^u, \mathcal{H}^u)$ ultrapower rep$\underline{n}$

Facts Γ am $\Leftrightarrow$ $\lambda : \Gamma \curvearrowright \ell_2\Gamma$ has app. invariant vectors ξ_n, $\|\xi_n\|_2 = 1$,
$\|\xi_n - \lambda_n \xi_n\|_2 \to 0$ $(g \in \Gamma)$

examples include: solvable, subexp. growth

$\lambda(\mu) = \sum \mu(g) \lambda_g$ self adj contraction

Γ finite $\Leftrightarrow$ 1 = eigenvalue of $\lambda(\mu)$

Γ am $\Leftrightarrow 1 \in sp(\lambda(\mu))$

$\Leftrightarrow \underset{\mu^{*2n}(e)}{\underset{\shortparallel}{\langle \lambda(\mu)^{2n} \delta_e, \delta_e \rangle}}^{1/2n} \to 1$ (Kesten)

$\langle \cdot \delta_e, \delta_e \rangle$ is a faithful tracial state on $C^*_\lambda(\Gamma)$

$\mu^{*n}(e) = \langle \lambda(\mu)^n \delta_e, \delta_e \rangle = \int_0^1 t^n \, d\nu(t)$

ν = prob. measure on $sp(\lambda(\mu)) \subseteq [0,1]$ with
$\nu(\{1\}) = 0$, $1 \in \operatorname{supp} \nu$

∴ $c_n(g) := \mu^{*n/2} - \lambda_g \mu^{*n/2} \in \ell_2\Gamma$

$$\|c_n\|^2 = \sum_s \|c_n(s)\|^2 = 2\mu^{*n}(e) - 2\mu^{*n+1}(e)$$

$$= 2\int_0^1 t^n(1-t)\,d\nu(t) > \frac{1}{2}\delta^n(\nu((\delta,1)) \qquad \frac{1}{2}<\delta<1$$

$b_n := c_n/\|c_n\|$

Define $b^u(g) := (b_n(g)) \in (\ell_2\Gamma)^u$ cocycle

We claim that b^u = harmonic

$$\Big\|\sum_x b^u(x)\Big\|^2 = \lim_u \frac{\|\mu^{*n/2} - \mu^{*n/2+1}\|^2}{\|c_n\|^2}$$

$$= \lim_n \frac{\mu^{*n}(e) - 2\mu^{*n+1}(e) + \mu^{*n+2}(e)}{\|c_n\|^2}$$

$\gamma(n) := \frac{1}{2}\|c_n\|^2$

$= \int_0^1 (1-t)t^n\,d\nu(t)$

$$= \lim_u \frac{\gamma(n) - \gamma(n+1)}{2\gamma(n)} = 0$$

∴ $\exists\, b$ = nonzero harmonic wrt some $(\pi, \underbrace{\mathrm{Sp}\, b(\Gamma)}_{\text{f.d. for pol. growth case}})$

In general

$$\pi = \underbrace{\oplus\, \text{f. dim subrep}\underline{\text{ns}}}_{\text{almost periodic}} \oplus \underbrace{\text{No non-zero f.d. summand}}_{\text{weakly mixing}}$$

$$= \pi_{ap} \oplus \pi_{wm}$$

$$b = b_{ap} \oplus b_{wm}$$

<u>Thm</u> Γ = amenable, b = harmonic, $b_{ap} \neq 0 \Rightarrow \Gamma$ has virtually $\mathbb{Z}$-quotient.

Pf/ we may assume π = f.d.

Conversely, if $U \subseteq G^{(0)}$ is open, for $I = \langle C_0(U) \rangle \trianglelefteq \underset{\overset{\shortparallel}{C_r^*(G|_U)}}{C_r^*(C)}$

we have $U(I) = U$, i.e. θ = onto.

[for $x \in G^{(0)}$ with $f(x) \neq 0$ for some $f \in I \cap C_0(G^{(0)})$

approximate f with $\varphi * \tilde{f} * \psi$; $\varphi, \psi \in C_c(G)$, $\tilde{f} \in C_c(U)$,

with $\varphi * \tilde{f} * \psi(x) \neq 0$; i.e.

$$0 \neq \varphi(h) \tilde{f}(h^{-1}g) \psi(g^{-1}) \quad \text{, some } g, h \in G^x$$

$$s(h) = s(h^{-1}g) \in U \to \underset{x}{\underset{\shortparallel}{r(h)}} \in U \;]$$

We ~~say~~ want that θ = inj.

This happens iff $I = \langle E(I) \rangle$, for $I \trianglelefteq C_r^*(G)$

for $E : C_r^*(G) \to C_0(G^{(0)})$

$$[\; I \cap C_0(G^{(0)}) \subseteq E(I) \subseteq \langle E(I) \rangle \cap C_0(G^{(0)}) = I \cap C_0(G^{(0)})$$

$$\therefore I = \langle E(I) \rangle = \langle I \cap C_0(G^{(0)}) \rangle$$

i.e. θ = inj

Conversely; $I = C_r^*(G|_U)$

$$C_r^*(G|_U) = \langle C_0(U) \rangle = \langle E(C_r^*(G|_U)) \rangle \;]$$

We say that G is inner-exact if

$$0 \to C_r^*(G|_U) \to C_r^*(G) \to C_r^*(G|_{G^{(0)} \setminus U}) \to 0$$

is exact, for every $U \subseteq G^{(0)}$ open G-invariant

Ex minimal gpds are inner-exact

amenable or exact gpds are inner-exact.

We say G has ~~approx.~~ intersection property (IP) if

$$C_0(G^{(0)}) \cap I \neq 0 \qquad (I \neq 0)$$

and virtual intersection property (VIP) if $G|_D$ has (IP) for $D \subseteq G^{(0)}$ G-inv. closed

We say that G is topologically principal $\overset{(TP)}{\text{if}}$

$$\{x \in G^{(0)} : G_x^x = \{x\}\}^- = G^{(0)}$$

and essentially Principal (EP) if $G|_D = TP$ for each $D \overset{\substack{G\text{-inv}\\ \text{Closed}}}{\subseteq} G$

Ex $\quad \underset{\text{top. free}}{\Gamma \curvearrowright X} \qquad \Gamma \times X = TP$

$(X, d) = \text{discrete} \qquad G_X = TP$

Note: $\quad (TP) \Rightarrow (IP)$

$(TP) \Leftarrow (IP)_{\text{full}}$ [where $(IP)_{\text{full}}$ is int. prop. for ideals of $C^*(G)$ instead of $C_r^*(G)$]

Thm $\;G = (TP)$, $\pi : C_r^*(G) \to A$ $*$-homo.

$$\pi|_{C_0(G^{(0)})} = \text{inj} \Rightarrow \pi = \text{inj}$$

Cor 1 $\exists x \in G^{(0)}\; G_x^x = \{x\}$

$C_r^*(G) = \text{simple} \Leftrightarrow G\text{ min.}$

$C^*(G) = \text{simple} \Leftrightarrow$

$C^*(G) = C_r^*(G)$, $G = \text{min} + TP$

Pf/ Claim

$$\|\pi(E(\cdot))\| \leq \|\pi(\cdot)\| \quad (\dagger)$$

If this holds; we have

$$\begin{tikzcd} C_r^*(G) \arrow[r, "\pi"] \arrow[d, "E"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi(C_r^*(G)) \arrow[d, "\psi\ (\text{by } \dagger)"] \\ C_0(G^{(0)}) \arrow[r, "\pi"'] & \pi(C_0(G)) \end{tikzcd}$$

▨

to prove the claim; for $\varepsilon > 0$,

$\exists x \in G^{(0)}$ with $G_x^x = \{x\}$ s.t. $|f(x)| > \|E(f)\|_\infty - \varepsilon$
$= \|\pi(E(f))\| - \varepsilon$

$f - E(f) = \sum_{i=1}^{n} f_i \begin{cases} \text{if } x \in s(\text{supp} f_i) \\ \text{if } x \notin s(\text{supp} f_i) \end{cases} : \exists V \quad h * f * h = h * E(f) * h$
$h \in C_c(V)$

$\text{supp}(f_i) \subseteq U_i = \text{open bisection} \subseteq G \setminus G^{(0)}$

<u>Case 1</u> $\pi(\Gamma)$ infinite $\xrightarrow[\text{Tits alt.}]{}$ $\pi(\Gamma)$ v. solvable

$\longrightarrow \exists$ v. $\mathbb{Z}$-quotient

<u>Case 2</u> $\pi(\Gamma)$ finite $\rightarrow [\Gamma : \ker\pi] < \infty$

$\rightarrow b|_{\ker\pi}$ = additive character $\neq 0$

$\rightarrow \exists$ v. $\mathbb{Z}$-quotient ▨

<u>Fact</u> $\pi : \Gamma \curvearrowright \mathcal{H}$

$$\pi_{ap} \neq 0 \iff (\mathcal{H} \otimes \mathcal{H})^{(\pi\otimes\pi)(\Gamma)} \neq 0$$

Pf/ $\mathcal{H}\otimes\mathcal{H} \simeq HS(\mathcal{H})$, $(\pi\otimes\pi)(g) \leftrightarrow Ad\,\pi(g) : T \mapsto \pi(g)T\pi(g)^*$

~~so~~ invariant vectors $\leftrightarrow \{T \in HS(\mathcal{H}) : [T, \pi(\Gamma)] = 0\}$

Take $T = U|T|$ with $|T| \in \pi(\Gamma)'$, then

$|T| = \sum \lambda_i E_i$ E_i = f. rank proj. $\in \pi(\Gamma)'$

Now $E_i\mathcal{H}$ = f. dim. subrep$\underline{\text{n}}$. ▨

<u>Thm</u> (Shalom 2004)

$$\begin{tikzcd} \Gamma = \text{v. nilpotent} \arrow[r, Leftrightarrow, "\text{Shalom}"] & H_{FD} \\ \Gamma = PG \arrow[u, Rightarrow, "\text{Gromov, 1981}"] \arrow[ur, Rightarrow, "\text{Ozawa}", "\cancel{\text{Shalom}}"'] & \end{tikzcd}$$

(i.e. every harmonic cocycle is a.p.)

(equivalently, b = harmonic $\Rightarrow b|_{Z(\Gamma)} = 0$)

<u>Gromov idea:</u>

$\Gamma = PG$

$\lim_{u} (\Gamma, \frac{1}{\lambda_n} d)$

= connected l.cpt

$(\lambda_n \to \infty)$, so

divide by normal subgp ~~with~~ to get a Lie gp

Shalom found alt. proof.

Bönicke

Christian (II)

Ideal structure,

G = étale ; $C_c(G^{(0)}) \to C_c(G)$

$f \mapsto f$ extends by zero

extends to

$$i : C_0(G^{(0)}) \hookrightarrow C_r^*(G)$$

restriction gives a faithful conditional exp.

$$E : C_r^*(G) \to C_0(G^{(0)})$$

Given $I \trianglelefteq C_r^*(G)$;

$$I \cap C_0(G^{(0)}) \trianglelefteq C_0(G^{(0)})$$

$A \subseteq G^{(0)}$ is G-inv. if $s(g) \in A \Leftrightarrow r(g) \in A \quad (g \in G)$

G is minimal if $\emptyset, G^{(0)}$ are the only closed G-inv. subsets of $G^{(0)}$.

Easy to see that

$$U(I) = \bigcup_{f \in I \cap C_0(G^{(0)})} f^{-1}(\mathbb{C} \setminus \{0\}) \subseteq G^{(0)}$$

is an open G-set

[if $s(g) \in U(I)$ $\exists f \in I \cap C_0(G^{(0)})$ $f(s(g)) \neq 0$

Pick open bisection $V \subseteq G$ with $g \in V$ and $\varphi \in C_c(V)$ s.t. $\varphi(g) = 1$. Then

$$\mathrm{supp}(\underbrace{\varphi * f * \varphi^*}_{\text{in } I \cap C_0(G^{(0)})}) \subseteq V G^{(0)} V^{-1} \subseteq G^{(0)}$$

$$(\varphi * f * \varphi^*)(r(g)) = f(s(g)) \neq 0 \to r(g) \in U(I) \;]$$

This gives $\theta : \mathcal{I}(C_r^*(G)) \to \mathcal{O}_G(G^{(0)})$

$$I \mapsto U(I).$$

Ozawa (Extra Talk)

- Connes Embedding Conj.

Any M = Type II_1-factor with sep. predual $\hookrightarrow \mathcal{R}^\omega$

$\mathcal{R}$ = hyperfinite II_1-factor

- Kirchberg Conj.

$$C^*\mathbb{F}_\infty \otimes_{max} C^*\mathbb{F}_\infty = C^*\mathbb{F}_\infty \otimes_{min} C^*\mathbb{F}_\infty$$

- Tsirelson Conj.

$$\overline{Q}_s = Q_c$$

where

$$\overline{Q}_s = cl\left\{ [\langle\psi|(P_i^k \otimes Q_j^l)\psi\rangle] : \begin{array}{l} (P_i^k) = \text{PVM on } \mathcal{X} \\ (Q_j^l) = \ " \quad " \ \mathcal{X} \\ \psi \in (\mathcal{H}\otimes\mathcal{X})_1 \end{array} \right\}$$

$$Q_c = \left\{ [\langle\psi|P_i^k Q_j^l \psi\rangle] \quad \begin{array}{l} " \\ [P_i^k, Q_j^l] = 0 \end{array} \right\}$$

$$C^*((P_i^k)_i) \cong \ell_\infty^m \quad \forall k \qquad C^*((P_i^k)_{i,k}) \cong \ell_\infty^m *_1 \cdots *_1 \ell_\infty^m$$

$$\ell_\infty^m = C^*(\mathbb{Z}_m) \qquad \ell_\infty^m *_1 \cdots *_1 \ell_\infty^m = C^*(\mathbb{Z}_m * \cdots * \mathbb{Z}_m)$$

$$\therefore \overline{Q}_s = \left\{ [\varphi(P_i^k \otimes Q_j^l)] : \varphi \in S(C^*(\mathbb{Z}_m^{*d}) \otimes_{min} C^*(\mathbb{Z}_m^{*d})) \right\}$$

$$Q_c = \left\{ [\qquad\qquad] : \varphi \in S(C^*(\mathbb{Z}_m^{*d}) \otimes_{max} C^*(\mathbb{Z}_m^{*d})) \right\}$$

hence

$\overline{Q}_s = Q_c \iff \otimes_{max} = \otimes_{min}$ for free gps.

for all m,d

[LHS is known only for $m = d = 2$]

For $[\alpha_{ij}] \in \mathbb{R}^{d\times d}$

$$\sup_{\substack{\|x_i\|\leq 1 \\ \|y_j\|\leq 1}} \Big\| \sum_{i,j=1}^{d} \alpha_{ij}\, x_i \otimes y_j \Big\| = \Big\| \sum \alpha_{ij}\, g_i \otimes g_j \Big\|_{C^*(\mathbb{Z}_2^{*d}) \otimes_{\min} C^*(\mathbb{Z}_2^{*d})}$$

If Connes Embedding Conj. is true
This is computable

Bönicke (III)

$a, b \in A_+$, $a \precsim b \overset{}{\underset{\text{Def}}{\iff}} \exists (v_n) \subseteq A$ $\|a - v_n b v_n^*\| \to 0$

$a \in A_+$ is infinite if $\exists\, \overset{0\neq}{b} \in A_+$ $\begin{bmatrix} a & \\ & b \end{bmatrix} \precsim a$

$0 \neq a \in A_+$ " prop " if $\begin{bmatrix} a & \\ & a \end{bmatrix} \precsim a$

A is purely inf if each $0 \neq a \in A_+$ is properly inf.

Facts For $a, b \in A_+$

(i) $\|a - b\| < \varepsilon \Rightarrow \exists d$ $d b d^* = (a-\varepsilon)_+$

(ii) $0 \neq a \in A_+$ properly inf $\iff$ $a + I \in A/I$ is infinite for any $a \notin I \trianglelefteq A$.

(iii) $b \in \overline{AaA}$ & $a \precsim b$ & a = prop. inf. $\Rightarrow$ b = prop. inf.

Now let G be a TP étale groupoid, Then for
$0 \neq a \in C_r^*(G)$
there is $0 \neq h \in C_0(G^{(0)})_+$ with $h \precsim a$ [for $\varepsilon > 0$, pick $f \in C_c(G^{(0)})_+$ with $\|f\|_\infty = 1$, $\|faf - fE(a)f\| < \varepsilon$, and $\|fE(a)f\|_\infty > \|E(a)\| - \varepsilon$
take $h = (fE(a)f - \varepsilon)_+ \in C_0(G^{(0)})_+$, Then $\|h\| \geq \|fE(a)f\| - \varepsilon > \|E(a)\| - \overset{>}{2\varepsilon}$
i.e. $h \neq 0$, by (i), there is $d \in C_r^*(G)$ $dfafd^* = h$; i.e. $h \precsim a$.]

Thm (Brown-Clark-Sierakowski) G = TP and minimal étale gpoid (in particular, $C_r^*(G)$ is simple) Then

$C_r^*(G)$ = purely inf $\iff$ Every $\overset{0\neq}{f} \in C_0(G^{(0)})^+$ is infinite in $C_r^*(G)$

∴ Pf/ Given $\overset{0\neq}{a} \in C_r^*(G)_+$, find $h \in C_0(G^{(0)})_+$ with $h \precsim a$, and h = inf. then by (ii), h is prop. inf, so by (iii), a is prop. inf. ☒

Def/ G is <u>ample</u> if it has a basis consisting of cpt open bisections.

G=étale: G = ample $\Leftrightarrow$ $G^{(0)}$ = totally disconnected

<u>Thm</u> (Li-Bönicke, 2017) G = ample + EP + inner exact, TFAE

(i) $C_r^*(G)$ = purely inf

(ii) Every $0 \neq p \in \mathrm{Proj}(C_0(G^{(0)}))$ is properly inf in $C_r^*(G)$

(iii) $\forall \underset{\substack{\text{closed}\\ G\text{-inv.}}}{D \subseteq G^{(0)}}$, every $0 \neq p \in \mathrm{Proj}(C_0(D))$ is inf in $C_r^*(G|_D)$.

Pf/ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Easy

(iii) $\Rightarrow$ (i) We need:

Kirchberg-Rørdam: if every nonzero hereditary subC*-alg. in every quotient A/I contains an inf proj. then A is purely inf.

Let $I \trianglelefteq C_r^*(G)$, then $C_r^*(G)/I \cong C_r^*(G_D)$, some $\underset{\text{closed inv.}}{D \subseteq G^{(0)}}$

Take $B \underset{her}{\leq} C_r^*(G|_D)$, $0 \neq b \in B_+$, then $G|_D$ is TP (since G = EP) so we may pick $0 \neq h \in C_0(D)_+$ with $h \precsim b$. Look @ the hereditary subalg $\overline{h\, C_0(D)\, h}$, then by (3), this contains an inf. proj. p with $p \precsim h \precsim b$, so take $x \in C_r^*(G|_D)$ with $p = x^*bx$, put $z = b^{1/2}x$, then $p = z^*z$, hence $q := zz^* = b^{1/2}xx^*b^{1/2} \in B$, $p \sim q$, and q is inf., and KR-thm applie ☒

Paradoxical Decomposition:

Ex $\mathbb{F}_2 = \langle a, b \rangle$

$w(x) = \{$reduced words starting @ $x\}$

$(x \in \{a, a^{-1}, b, b^{-1}\})$

~~$\mathbb{F}_2 = w(a) \cup a w(a^{-1})$~~

$$\mathbb{F}_2 = w(a) \cup a w(a^{-1}) = w(b) \cup b w(b^{-1})$$
$$= w(a) \sqcup a^{-1} a w(a^{-1}) \sqcup w(b) \sqcup b^{-1} b w(b^{-1})$$

Let G = étale and let $V \subseteq G$ be a cpt+open bisection, then $\exists$ $s(V) \xrightarrow[\text{homeo}]{\alpha_V} r(V)$ and these are used (instead of the group elements) to move things around.

Def/ G = étale + ample

A compact open subset $A \subseteq G^{(0)}$ is paradoxical if for $i=1,2$, $\exists n_i \in \mathbb{N}$ $\exists$ cpt open bisections $V_{i1}, \dots, V_{i,n_i}$ $\ni$

$$\bigsqcup_{j=1}^{n_i} s(V_{i,j}) = A \quad i=1,2$$

$$r(V_{i,j}) \subseteq A \text{ pairwise-disjoint.}$$

Fact $A \subseteq G^{(0)}$ cpt+open. is paradoxical $\rightsquigarrow$ $1_A \in C_0(G^{(0)})$ is properly inf proj in $C_r^*(G)$ [Take $A = \bigsqcup_1^n s(V_i) = \bigsqcup_{n+1}^{n+m} s(V_i)$ with $r(V_i)$'s pairwise-disjoint. Take $f_1 = \sum_1^n 1_{V_i}$, $f_2 = \sum_{i=n+1}^{n+m} 1_{V_i}$, then since $V_i^{-1} V_j = \emptyset$ $(i \neq j)$ $V_i^{-1} V_i = s(V_i)$ we get

$$\left.\begin{array}{l} f_1^* * f_1 = 1_A = f_2^* * f_2 \\ f_1 * f_1^* + f_2 * f_2^* \leq 1_A \end{array}\right\} \to 1_A = \text{prop. inf.} \ \boxed{/}$$

Def/ G = étale + ample, $k > l > 0$,

A = cpt + open $\subseteq G^{(0)}$ is (k,l)-paradoxical

if for $i = 1, 2, \dots, k$ $\exists n_i \in \mathbb{N}$ cpt open bisections

$V_{i1}, \dots, V_{i,n_i}$ and $m_{i,1}, \dots, m_{i,n_i} \in \{1, \dots l\}$ $\ni$

$$\bigsqcup_{j=1}^{n_i} s(V_{ij}) = A \qquad j = 1, 2, \dots, k$$

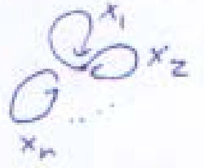
$r(V_{i,j}) \times \{m_{ij}\}$ pairwise disj. on $A \times \{1 \cdots, l\}$

Fact $A = (k,l)$-paradoxical $\rightsquigarrow$ $1_k \otimes 1_A \in M_k(C_r^*(G))$ is an infinite projection.

Cor G = ample + étale

$\mathcal{B}$ = a basis consisting of cpt open $\overset{(2,1)\text{-paradoxical}}{\vee}$ bisections for G,

G = EP + inner_exact $\Rightarrow$ $C_r^*(G)$ is purely inf.

Ex (1) $\overset{(V,E)}{\overset{\shortparallel}{\Lambda_n}}$ =  $G_n = G_{\Lambda_n}$ $\qquad Z(\alpha) = \{\alpha x : x \in E^\infty\}$, $\alpha \in E^{<\infty}$

Cuntz gpoid

for $\beta_i = \alpha x_i$, $s(Z(\beta_i, \alpha)) = Z(\alpha)$

$r(\quad \text{"} \quad)$ pairwise disjoint $\qquad \therefore (n,1)$-parad. decomp.

$\therefore C_r^*(G_n) = \mathcal{O}_n$ = purely inf.

Liao (III)

- $\dim_{asy}(\mathcal{X}) < \infty \Leftrightarrow \Gamma$ = exact
- $\dim_{asy}(\mathcal{X}) < \infty \Leftrightarrow \mathcal{X}$ = property (A)
- $\dim_{asy}(X) = \dim_{asy}(Y) \Leftarrow X \simeq_{QI} Y$.

- $\dim_{asy}(\mathbb{F}_2) = 1$ but $\mathbb{F}_2 \neq$ am.

- $\mathbb{Z} \wr \mathbb{Z} = (\bigoplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$ = amenable $\underset{\forall n}{\geqslant} \mathbb{Z}^n$

  $\therefore \dim_{asy}(\mathbb{Z} \wr \mathbb{Z}) = +\infty$

Def/ X = metric space, then X has <u>b. geometry</u> if

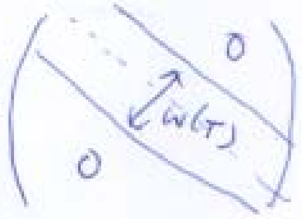
$$\sup_{x \in X} |B(x,R)| < \infty \quad \forall R > 0$$

<u>Rem</u> (i) $(\overset{\text{f.g.}}{\Gamma}, |\cdot|)$ has always b·geom.

(ii) If X = b.geom. then X = discrete + countable.

Def/(Unif. Roe algebra) For $T \in \mathbb{B}(\ell_2 X)$, the <u>propagation</u> or <u>width</u> of T is

$$\omega(T) = \sup_{\langle T\delta_y, \delta_x \rangle \neq 0} d(x,y) \qquad T =$$



$\mathbb{C}_u[X] = \{T \in \mathbb{B}(\ell_2 X) : u(T) < \infty\}$ $\quad \omega(T) < \infty$

$C^*_u(X) := \mathbb{C}_u[X]^- \leqslant \mathbb{B}(\ell_2 X)$ unif. Roe alg.

<u>Facts</u> $C^*_u(X) \supseteq \ell^\infty(X)$ as diagonal <u>ops</u>

$C^*_u(X) \supseteq \mathcal{K}(\ell^2(X))$

unless $|X| < \infty$ ↙

In particular, $C^*_u(X)$ is neither simple nor separable

$$|X| < \infty \to C^*_u(X) = \mathbb{B}(\ell_2 X) = M_{|X|}(\mathbb{C})$$

$$\overset{\text{f.g.}}{\Gamma} \rightsquigarrow C^*_u(\Gamma) = \ell^\infty(\Gamma) \rtimes_r \Gamma$$

<u>Thm</u> (Ozawa) $C^*_u(G)$ = nuclear $\Leftrightarrow$ G = exact.

Prop (Skandalis - Tu - Yu) X, Y = metric space with bdd geom.

$$X \underset{\text{Course}}{\sim} Y \Rightarrow C_u^*(X) \underset{\text{m.E.}}{\sim} C_u^*(Y)$$

If $X \simeq Y$ then $C_u^*(X) \cong C_u^*(Y)$ [just use $U(\delta_x) = \delta_{f(x)}$]

or Course equiv. ← QI bijective

Thm (Špakula - Willett) X, Y = Prop. (A)

$$C_u^*(X) \underset{ME}{\sim} C^*(Y) \Rightarrow X \underset{\text{Course}}{\sim} Y$$

If X, Y are known to non-amenable then
+ X, Y = prop. (A)

$$C_u^*(X) \cong C_u^*(Y) \Rightarrow X \underset{\text{bi Lip}}{\sim} Y \quad (\Leftrightarrow X \overset{\text{biject}}{\underset{QI}{\simeq}} Y)$$

Thm (Li - Liao) $C_u^*(G) \cong C_u^*(H) \Rightarrow G \overset{\text{bij.}}{\underset{\text{coarse}}{\sim}} H$

when G, H have $\dim_{asy} = 0$. $\left(G, H = \text{countable gps}\right)$

Next we explore $\dim_{nuc} C_u^*(X)$.

Lemma 1 X = bdd geom. $[\langle T\delta_y, \delta_x\rangle] = T \in \mathbb{C}_u[X]$ with $w(T) \le S$, $\sup_{x,y} |\langle T\delta_y, \delta_x\rangle| \le M$, then

$$\|T\| \le b(T)M \text{, with } b(T) = \sup_x |B(x,S)|$$

Lemma 2 $(n_k)_{k \in K}$ banded,

non sep C^* algebra ← $A = \prod_{k \in K} M_{n_k} = AF$ (i.e. $F \subset\subset A$ is within ε of some f.d. subalgebra)

Lemma 3 $\dim_{asy} X \le n$: $\forall R > 0\ \exists (h_j^{(i)})$ = p.o.u. $i = 0, \dots, n$, $j \in J_{(i)}$ with $\text{supp}(h_j^{(i)})$ = R-disj ; $h_j^{(i)} = \frac{1}{R}$-Lip.

← unif. bdd

∵ Thm (Winter-Zacharias) $\dim_{nuc}(C_u^*(X)) \le \dim_{asy}(X)$.

Pf/ $T =$

$\prod M_r = AF = A^{(0)}$ $\prod M_{r'} = AF = A^{(1)}$

$$\begin{tikzcd}
C_u^*(X) \arrow[rr, "id"] \arrow[dr, "\varphi = \oplus\varphi^{(i)}"'] & & C_u^*(X) \\
& A^{(0)} \oplus A^{(1)} \arrow[ur, "\psi : (x,y) \mapsto x+y"'] &
\end{tikzcd}$$

$\varphi^{(i)}(a) = \sum_{j \in J_{(i)}} h_j^{(i)\,1/2} a h_j^{(i)\,1/2}$

by Lemma 1

$\|[h^{(i)},a]\| \le b([h^{(i)},a])M$

$= b(a) M_{[h^{(i)},a]}$

$= b(a)(\sup_{d(x,y)<S} |h^{(i)}(x) - h^{(i)}(y)|) M_a$

$\le b(a)\frac{S}{R} M_a$

$h^{(i)}, = \sum_j h_j^{(i)} = \frac{1}{R}\text{-Lip}$

$\langle [h^{(i)},a]\delta_y, \delta_x\rangle =$

$(h^{(i)}(x) - h^{(i)}(y))\langle a\delta_y, \delta_x\rangle$

$\therefore \omega([h^{(i)},a]) = \omega(a)$

$\|\varphi\circ\psi(a) - a\| =$

∴ but now $\sum_i \sum_j \|a - h_j^{(i)\,1/2} a h_j^{(i)\,1/2}\| = \sum_i \|[h^{(i)},a]\| < \varepsilon$

for large R. ▨

Taka (III)

Def/ $\Gamma = H_{FD} \overset{}{\underset{Def}{\Leftrightarrow}} \forall b = \text{cocycle}$ $b = \text{harmonic} \Rightarrow b = \text{ap}.$

$$\begin{tikzcd}
\pi = \pi_{ap} \arrow[d] & \oplus & \pi_{wm} \arrow[d] \\
\oplus \text{ f.d. repn} & & \text{no f.d. summand}
\end{tikzcd}$$

Thm (Shalom 2004) Amenable + H_{FD} is QI-inv.

<u>Ex</u> The following gps are H_{FD}

- Nilpotent
- Polycyclic (successive extension of cyclic gps)
  - ↳ like lattices in simply connected solvable Lie gps.
- $(\bigoplus_{\mathbb{Z}} \mathbb{Z}_2) \rtimes \mathbb{Z}$
- gps with (T) Kazhdan prop.

<u>Non Ex</u> The following are <u>not</u> H_{FD}

- $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$, $(\bigoplus_{\mathbb{Z}^d} \mathbb{Z}) \rtimes \mathbb{Z}^d$ $(d \geq 3)$
  $d = 2$; open
- finitely generated + torsion + simple.
  These include some amenable gps
- $\mathbb{F}_2$

<u>Criterion</u> (using random walks)

$X_n = s_1 \cdots s_n \in \Gamma$ s_i's = random & indep.

b = harmonic $\iff$ $b(X_n)$ = martingale, i.e.
$\mathbb{E}(b(X_{n+1})|X_1, \ldots, X_n] = b(X_n)$

b = harmonic $\Rightarrow$
$$\mathbb{E}(\|b(X_n)\|^2) = \mathbb{E}\left[\sum_t \|b(X_{n-1}t)\|^2\right]$$
$$= \mathbb{E}\left[\sum_t \|b(X_{n-1}) + \pi_{X_{n-1}} b(t)\|^2\right]$$
$$= \mathbb{E}\ \|b(X_{n-1})\|^2 + \sum_t \underbrace{\|b(t)\|^2}_{\|b\|^2}$$
$$= \cdots = n\|b\|^2.$$

$\therefore$ $\|b(X_n)\| \sim \sqrt{n}\|b\|$ in average

<u>Prop</u> (Martingale Central Limit Thm)

b = harmonic cocycle

$$\frac{1}{\sqrt{n}} \langle b(X_n), \xi \rangle \xrightarrow{dist} N(0, q(\xi)) \qquad \xi \in \mathcal{H}$$

$\therefore$ where

$$q(\xi) = \lim_n \frac{1}{n} \mathbb{E}\left[\langle b(x_n), \xi \rangle^2 \right]$$
$$= \lim_n \frac{1}{n} \mathbb{E} \, \langle (b\otimes b)(x_n), \xi\otimes\xi \rangle$$

$$\mathbb{E}\,(b\otimes b)(x_n) = \mathbb{E} \sum_t (b\otimes b)(x_{n-1}t)$$
$$= \mathbb{E}\left[(b\otimes b)(x_{n-1}) + \sum_t (\pi\otimes\pi)_{x_{n-1}} (b\otimes b)(t) \right]$$
$$= \mathbb{E}\left[(b\otimes b)(x_{n-1}) \right] + T^{n-1} W$$
$$= \cdots = (1 + T + \cdots + T^{n-1}) W$$

where; $\overset{\text{s.-adj contraction}}{\uparrow}$

$T := \sum_t (\pi\otimes\pi)_t$

$W := \sum_t (b\otimes b)(t)$

$$\therefore \quad q(\xi) = \lim_n \frac{1}{n} \langle (1+T+\cdots+T^{n-1})W, \xi\otimes\xi \rangle$$
$$= \langle \underbrace{\chi_{\{1\}}(T)}_{P} W, \xi\otimes\xi \rangle$$
$$\doteq \langle PW \xi \mid \xi \rangle$$

$P = \mathrm{Proj}_{(\mathcal{H}\otimes\mathcal{H})^{(\pi\otimes\pi)(\Gamma)}}$

$PW \in \pi(\Gamma)'$

Let $\lambda_1 \geq \lambda_2 \geq \cdots$ $\overset{\text{nonzero}}{\text{eigenvalues}}$ of PW

with $\zeta_1, \zeta_2, \ldots$ eigenvectors

$$\frac{1}{\sqrt{n}} \langle b(x_n), \zeta_i \rangle \xrightarrow{\text{dist}} N(0, q(\zeta_i)) = \lambda_i^{1/2} g_i \quad (g_i \sim N(0,1))$$

$$\sum_i^{\text{finite}} \alpha_i \frac{1}{\sqrt{n}} \langle b(x_n), \zeta_i \rangle \longrightarrow \Big(\sum_i^{\text{finite}} \alpha_i^2 \lambda_i\Big)^{1/2} N(0,1)$$

i.e. g_i = independent (asymptotically)

$\sum_i \lambda_i = \mathrm{Tr}(PW)$
$= \|b_{ap}\|^2$

$$\left\| \frac{1}{\sqrt{n}} b(x_n) \right\|^2 = \frac{1}{n} \sum_i \langle b(x_n), \zeta_i \rangle^2 + \text{the term for } \ker(PW)$$
$$\xrightarrow{\text{dist}} \sum_i \lambda_i g_i^2 + g_0$$

Cor (Erschler-Ozawa) b = harmonic

$$\frac{1}{n}\|b(X_n)\|^2 \xrightarrow{dist} \sum \lambda_i \xi_i + \theta$$

$$\sum \lambda_i = \|b_{ap}\|^2, \quad \theta = \|b_{wm}\|^2$$

Cor b = harmonic

$$b = \text{ap.} \Longleftrightarrow \forall \varepsilon > 0 \quad \limsup_n \mathbb{P}\left(\|b(X_n)\| < \varepsilon\sqrt{n}\right) > 0$$

Cor If $\exists \mu$ $\quad \limsup_n \mathbb{P}\left(|X_n| < c\sqrt{n}\right) > 0$ $\quad \forall c > 0$

then $\Gamma = H_{FD}$

Remark (i) Γ = non-amenable

$$\frac{|X_n|}{n} \xrightarrow{dist} \text{const} > 0$$

(ii) If

continuous case (1) $\limsup_n \mathbb{P}\left(\max_{1 \le k \le n} |X_k| < c\sqrt{n}\right) > 0$ $\quad \forall c > 0$

$$\Downarrow$$

$H_{FD} \Leftarrow$ (2) $\limsup_n \mathbb{P}\left(|X_n| < c\sqrt{n}\right) > 0$ $\quad \forall c > 0$

$$\Downarrow$$

"diffuse case" (3) $\limsup_n \mathbb{P}\left(|X_n| < c\sqrt{n}\right) > 0$ $\quad$ some $c > 0$

Open Problem Session

Christian Bönicke

(1) A = purely inf. if every nonzero hereditary subalg in every $^{0\neq}$quotient contains an infinite projection (Kirchberg-Rørdam)

Q Can we replace "proj" with "pos. element"

If yes, we could remove "ample" from the result in my lecture (III).

The converse is known to be true.

(2) $$0 \to \mathbb{T} \times G^{(0)} \to \Sigma \to G \to 0$$

$C_r^*(G, \Sigma)$ = twisted group C^*-alg.

non-comm. Tori is an example

Q Can we repeat results on $C_r^*(G)$ for the twisted case?

Q Do we know when $C_r^*(G, \Sigma)$ = simple?

(3) G = am. $\Rightarrow$ $C_r^*(G) = C^*(G)$

$\not\Leftarrow$ exple: R. Willett

Q ($\Leftarrow$) holds when G = exact?

G = inner-exact?

(4) G = exact group bundle $\Rightarrow$ $\forall x$ G_x = exact

Q ($\Leftarrow$)?

When G is also inner-exact $\Leftrightarrow$ $0 \to C_r^*(G)_{G^{(0)}\setminus\{x\}} \to C_r^*(G) \to C_r^*(G_x) \overset{0}{\uparrow}$ is exact.

Narutaka Ozawa

(1) Is every maximal ideal in a C^*-algebra A closed?

True: if A is unital. or A comm.
if A nonunital, there are many dense ideals.

(2) A = unital nuclear C^*-alg.

$I(A)$ = isometries

When $I(A)$ = am? (i.e. $\forall \varepsilon > 0$ $\forall E \subset\subset I(A)$ $\exists F \subset\subset I(A)$
$\forall s \in E$ $\exists \pi_s \in \mathfrak{S}(F)$

$$\left| \{ x \in F : \| x s - \pi_s(x) \| < \varepsilon \} \right| > (1-\varepsilon)|F| \;)$$

When A = finite $I(A) = \mathcal{U}(A)$

$$\frac{1}{|F|} \sum_{u \in F} \varphi(u \cdot u^*) \xrightarrow[(E,\varepsilon)]{} \text{tracial state}$$

but not all finite nuclear C^*-algebras are stably finite, so the answer is no for A = finite.

<u>Conj</u>: $\mathcal{Z}$-stable $\Rightarrow$ $I(A)$ am.

We know it is true purely-inf. simple + ASH case
AH case is trivial

$$\text{unital } A = \text{nuclear} \iff \begin{tikzcd} & M_n \arrow[dr] & \\ A \arrow[ur] \arrow[rr, "id"'] \arrow[rr, phantom, bend left=20, "\circlearrowleft"] & & A \end{tikzcd}$$

$\iff$ any $E \subset\subset A_1$, $\forall \varepsilon > 0$ $\exists \varphi : M_n \to A$ ucp

$$d(\overset{s}{a}, M_n)_1) < \varepsilon \quad (\overset{s}{a} \in E).$$

this passes to quotient

∴ by KSGNS-construction

$$\exists \pi : \mathbb{M}_n \to M(A \otimes K) \quad *\text{-homo}$$

$$\varphi(\cdot) = \pi(\cdot)_{11}$$

$\rightsquigarrow$ $\exists x_s \in \mathcal{U}(\mathbb{M}_n)$ s.t.

$$\operatorname{dist}(s, \varphi(x_s)) < \varepsilon'$$

$$\underset{\delta\text{-net}}{\tilde{F} \subset\subset} \mathcal{U}(n) \quad \delta \ll \varepsilon$$

$$\exists \pi_s \quad \| a s - \pi_s a \| < 2\delta \quad (a \in \tilde{F})$$

$$M(A \otimes K) = \mathbb{M}_\infty(A) \quad \text{acting on } \ell_2 \otimes A$$

$$\hat{F} = \left\{ \pi(x) \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} : x \in \tilde{F} \right\} \subseteq \ell_2 \otimes A$$

$$s \simeq \pi(x_s)_{11}$$

$$\therefore \pi(x_s) \approx \left[\begin{array}{c|c} s & * \\ \hline 0 & * \end{array} \right]$$

$$\therefore \pi(a) \begin{bmatrix} s \\ 0 \\ \vdots \end{bmatrix} \approx \pi(a)\pi(x_s) \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$$

$$\approx \pi(a x_s) \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} \approx \pi(\pi_s(a)) \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$$

$$y(a) := \pi(a) \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} \in \ell_2^{\mathbb{N}} \otimes A \qquad \approx y(\pi_s(a))$$

∴ if $\mathcal{O}_\infty \hookrightarrow A$ then $I(A)$ is amenable.

$I(A)$ amenable $\Rightarrow$ A = nuclear

[$F \subset\subset I(\ell_2 \otimes A)$ $\quad \mathbb{B}(H) \ni a \mapsto \frac{1}{|F|} \sum_i x_i^* a x_i$ in limit it gives cond. exp to $A' \subseteq \mathbb{B}(H)$ $\rightsquigarrow$ A' = inj]

Γ exact $\iff$ property (A)

· $\Lambda \leq \Gamma$ $\quad$ Γ/Λ coarse metric space

<u>Q</u> When Γ/Λ = property (A) ?

$\Gamma \times \Gamma / \Delta(\Gamma)$ has property (A) for

$\Gamma = SL_2\mathbb{Z}$

but not for $\Gamma = SL_2\mathbb{Z}[1/p]$

Sakai Kishimoto : arxiv 2020

heiki Suzuki $\quad$ $A \subseteq N_n$ $\quad$ $N_n \nmid$ nuclear

↓

in Nagoya !

Ozawa (IV)

We show that Γ with $\gamma_\Gamma(n) \ll e^{\sqrt{n}}$ has virtually $\mathbb{Z}$-quotients (H_{FD}).

$\Gamma = \langle S \rangle$, μ = symm. prob. measure with $\operatorname{supp}\mu = S$

(0) Γ has controlled Følner sets

$$\begin{array}{ll}
 & (0) \\
 & \Downarrow \\
\text{"Continuous"} & (1)\ \limsup_n \mathbb{P}\left(\max_{1\le k\le n} |X_k| < c\sqrt{n}\right) > 0 \quad \forall c>0 \\
 & \Downarrow \\
 & (2)\ \limsup_n \mathbb{P}\left(|X_n| < c\sqrt{n}\right) > 0 \quad \forall c>0 \Rightarrow H_{FD} \\
 & \Downarrow \\
\text{"diffused"} & (3)\ \limsup_n \mathbb{P}\left(|X_n| < C\sqrt{n}\right) > 0 \quad \text{some } C>0
\end{array}$$

Q $(0) \Leftrightarrow (1) \Leftrightarrow (2) \Leftrightarrow (3)$?

$\gamma_\Gamma(n) \ll e^{\sqrt{n}} \Rightarrow (3)$?

Ex
- Γ = abelian $\Rightarrow$ (1) (central limit thm)
- Γ = Lamplighter gp $= \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}_2\right) \rtimes \mathbb{Z} \Rightarrow (1)$

  μ = switches $\left.\begin{array}{l} 1 \text{ to } 0 \\ 0 \text{ to } 1 \end{array}\right\}$ $1/2$ = probabilities

  shifts to right $1/4$ = "

  " " left $1/4$ = "
- $\Gamma = \left(\bigoplus_{\mathbb{Z}^2} \mathbb{Z}_2\right) \rtimes \mathbb{Z}^2$ does not satisfy (1)

Open Problem $\left(\bigoplus_{\mathbb{Z}^2} \mathbb{Z}_2\right) \rtimes \mathbb{Z}^2 \not\equiv H_{FD}$? (open problem)

We know: $\left(\bigoplus_{\mathbb{Z}^d} \mathbb{Z}_2\right) \rtimes \mathbb{Z}^d \neq H_{FD}$ for $d \geq 3$.

Def/ Γ has controlled Følner sets $\underset{\text{Def}}{\Leftrightarrow}$ $\exists \delta, K > 0\ \forall N$

$\exists n \geq N\ \exists F \subseteq B(0,n) = \bigcup_{k=0}^{n} S^k$ s.t.

$$|\{x : d(x,F) \leq \delta n\}| \leq K|F|.$$

Ex $\Gamma = \mathbb{Z}^d$

$$F = [0,n]^d \subseteq B(0,\delta n) \qquad |F| = (n+1)^d$$

take $K = (\delta+1)^d$.

In general;

Følner set $\Longleftrightarrow \exists F \quad \max_{g\in S} \dfrac{|gF \Delta F|}{|F|} < \varepsilon$

controlled " " $\Longleftrightarrow \exists F \subseteq B(0,n) \quad \max_{g\in S} \dfrac{|gF\Delta F|}{|F|} < \dfrac{k}{n}$

for inf. many n.

Ex . Γ = with polynomial growth

$$|B(0,n)| \leq Cn^d \rightsquigarrow |B(0,\delta n)| \leq C\delta^d n^d.$$

$\Rightarrow \Gamma$ has controlled Følner sets.

- $\Gamma = \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}_2\right) \rtimes \mathbb{Z}$ has controlled Følner sets
- Γ = polycyclic gps (Tessera)

Facts (1) Γ has controlled Følner set Then

$$\exists K \qquad \inf_{\substack{0 \neq \xi \in \ell_2\Gamma \\ \operatorname{supp}\xi \subseteq B(0,n)}} \frac{\sum_g \|\xi - \lambda_g \xi\|^2}{\|\xi\|^2} \leq K/n^2 \quad \text{for inf. many } n,$$

Remark $\max_{g\in S} \dfrac{|gF\Delta F|}{|F|} < \dfrac{K}{n}$ for $\xi = \chi_F$ only gives

the above value $\leq K/n$ not K/n^2!

Pf of (2) $\xi(x) :\equiv d(x, \{y : d(y,F) > \delta n\})$

$$\operatorname{supp}(\xi) \subseteq \{y : d(y,F) \leq \delta n\} \subseteq B(0,(1+\delta)n)$$

$$\|\xi - \lambda_g \xi\|^2 \leq 2\,|\operatorname{supp}(\xi)| \leq 2K|F|$$

$$\|\xi\|^2 \geq (\delta n)^2 |F|$$

Proof of $(0) \Rightarrow (1)$ (Erschler - Zheng) $\quad m = c\sqrt{n}$

$$\alpha = \mathbb{P}\left(\max_{1 \leq k \leq m} |X_k| \leq m \right\} = ?$$

$$T = \sum \lambda_g \in \mathbb{B}(l_2 \Gamma) \quad , \quad P : l_2\Gamma \to l_2 B(0,m)$$

then

$$\alpha = \sum_y \langle (PTP)^n \delta_y , \delta_e \rangle$$

PTP has largest eigenvalue λ with nonzero eigenvector $\xi \geq 0$ (Perron_ Frobenius)

Take $x_0 \in B(0,m)$ $\quad \xi(x_0) = \|\xi\|_\infty > 0$

$$\sum_y \langle (PTP)^n \delta_y , \delta_e \rangle = \sum_y \langle (PTP)^n \delta_y , \delta_{x_0} \rangle \, \xi(x_0)$$

$$\geq \sum_y \langle (PTP)^n \delta_y , \delta_{x_0} \rangle \, \xi(y) = \lambda^n \xi(x_0)$$

$$\therefore \mathbb{P}\left(\max_{1 \leq k \leq n} |x_0 X_k| \leq m \right) \geq \lambda^n$$

$$\therefore \mathbb{P}\left(\max_{1 \leq k \leq n} |X_k| \leq 2m \right) \geq \lambda^n$$

By the above fact

$$\frac{1}{2} \inf_{\substack{\xi \in l_2 B(0,m) \\ \|\xi\|_2 = 1}} \sum_g \| \xi - \lambda_g \xi \|^2 \leq K/m^2$$

$$LHS = \inf \langle (1 - T)\xi , \xi \rangle = \inf \langle (1 - PTP)\xi , \xi \rangle$$

$= $ smallest eigenvalue of $P(1-T)P$

$$\therefore \lambda \geq 1 - K/m^2$$

$$\therefore \ \lambda^n \geqslant \left(1-\frac{k}{m^2}\right)^n = \left(1-\frac{K}{(\frac{1}{2}c\sqrt{n})^2}\right)^n$$

$$= \left(1-\frac{4k}{c^2 n}\right)^n \approx e^{-4k/c^2} > 0$$

<u>Conjecture</u> (3) $\Rightarrow H_{FD}$

Suppose b = harmonic cocycle

π = mixing (i.e. $\langle \pi_g \xi, \eta \rangle \xrightarrow[g\to\infty]{} 0$ $(\xi, \eta \in \mathcal{H})$)

Then

$$X_n = g_1 \cdots g_\ell \qquad \ell = |X_n|$$

$$b(X_n) = b(g_1) + \pi_{g_1} b(g_2) + \pi_{g_1 g_2} b(g_3) + \cdots + \pi_{g_1 \cdots g_{\ell-1}} b(g_\ell)$$

$$\therefore \ \|b(X_n)\|^2 \ll \ell^2 = |X_n|^2 \ll n \quad \text{for } |X_n| < c\sqrt{n}$$

but $\dfrac{\|b(X_n)\|}{\sqrt{n}} \xrightarrow{\text{dist}} \theta = \text{constant} \quad (b_{ap} = 0)$

$\therefore \ \theta = 0.$

We need to have this for π = "weakly mixing"

i.e. $\langle \pi_g \eta, \xi \rangle \underset{\text{in average}}{\longrightarrow} 0$

<u>Q</u> State exactness of Γ in terms of Random Walks!

Liao (IV)

Ref. (Špakula – Tikuisis) Relative commutant picture of uniform Roa algebra, arxiv

Let X = bdd geometry metric space, $R>0$

Lemma $\omega(T)\leq R \iff fTf'=0$ $\forall f,f'\in \ell^\infty X$ with R-disj. supports

Pf/ ($\Leftarrow$) Take $x,y\in X$, $d(x,y)>R$,

$|\langle T\delta_y,\delta_x\rangle| = |\langle 1_{\{x\}}T1_{\{y\}}\delta_y,\delta_x\rangle| = 0$, by assumption.

($\Rightarrow$) Given f,f' as above,

$|\langle fTf'\delta_y,\delta_x\rangle| = |f'(y)f(x)\langle T\delta_y,\delta_x\rangle| = 0$ one of the three terms is zero ▨

Def/ T has ε-propagation at most R if $\|fTf'\|<\varepsilon$ for any $f,f'\in\ell^\infty X$ with R-disjoint supports

T is quasi-local if T has finite ε-propagation for any $\varepsilon>0$

Norm limits of q-local T's is q-local

$C_u^*(X)\leq QL(\ell_2X)\leq B(\ell_2X)$ are C*-algebras

Q (Open Problem) Is $C_u^*(X)=QL(\ell_2X)$?

The above reference shows the equality for $\dim_{asy}X<\infty$.

Def/ A bdd sequence $(f_n)\subseteq\ell^\infty(X)$ is very Lipschitz if

$\forall L\geq 0$ $\exists n_0$ $\forall n\geq n_0$ $f_n = L$-Lipschitz

$VL(X) := \{(f_n) : (f_n) \text{ is } VL\}\subseteq\ell^\infty(\mathbb{N},\ell^\infty(X))$

$\therefore VL(X)$ is a C^*-algebra $\qquad VL_0(X) = \{(f_n) : \|f_n\| \to 0\}$

$$VL_\infty(X) := \frac{VL(X)}{VL_0(X)} = \overset{\text{abelian}}{C^*\text{-algebra}} \leqslant \ell^\infty(X)_\infty < B(\ell_2 X)_\infty$$

<u>Observation</u> $T \in B(\ell^2 X)$

$T \in QL(\ell_2 X) \Longleftrightarrow [T, VL_\infty(X)] = 0$ i.e.

$$QL(\ell_2 X) = VL_\infty(X)' \cap B(\ell^2 X)_\infty$$

When $\dim_{asy} X < \infty$

$$QL(\ell_2 X) = VL_\infty(X)' \cap B(\ell^2 X)_\infty = C_u^*(X).$$

Also

$$\dim_{nuc}(VL_\infty(X)) \leqslant \dim_{nuc}(VL(X))$$

$$\leqslant \dim_{asy} X$$

when $\dim_{asy} X < \infty$, then all of the above are "equality".

Pf/ Let $n = \dim_{asy} X < \infty$, fix $m \geqslant 1$, Pick $(h_j^{(i)})$ = p.o.u. as before, with disjoint supports, $\frac{1}{m}$-Lipschitz, and unif. bound D on diameters of supports.

Choose points $x_j^{(i)}$ in supp $h_j^{(i)}$ and observe that for

$$F = \ell^\infty(J^{(0)}) \oplus \cdots \oplus \ell^\infty(J^{(n)}) = AF\text{-alg.}$$

as functions of j

$$\psi(f) = (f(x_j^{0}), \ldots, f(x_j^{n}))$$

$$\varphi^{(i)}(g) = \sum_{j \in J^{(i)}} g(j) h_j^{(i)}$$

$$\begin{tikzcd}
\ell^\infty(X) \arrow[rr] \arrow[dr, "\psi"'] & \arrow[d, phantom, "\circlearrowleft"] & \ell^\infty(X) \\
& F \arrow[ur, "\varphi = \sum_i \varphi^{(i)}"'] &
\end{tikzcd}$$

with functions in $\mathrm{Im}(\varphi)$ being $\frac{2(n+1)}{m}$-Lipschitz.

$$|\varphi\psi(f)(x) - f(x)| = \left|\sum_i \sum_j f(x_j^{(i)}) h_j^{(i)}(x) - \sum_i \sum_j f(x) h_j^{(i)}(x)\right|$$

$$\leqslant \varepsilon \quad \text{for } f = \frac{\varepsilon}{(n+1)D}\text{-Lipschitz}$$

Next, we allow m to vary, the same holds for $f = (f_m) \in VL(X) \leq \ell^\infty(\mathbb{N}, \ell^\infty X)$. We may assume that each f_m is $\frac{\varepsilon}{(n+1)D_m}$-Lipschitz;

$$\Psi((f_m)) = (\overset{\text{as before}}{\psi_m}(f_m))$$

$$\begin{tikzcd} VL(X) \arrow[rr] \arrow[dr, "\Psi"'] & \arrow[d, phantom, "2"] & VL(X) \\ & \prod\limits_{m\in\mathbb{N}}\left(\bigoplus\limits_{i=0}^{n} \ell^\infty(J(i,m))\right) \arrow[ur, "\Phi=\sum_i \Phi^{(i)}"'] & \end{tikzcd}$$

$$\Phi^{(i)}((g_m)) = (\overset{\text{as be}}{\Phi_m^i}(g_m))$$

$$\prod_{m\in\mathbb{N}}\left(\bigoplus_{i=0}^{n} \ell^\infty(J(i,m))\right) = \bigoplus_{i=0}^{n}\left(\prod_m \ell^\infty(J(i,m))\right)$$

$$\|\Phi\psi((f_m)) - (f_m)\|_{\ell^\infty(\mathbb{N},\ell^\infty X)} = \sup_m \|\phi_m(\psi_m(f_m)) - f_m\|_\infty \leqslant \varepsilon$$

The reverse inequality is even harder! ▨

Bönicke (IV)

Type Semigroup f ample gpoids

$$S(G) = \{ \bigcup_{i=1}^{n} A_i \times \{i\} \mid A_i \subseteq G^{(0)} \text{ cpt-open} \}/\sim$$

$A \sim B \iff \exists \ell \ \exists V_1 \cdots V_\ell$ cpt open bisections

$\exists n_1 \cdots n_\ell, m_1 \cdots m_\ell$

$$A = \bigsqcup_i s(V_i) \times \{n_i\}$$

$$B = \bigsqcup_i r(V_i) \times \{m_i\}$$

$$[A] + [B] = [A \cup B]$$

$$0 = [\emptyset]$$

$x \leq y \iff y = x + z$, some z $\quad (x, y \in S(G))$

<u>Facts</u> G = ample, $A \subseteq G^{(0)}$ cpt-open

$A = (k,\ell)$-paradoxical $\iff k[A] \leq \ell[A]$.
$k > \ell$

<u>Thm</u> (Tarski) Let $\overset{\overset{x}{\cap}}{S}$ = abelian monoid, TFAE

(i) $(n+1)x \not\leq nx \quad \forall n \in \mathbb{N}$
(ii) $\exists$ additive map $f : S \to [0, \infty]$ $f(x) = 1$.

<u>Prop</u> G = ample; $G^{(0)}$ = cpt, Then
$\exists$ faithful, order-preserving surjective monoid homo $\rho : C(G^{(0)}, \mathbb{Z})^+ \to S(G)$

$\forall V$ = cpt-open bisection $\forall f$ $\operatorname{supp} f \subseteq r(V)$

$$\rho(f) = \rho(f \circ \alpha_V) \qquad \alpha_V : s(V) \to r(V)$$

Pf/ Since $G^{(0)}$ = totally disconnected, need to check this for $f = \sum_1^n 1_{A_i}$; $\rho[\sum 1_{A_i}] = [\cup A_i \times \{i\}]$. ▨

no nontrivial closed G-inv. subsets

Prop G=ample, $G^{(0)}$=cpt, $\overset{G=}{\text{minimal}}$, take

$\varphi: S(G) \to [0,\infty)$ faithful state

i.e. $\varphi[G^{(0)}]=1$, then φ lifts to a tracial state

$\tau : C^*_r(G) \to \mathbb{C}$ $\quad \tau([1_A]) = \varphi([A])$.

Pf/ Consider the decomposition $\tilde{\tau} = \varphi \circ \rho$ and extend to $\tilde{\tau} : C(G^{(0)}, \mathbb{Z}) \to \mathbb{R}$. This is a state on $K_0(C(G^{(0)}))$.

By Blackadar–Rørdam, $\tilde{\tau}$ lifts to a state $\tau_0 : C(G^{(0)}) \to \mathbb{C}$. Put $\tau = \tau_0 \circ E \in S(C^*_r(G))$. ▨

Def/ A=C*-alg is stably finite if $1_k \otimes 1_A$ is a finite projection in $M_k(A)$, for $k \geq 1$.

Thm G=ample, minimal, $G^{(0)}$=cpt., TFAE

(1) $C^*_r(G)$ has faithful tracial state

(2) $C^*_r(G)$ stably finite

(3) Every clopen $A \subseteq G^{(0)}$ is non-paradoxical for any (k,l).

(4) $\exists$ faithful state $\varphi: S(G) \to [0,\infty)$ with $\varphi([G^{(0)}]) = 1$.

Pf/ Seen before, plus Tarski: (3) ⇒ (4). ▨

Def/ $B \subseteq A$ = C*-alg. is called a Cartan if

(i) B is an abelian C^*-subalgebra

(ii) B contains a bai of A

(iii) $B = $ MASA

(iv) $B =$ regular

(v) $\exists\, \mathbb{E} : A \to B$ a conditional expectation.

We are interested in $B = C_0(G^{(0)}) \leq A = C_r^*(G)$

Consider $j : C_r^*(G) \to C_0(G)$ (extending $\mathrm{id}_{C_c(G)}$)

put $\mathrm{supp}^1(a) := \{g : j(a)(g) \neq 0\}$ $(a \in C_r^*(G))$

Prop (i) $a \in C_0(G^{(0)})' \Leftrightarrow \mathrm{supp}^1(a) \subseteq \mathrm{Iso}(G) = \{g : s(g) = r(g)\}$

(ii) $C_0(G^{(0)}) = $ MASA $\Leftrightarrow \mathrm{Iso}(G) = G^{(0)}$

(iii) $G = TP \Rightarrow \mathrm{Iso}(G) = G^{(0)}$.

Pf/ (i) $j(ab)(g) = j(a)(g)\, b(s(g))$ $\qquad b \in C_0(G^{(0)}),\ a \in C_r^*(G)$

$j(ba)(g) = b(r(g))\, j(a)(g)$

$\mathrm{supp}^1(a) \subseteq \mathrm{Iso}(G) \rightsquigarrow j(ab) = j(ba)$

$\rightsquigarrow ab = ba$

$\rightsquigarrow a \in C_0(G^{(0)})'$

the converse is similar.

(ii) If $\mathrm{Iso}(G) \neq G^{(0)}$, then $\overset{\emptyset}{\overset{\neq}{U}} \underset{\text{open}}{\subseteq} \mathrm{Iso}(G) \setminus G^{(0)}$ gives

$s(U) \cap \{x \in G^{(0)} : G_x^x = \{x\}\} = \emptyset$ $\rightarrow\leftarrow$ ▨

Def/ $B \leq A$ $\quad \mathcal{N}(B) = \{n \in A : nBn^* \leq B,\ n^*Bn \leq B\}$

B is regular if $A = C^*(\mathcal{N}(B))$

Prop/ $a \in C_r^*(G)$ $\operatorname{supp}^1(a) \subset G$ open bisection, then $a \in N(C_0(G^{(0)})$

If $G = TP$, The converse also holds.

Cor $G = TP \Rightarrow C_0(G^{(0)}) \leq C_r^*(G)$ Cartan.

Thm (Renault 2008, Kumjian) $B \leq A$ Cartan, then $\exists!$ top. principal étal groupoid G and a twist Σ

$$0 \to G^{(0)} \times \mathbb{T} \to \Sigma \to G \to 0$$

(central ext. of l.cpt. gpoids)

with $C_r^*(G,\Sigma) \simeq A$ mapping $C_0(G^{(0)})$ onto B.

The uniqueness is important in questions of rigidity. This has applications in top. dynamics and Roe algebras

Pf of Renault's thm

If $n \in N(B)$ then $nn^*, n^*n \in B$. Choose open subsets

$$\operatorname{dom}(n) = \{x \in \widehat{B} : n^*n(x) > 0\}$$
$$\operatorname{ran}(n) = \{x \in \widehat{B} : nn^*(x) > 0\}$$

$\alpha_n : \operatorname{dom}(n) \xrightarrow[\text{homeo}]{} \operatorname{ran}(n)$

Put $G := \{[\alpha_n, x] : n \in N(B), x \in \operatorname{dom}(n)\}$

$(\alpha_n, x) \sim (\alpha_m, y) \Leftrightarrow x = y,\ \alpha_n|_V = \alpha_m|_V$ some nbd V_x.

▨

The. UCT

A sep C^*-alg A satisfies UCT if

$\forall B$ = separable,

$$0 \to \mathrm{Ext}_*(K_*A, K_{*-1}B) \to KK(A,B) \to \mathrm{Hom}(K_*A, K_*B) \to 0$$

exact.

Thm (Tu) $C^*_r(G)$ satisfies UCT for G = am. + 2^{nd} countable

Thm (Barlak, Li, 2016) $C^*_r(G, \Sigma)$ satisfies UCT for G = am. étale

Cor If A = sep. nuclear admitting Cartan $\Rightarrow A \in UCT$.

Now we know that

A = simple + sep. + unital, $\dim_{nuc} A < \infty$ admits Cartan $\Leftrightarrow A \in UCT$.

Open Problems

Hung-Chung Liao

(1) $\dim_{nuc} C_u^*(X) = ?$

for $X = \mathbb{Z}^2$?

for $X=\mathbb{Z}^2$, it is either 2 or 1, which one?
we know it is 1 for $X = \mathbb{Z}$.

(2) How to read $\dim_{asy} X$ from $C_u^*(X)$?

Def/ $B \leq A$ Cartan, $\dim_{nuc}(A,B) \leq n \overset{}{\underset{Def}{\Longleftrightarrow}} \forall \mathcal{F} \subset\subset A \; \forall \varepsilon > 0$

$\exists F =$ f.d. C^*-alg. $\exists$ c.p. maps

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\psi"'] & & A \\ & F \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$\varphi\psi(a) \underset{\varepsilon}{\approx} a \quad (a \in \mathcal{F})$

$F = F^{(i)} \oplus \cdots \oplus F^{(n)}$

$\varphi|_{F^{(i)}} = \text{cpc} \perp$ and

$\psi(B) \subseteq D_F$ = diagonal subalgebra of F

$\varphi(D_F) \subseteq B$

and

for any matrix unit $e \in F$ w.r.t. D_F, $\varphi(e) \in N(B)$

$$\begin{tikzcd} A \arrow[r, "\psi"] & F \arrow[r, "\varphi"] & A \\ B \arrow[u, hook] \arrow[r, "\psi"'] \arrow[ur, phantom, "\swarrow"] & D_F \arrow[u, hook] \arrow[r, "\varphi"'] \arrow[ur, phantom, ")"] & B \arrow[u, hook] \end{tikzcd}$$

<u>Thm</u> (Winter, Liao)

$\dim_{nuc}(C_u^*(X), \ell^\infty(X)) = \dim_{asy} X.$

(3) $VL_\infty(X) \cong VL_\infty(Y) \Rightarrow X \underset{CE}{\sim} Y$?

CE = coarse equivalent

$$\dim_{nuc} VL_\infty(X) = \dim_{nuc}(VL(X))$$
$$= \dim_{asy}(X) ?$$

We know this when $\dim_{asy}(X)$ is known to be finite.

(4) X = proper metric space (proper means closed + bdd → cpt)
~~with bdd geometry~~

Def/ $g \in C(X)$ is Higson (slowly oscillating) s.o. if $\forall R>0, \varepsilon>0 \ \exists K \subseteq X$ cpt.

$$|g(x) - g(y)| < \varepsilon$$
$$(x,y \in K^c, \ d(x,y) < R)$$

$$C_0(X) \le C_{s.o}(X) \le \ell^\infty(X) = C(\beta X)$$

C*-subalg.

$$C_{s.o}(X) = C(hX)$$

hX = a compactification of X

Write

$$\nu X := hX \setminus X \quad \text{(Higson Corona)}$$

Thm (Dranizhnikov. Keesling - Uspenskij) 1998

$$\dim_{top}(hX\setminus X) \le \dim_{asy}(X)$$

with equality when $\dim_{asy} X < \infty$ (Dranishnikov, 2000)

Q. (Špakula - Tikuisis) $\dim(hX\setminus X) = \dim_{nuc}(VL(X))$?

(5) Q. $\dim_{nuc} C(hX) \le \dim_{asy} X$?

Q

$$RR(C_u^*(\mathbb{Z})) = 0 ?$$

R. Willett ↓

We know: $\dim_{asy} X = 0 \Rightarrow rr(C_u^*(X)) = 0$, also $rr(C_u^*(\mathbb{Z}^2)) \neq 0$

(5) Continued

$$rr(C^*_u(\mathbb{Z})) \begin{cases} \neq 0 \rightsquigarrow s^{rr}(C^*_u(X)) = 0 \Leftrightarrow \dim_{asy} X = 0 \\ = 0 \rightsquigarrow A = C^*_u(\mathbb{Z}) \text{ is a } C^*\text{-alg.:} \end{cases}$$

Stably finite, rr0, but
no cancellation
(this was asked
by B. Blackadar).

(6)

$\mathcal{A}_\theta$ = irrational rotation algebra,

There is some action ~~Aff~~ $SL_2(\mathbb{Z}) \curvearrowright A_\theta$ s.t.

$$\underset{\text{finite}}{F \lneq} SL(2,\mathbb{Z}) \rightsquigarrow F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$$

Then $F \curvearrowright A_\theta$ has tracial Rokhlin property.

<u>Thm</u> (Echterhoff-Lück-Phillips-Walters) $A_\theta \rtimes F = AF$.

<u>Q</u> (in above paper) How to realize the inductive structure explicitly?

(7)

Study $C^*_u(X)$ for $X \neq$ property (A)?

We know $C^*_u(\Gamma)$ = nuclear $\Leftrightarrow$ $C^*_u(\Gamma)$ exact $\Leftrightarrow$ $C^*_r(G)$ exact $\Leftrightarrow$ G exact.

Sako, 2012:

$C^*_u(X)$ = nuclear $\Leftrightarrow$ $C^*_u(X)$ exact $\Leftrightarrow$ X = property (A).
$\Leftrightarrow$ $C^*_u(X)$ locally reflexive.