Function spaces on LCQGs

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(January, 2019)

Outline

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- Locally Compact Quantum Group

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A pair (M, Δ) is called a Hopf-von Neumann algebra if

- M is a von Neumann algebra,
- $\Delta : M \to M\overline{\otimes}M$ is a non-degenerate $*-$homomorphism with the property $(\Delta \otimes i) \circ \Delta = (i \otimes \Delta) \circ \Delta$.

Definition

- A weight on a von Neumann algebra M is the map $\varphi : M_+ \to [0, \infty]$ with the properties

$$\varphi(x + y) = \varphi(x) + \varphi(y), \quad \varphi(\lambda x) = \lambda\varphi(x), \quad (x, y \in M_+, \ \lambda \geq 0).$$

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- φ is said to be semifinite if $p_\varphi = \{x \in M_+ : \varphi(x) < +\infty\}$ is dense in M_+ with respect to the $wo-$topology (or equivalently, $\overline{n_\varphi}^{wo} = M$, where $n_\varphi = \{x \in M : \varphi(x^*x) < \infty\}$).

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- φ is said to be faithful if for each $x \in M_+$, $\varphi(x) \neq 0$.
- φ is said to be normal if for each increasing bounded net (x_i) in M_+

$$\varphi(\sup x_i) = \sup \varphi(x_i).$$

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- φ and ψ are left and right Haar weight, i.e. n.s.f weights with

$$\varphi((\omega \otimes i)\Delta(x)) = \omega(1)\varphi(x), \quad x \in m_\varphi, \omega \in M_*$$
$$\psi((i \otimes \omega)\Delta(x)) = \omega(1)\psi(x), \quad x \in m_\psi, \omega \in M_*.$$

Multipilicative unitary

Theorem

There exists a unitary $W \in B(H_\varphi \otimes H_\varphi)$ *such that*

$$W^*(\lambda_\varphi(a) \otimes \lambda_\varphi(b)) = (\lambda_\varphi \otimes \lambda_\varphi)(\Delta(b)(a \otimes 1)), \quad a, b \in n_\varphi$$

where λ_φ *is the GNS map induced by* φ *and* H_φ *is the Hilbert space that forms by GNS map. Moreover*

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Theorem

There exists a unitary $V \in B(H_\psi \otimes H_\psi)$ *such that*

$$V(\lambda_\psi(a) \otimes \lambda_\psi(b)) = (\lambda_\psi \otimes \lambda_\psi)(\Delta(a)(1 \otimes b)), \quad a, b \in n_\psi$$

where λ_ψ *is the GNS map induced by* ψ *and* H_ψ *is the Hilbert space that forms by GNS map. Moreover*

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

Example

$$\Delta_a : L^\infty(G) \to L^\infty(G)\overline{\otimes}L^\infty(G) = L^\infty(G \times G)$$
$$\Delta_a(f)(s,t) = f(st), \quad f \in L^\infty(G), s,t \in G.$$

Δ_a is a comultuplication on $L^\infty(G)$ and $\mathbb{G}_a = (L^\infty(G), \Delta_a, \varphi_a, \psi_a)$ is a locally compact quantum group where φ_a and ψ_a are the left and the right Haar measures on G, respectively.
Moreover, each commutative locally compact quantum group $\mathbb{G}$ (i.e. the underlying von Neumann algebra M is commutative) is of the form $\mathbb{G}_a$ for some locally compact group G.

Classic Case

For a locally compact group G, W and V are characterized by the following rules.

$$W : L^2(G) \otimes_2 L^2(G) = L^2(G \times G) \to L^2(G \times G)$$

$$W(\zeta)(r,s) = \zeta(r, r^{-1}s), \quad \zeta \in L^2(G \times G), r, s \in G$$

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$$V : L^2(G) \otimes_2 L^2(G) = L^2(G \times G) \to L^2(G \times G)$$
$$V(\zeta)(r,s) = \zeta(rs, s), \quad \zeta \in L^2(G \times G), r, s \in G.$$

Example

Let G be a locally compact group.

$$\widehat{\Delta_a} = \Delta_s : \mathsf{vN}(G) \to \mathsf{vN}(G)\overline{\otimes}\mathsf{vN}(G) = \mathsf{vN}(G \times G)$$
$$\Delta_s(\lambda_g) = \lambda_g \otimes \lambda_g.$$

Δ_s is a comultiplication on $\mathsf{vN}(G)$. There exists a n.s.f. weight φ_s on $\mathsf{vN}(G)$ which is right and left invariant. $(\mathsf{vN}(G), \Delta_s, \varphi_s)$ is a locally compact quantum group.
Moreover each cocommutative locally compact quantum group $\mathbb{G}$ (i.e. $\sigma \circ \Delta = \Delta$, in which σ is flip map) is of the form $\mathbb{G} = \mathbb{G}_s = (\mathsf{vN}(G), \Delta_s, \varphi_s)$ for some locally compact group G.

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- $M(\mathbb{G}) = C_0(\mathbb{G})^*$
- $M(C_0(\mathbb{G})) = C_b(\mathbb{G})$.

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$$luc(G) = \{f \in C_b(G) : x \in G \mapsto L_x f \in C_b(G) \text{ is } \|\cdot\| - \text{continuous}\}.$$

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where $L_f : L^1(G) \to L^\infty(G)$ is defined by $L_f(\mu) = f \star \mu$.

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- All above function spaces are operator system.

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- $luc(G) = L^\infty(G)$ if and only if G is discrete.

Recent Developments

- M. Daws (2010): $ap(M(G))$ and $wap(M(G))$ are C^*-algebras.

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- P. Salmi (2010): If V is regular then, $luc(L^1(\mathbb{G})) = \{x \in C_b(\mathbb{G}) : \Delta(x) \in M(C_0(\mathbb{G}) \otimes C_b(\mathbb{G}))\}$ and so $luc(L^1(\mathbb{G}))$ is a C^*-algebra.

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- Runde (2010): Let $\mathbb{G}$ be a coamenable locally compact quantum group. Then $luc(L^1(\mathbb{G})) = \{x : \Delta(x) \in QM(C_0(\mathbb{G}) \otimes C_b(\mathbb{G}))\}$.

Recent Developments

- Runde (2010): Let $\mathbb{G}$ be a coamenable locally compact quantum group such that $C_0(\mathbb{G})$ has a bounded approximate identity in it's center. Then $luc(L^1(\mathbb{G}))$ is a C^*-algebra.

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- Runde (2012): Runde introduced completely almost periodic functions on a Hopf-von Neumann algebra, denoted by $cap(M_*)$ and proved that for an injective Hopf-von Neumann algebra $cap(M_*) = \{x \in M : \Delta(x) \in M \otimes M\}$.

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- Runde (2012): If (M, Δ) is a subhomogeneous Hopf-von Neumann algebra, then $wap(M_*)$ is a C^*-algebra.

Recent Developments

- M. Daws (2016): $\{x \in wap(M_*) : x^*x, xx^* \in wap(M_*)\}$ is the biggest C^*-algebra in $wap(M_*)$.

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- H. R. Ebrahimi Vishki, M. Ramezanpour, M. Neufang, Z.-J. Ruan, Zh. Hu

Completely weakly almost periodic

Definition

- Let I be a cardinal. An element $x \in L^\infty(\mathbb{G})$ is called $(I,1)-$weakly almost periodic if $L_x^{(I,1)} : K_{I,1}(M_*) \to K_{I,1}(M)$ is weakly compact. $(1,I)-$weakly almost periodicity is defined simlarly. x is called completely weakly almost periodic if it is $(1,I)-$weakly almost periodic and $(I,1)-$weakly almost periodic for all cardinal I.

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- We set $cwap(M_*)$ to be the set of all completely almost periodic element of M.

Theorem

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Corollary

Let (M, Δ) *be a commutative Hopf-von Neumann algebra. Then* $cwap(M_*) = wap(M_*)$.

Definition

Closed subspace $X \subseteq C_b(\mathbb{G})$ is called

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Definition

- A right $L^1(\mathbb{G})$-module $X \subseteq L^\infty(\mathbb{G})$ is called left introverted if $\nu \star x \in X$ for every $\nu \in X^*$ and $x \in X$, where $\langle \nu \star x, \mu \rangle = \langle \nu, x \star \mu \rangle, \ (\mu \in L^1(\mathbb{G}))$.

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- In this case one can construct a multipilcation on X^* by $\langle \nu \star \nu', x \rangle = \langle \nu, \nu' \star x \rangle$.
- For a left $L^1(\mathbb{G})$-module $X \subseteq L^\infty(\mathbb{G})$ which is right introverted, one can construct multiplication $*$ on X^* by $\langle \nu * \nu', x \rangle = \langle \nu', x * \nu \rangle$.

Theorem

Let $\mathbb{G}$ be a coamenable locally compact quantum group. The following statements hold:

- $ap(L^1(\mathbb{G})) = \{x : \Delta(x) \in QM(C_0(\mathbb{G}) \otimes ap(L^1(\mathbb{G}))) \cap QM(ap(L^1(\mathbb{G})) \otimes C_0(\mathbb{G}))\}$

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- $wap(L^1(\mathbb{G})) = \{x : \Delta(x) \in QM(C_0(\mathbb{G}) \otimes wap(L^1(\mathbb{G}))) \cap QM(wap(L^1(\mathbb{G})) \otimes C_0(\mathbb{G}))\}$

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- $luc(L^1(\mathbb{G})) = \{x : \Delta(x) \in QM(C_0(\mathbb{G}) \otimes C_b(G))\}$.

Theorem

Let $\mathbb{G}$ be a coamenable locally compact quantum group such that either V is regular or $C_0(\mathbb{G})$ has a bounded approximate identity in it's center, then the following hold.

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- $ap(L^1(\mathbb{G})) = \{x : \Delta(x) \in M(C_0(\mathbb{G}) \otimes ap(L^1(\mathbb{G}))) \cap M(ap(L^1(\mathbb{G})) \otimes C_0(\mathbb{G}))\}$
- $wap(L^1(\mathbb{G})) = \{x : \Delta(x) \in M(C_0(\mathbb{G}) \otimes wap(L^1(\mathbb{G}))) \cap M(wap(L^1(\mathbb{G})) \otimes C_0(\mathbb{G}))\}$

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Amenablity of $wap(L^1(\mathbb{G}))$

Definition

We say $\mathbb{G}$ has property (WS) if for each idempotent states $\nu, \omega \in wap(L^1(\mathbb{G}))^*$ the equation $\nu \star \omega = \omega$ implies $\omega \star \nu = \omega$.

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Theorem

Let $\mathbb{G}$ has property (WS) then $wap(L^1(\mathbb{G}))$ is amenable.

Theorem

Let $\mathbb{G}$ be a coamenable locally compact quantum group such that there exists $a \in Z(C_0(\mathbb{G}))$ with $\epsilon(a) \neq 0$. Then $luc(L^1(\mathbb{G}))$ is a C^*-algebra.

Theorem

Let $\mathbb{G}$ be a locally compact quantum group and $L^1(\mathbb{G})$ be strongly Arens irregular. The following statements hold:

1. $\mathbb{G}$ *is compact if and only if* $wap(L^1(\mathbb{G})) = luc(L^1(\mathbb{G}))$.

Theorem

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1. *$\mathbb{G}$ is compact if and only if $wap(L^1(\mathbb{G})) = luc(L^1(\mathbb{G}))$.*
2. *$\mathbb{G}$ is discrete if and only if $luc(L^1(\mathbb{G})) = L^\infty(\mathbb{G})$.*

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3. *$\mathbb{G}$ is finite if and only if $wap(L^1(\mathbb{G})) = L^\infty(\mathbb{G})$.*

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$L^1(\mathbb{G})$ is weakly Arens irregular if $Z(L^1(\mathbb{G})^{**}) \subseteq L^1(\mathbb{G})^{**} \cdot C_0(\mathbb{G})$.

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Theorem

If $luc(L^1(\mathbb{G})) \subseteq wap(L^1(\mathbb{G}))$, $L^1(\mathbb{G})$ is weakly Arens irregular and $wap(L^1(\mathbb{G}))$ is amenable, then $\mathbb{G}$ is compact.

ThAnk yOu for youR atTenTiOn

THANK YOU FOR YOUR ATTENTION

A Kawada-Itô theorem for locally compact quantum groups

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The 6ht Workshop on Operator Algebras and their Applications
January 6, 2019

A brief history

Let $M(G) \cong C_0(G)^*$ be the space of complex Radon measures on a locally compact group G. We define the convolution of two measures $\mu, \nu \in M(G)$ as follows:

$$\mu * \nu(f) = \int_G f(xy)d\mu(x)d\nu(y)$$

A state $\mu \in C_0(G)^*$ is called idempotent state if $\mu * \mu = \mu$.

Example

Let G be a compact group. Then the Haar state of G is an idempotent state.

Kawada-Itô theorem

Theorem (Kawada-Itô, 1940)

Let G be a compact group. Then every idempotent state $\omega \in M(G)$ arises as a Haar state of a closed subgroup.

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Theorem (Generalized Kawada-Itô theorem)

Let G be a locally compact group. Then every idempotent state $\omega \in M(G)$ arises as a Haar state of a compact subgroup.

Locally compact quantum group

The theory of locally compact quantum group in a language of Operator Algebra have been successfully introduced and studied by J. Kustermans and S.Vaes in 2000. There are three different approaches to the theory of locally compact quantum groups:

1. von Neumann algebraic approach,
2. reduced C^*-algebraic approach,
3. universal C^*-algebraic approach.

But they are equivalent in the sense that they study a same object which we will denote it by $\mathbb{G}$. There are also a standard procedures to pass from one to the other.

Locally compact quantum groups - Von Neumann algebraic version

A von Neumann algebraic locally compact quantum group is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta_\mathbb{G}, \varphi_\mathbb{G}, \psi_\mathbb{G})$, where

- $L^\infty(\mathbb{G})$ is a von Neumann algebra,
- $\Delta_\mathbb{G} \colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$, is an injective, unital, $*$-homomorphism which satisfies the coassociativity condition, i.e. $(1 \otimes \Delta_\mathbb{G})\Delta_\mathbb{G} = (\Delta_\mathbb{G} \otimes 1)\Delta_\mathbb{G}$
- $\varphi_\mathbb{G}$ and $\psi_\mathbb{G}$ are normal semifinite faithful weights on $L^\infty(\mathbb{G})$ such that:

$$(\mathrm{id} \otimes \varphi_\mathbb{G})\Delta_\mathbb{G} = 1\varphi_\mathbb{G}$$
$$(\psi_\mathbb{G} \otimes \mathrm{id})\Delta_\mathbb{G} = 1\psi_\mathbb{G}$$

$\varphi_\mathbb{G}$ and $\psi_\mathbb{G}$ are called left and right Haar weights respectively.

Locally compact quantum groups- Reduced C^*-algebraic version

A reduced C^*-algebraic locally compact quantum group is a quadruple $\mathbb{G} = (\mathrm{C}_0(\mathbb{G}), \Delta_{\mathbb{G}}, \varphi_{\mathbb{G}}, \psi_{\mathbb{G}})$, where $\mathrm{C}_0(\mathbb{G})$ is a C^*-algebra with a coassociative map

$$\Delta_{\mathbb{G}} : \mathrm{C}_0(\mathbb{G}) \to \mathrm{M}(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G})),$$

such that

$$\begin{aligned} \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G}) &= (\mathbb{1} \otimes \mathrm{C}_0(\mathbb{G}))\Delta_{\mathbb{G}}(\mathrm{C}_0(\mathbb{G})) \\ &= (\mathrm{C}_0(\mathbb{G}) \otimes \mathbb{1})\Delta_{\mathbb{G}}(\mathrm{C}_0(\mathbb{G})), \end{aligned}$$

and $\varphi_{\mathbb{G}}$ and $\psi_{\mathbb{G}}$ are left and right invariant faithful, proper, KMS-weights on $\mathrm{C}_0(\mathbb{G})$ respectively.

Locally compact quantum groups - Universal C*-algebraic version

The universal version $C_0^u(\mathbb{G})$ of $\mathbb{G}$ is equipped with a comultiplication such that

$$\begin{aligned} C_0^u(\mathbb{G}) \otimes C_0^u(\mathbb{G}) &= (\mathbb{1} \otimes C_0^u(\mathbb{G}))\Delta_{\mathbb{G}}(C_0^u(\mathbb{G})) \\ &= (C_0^u(\mathbb{G}) \otimes \mathbb{1})\Delta_{\mathbb{G}}^u(C_0(\mathbb{G})), \end{aligned}$$

Left and right Haar weights on are not faithful. But admits a $*$-homomorphism $\epsilon :\to \mathbb{C}$ such that

$$(\epsilon \otimes \mathrm{id})\Delta_{\mathbb{G}}^u = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta_{\mathbb{G}}^u$$

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The comultiplication $\Delta_{\mathbb{G}}^u$ yields an algebra structure on $C_0^u(\mathbb{G})^*$ in fact for $\mu, \nu \in C_0^u(\mathbb{G})^*$,

$$\mu * \nu := (\mu \otimes \nu)\Delta_{\mathbb{G}}^u$$

Theorem

A locally compact quantum group $\mathbb{G}$ is a compact quantum group if one of the following equivalent conditions is satisfied:

1. *the Haar weights are finite,*
2. $C_0(\mathbb{G})$ *is unital,*
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Example

The Haar state of a compact quantum group $\mathbb{G}$ is an idempotent state.

Definition

Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact quantum groups. Then $\mathbb{H}$ is said to be a closed quantum subgroup of $\mathbb{G}$ in the sense of Woronowicz if there exists a surjective $*$-homomorphism $\pi : C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H})$ which commutes the comultiplications, i.e.

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Can we characterize idempotent states on locally compact quantum groups with Haar states of compact quantum subgroups?

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Can we characterize idempotent states on locally compact quantum groups with Haar states of compact quantum subgroups? No

Counterexample

Example (Pal, 1996)

Let $\mathbb{G} = (A, \Delta)$ be the 8-dimensional Kac-Paljutkin finite quantum group, where $A = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$.

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$$e_k = \delta_{1k} \oplus \delta_{2k} \oplus \delta_{3k} \oplus \delta_{4k} \oplus \begin{bmatrix} \delta_{5k} & \delta_{8k} \\ \delta_{7k} & \delta_{6k} \end{bmatrix} \quad , \quad k = 1, \cdots, 8,$$

where δ denotes the Kronecker delta. Then $\{e_1, e_2, \cdots, e_8\}$ form a basis for A.

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$$\begin{aligned} J_\omega &= \{x \in A \,|\, \omega(x^*x) = 0\} = \langle e_2, e_3, e_5, e_7 \rangle, \\ J_\omega^* &= \{x \in A \,|\, \omega(xx^*) = 0\} = \langle e_2, e_3, e_5, e_8 \rangle. \end{aligned}$$

Haar idempotents

Definition

Let ω be an idempotent state on $C_0^u(\mathbb{G})$. The ω is called Haar idempotent if there exists a compact quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ with an associated map $\pi : C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H})$, such that $\omega = \boldsymbol{h}_{\mathbb{H}} \circ \pi$.

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Theorem (Salmi-Skalski, 2012)

Let ω be an idempotent state on $C_0^u(\mathbb{G})$. Then the following are equivalent:

- *ω is Haar idempotent,*
- *The set $J_\omega = \{x \in C_0^u(\mathbb{G}) \,|\, \omega(x^*x) = 0\}$ is an ideal.*

Compact quantum hypergroup

Definition (Chapovsky-Vainerman, 1999)

A quadruple (A, Δ, ϵ, R) is a hypergroup structure on a unital C^*-algebra A if

1. $\Delta : A \to A \otimes_{\min} A$ is a unital, $*$-preserving, positive, coassociative map,
2. $\epsilon : A \to \mathbb{C}$ is a linear homomorphism,

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id},$$

3. $R : A \to A$ is an anti-linear $*$-automorphism with $R^2 = \mathrm{id}$,

such that

$$\Delta \circ R = \boldsymbol{\sigma} \circ (R \otimes R) \circ \Delta.$$

The $*$-algebraic structure on the dual space of the C^*-algebra A of a hypergroup structure is given by

$$\xi \cdot \eta(a) := (\xi \otimes \eta) \circ \Delta(a)$$
$$\xi^\sharp(a) := \overline{\xi(R(a))}.$$

and $(A^*, \cdot, \sharp)$ is a Banach $*$-algebra.

For a hypergroup structure (A, Δ, ϵ, R), a state $\varphi \in A^*$ is called a *Haar state* if

$$(\varphi \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varphi) \circ \Delta = 1_A \varphi.$$

An element $a \in A$ is called positive definite, if for all $\xi \in A^*$,

$$\xi \cdot \xi^\sharp(a) \geq 0.$$

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$$\xi \cdot \xi^\sharp(a) \geq 0.$$

Theorem (Chapovsky-Vainerman, 1999)

Let (A, Δ, ϵ, R) be a hypergroup structure on a C^*-algebra A. Suppose that the linear space spanned by the positive definite elements is dense in A. Then there exists a unique Haar state ν on A.

Compact quantum hypergroup

Definition (Chapovsky-Vainerman, 1999)

Let (A, Δ, ϵ, R) be a hypergroup structure. Then $\mathcal{G} = (A, \Delta, \epsilon, R, \tau_t)$ is called a compact quantum hypergroup if

1. Δ is a completely positive map and the linear span of positive definite elements is dense in A,
2. $(\tau_t)_{t\in\mathbb{R}}$ is a continuous one-parameter group of automorphisms of A such that:
   - there exist dense $*$-subalgebras $A_0 \subset A$ and $\tilde{A}_0 \subset A \otimes A$ such that the one-parameter groups $(\tau_t)_{t\in\mathbb{R}}$, $(\tau_t \otimes \mathrm{id})_{t\in\mathbb{R}}$ and $(\mathrm{id} \otimes \tau_t)_{t\in\mathbb{R}}$ can be extended to complex one-parameter groups $(\tau_z)_{z\in\mathbb{C}}$, $(\tau_z \otimes \mathrm{id})_{z\in\mathbb{C}}$ and $(\mathrm{id} \otimes \tau_z)_{z\in\mathbb{C}}$ of automorphisms of the algebras A_0 and $\tilde{A}_0$, respectively;

Definition-continued

2
- $R(A_0) \subset A_0$ and $\Delta(A_0) \subset \tilde{A}_0$;
- the following relations hold on A_0 for all $z \in \mathbb{C}$,

$$\Delta \circ \tau_z = (\tau_z \otimes \tau_z) \circ \Delta,$$
$$\boldsymbol{h}\tau_z = \boldsymbol{h};$$

- there exists $z_0 \in \mathbb{C}$ such that the Haar measure $\boldsymbol{h}$ satisfies the following strong invariance condition, for all $a, b \in A_0$,

$$(\mathrm{id} \otimes \boldsymbol{h})\big((1 \otimes a)\Delta(b)\big) = (\mathrm{id} \otimes \boldsymbol{h})\Big(\big((\tau_{z_0} \circ R \otimes \mathrm{id}) \circ \Delta(a)\big)(1 \otimes b)\Big);$$

3 the Haar measure $\boldsymbol{h}$ is faithful on A_0.

Definition

Let $\mathbb{G}$ be a locally compact quantum group. A compact quantum hypergroup $\mathcal{G} = (A, \Delta, \epsilon, R, \tau_t)$ is called a compact quantum subhypergroup of $\mathbb{G}$, if there exists a surjective, completely positive map $\pi : C_0^u(\mathbb{G}) \to A$ which commutes the coproduct.

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Example

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Conjecture (Franz-Skalski, 2009)

All idempotent states on locally compact quantum group $\mathbb{G}$ arise (in a canonical way) as Haar states on compact quantum subhypergroups.

Definition

Let $\mathbb{G}$ be a locally compact quantum group. A compact quantum hypergroup $\mathcal{G} = (A, \Delta, \epsilon, R, \tau_t)$ is called a compact quantum subhypergroup of $\mathbb{G}$, if there exists a surjective, completely positive map $\pi : \mathrm{C}_0^u(\mathbb{G}) \to A$ which commutes the coproduct.

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Theorem (Franz-Skalski, 2009)

Let A be a finite quantum group and let $\omega \in A^*$, be an idempotent state. Then ω arises as a Haar state of a finite quantum subhypergroup of A (in a canonical way).

Operator system

Definition

An operator system is a (norm-closed) unital subspace S of a unital C^*-algebra A which is self-adjoint, that is, $x^* \in S$ if and only if $x \in S$.

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An operator system is a (norm-closed) unital subspace S of a unital C^*-algebra A which is self-adjoint, that is, $x^* \in S$ if and only if $x \in S$.

A state on an operator system S is a completely positive linear map $s : S \to \mathbb{C}$, such that $s(1) = 1$.
Given two operator systems $S \subset A$ and $S' \subset A'$, one can define their minimal tensor product as the completion of the algebraic tensor product $S \otimes S' \subset A \otimes A'$ under the following norm

$$\|t\|_{\mathrm{min}} := \|t\|_{\mathrm{C}^*-\mathrm{min}}, \qquad t \in S \otimes S'.$$

Open projections in C*-algebras

Definition

A projection $p \in A^{**}$ is called open if there exists a net $\{a_\alpha\} \subset A$ such that $0 \leq a_\alpha \uparrow p$ in the weak*-toplology of A^{**}.

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A projection $p \in A^{**}$ is said to be closed if its complement $1 - p \in A^{**}$ is an open projection.

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Definition

A projection $p \in A^{**}$ is called open if there exists a net $\{a_\alpha\} \subset A$ such that $0 \leq a_\alpha \uparrow p$ in the weak*-toplology of A^{**}.

A projection $p \in A^{**}$ is said to be closed if its complement $1 - p \in A^{**}$ is an open projection.

Theorem (Akemann-Pedersen-Tomiyama, 1973)

Let A be a C*-algebra and p a closed projection in A^{**}. Let $J = \{a \in A \,|\, ap = 0\}$ and define $\Phi : A^{**} \to A^{**}$ by $\Phi(a) = pap$. Then $A \cap \ker \Phi = J + J^*$ and pAp is isometrically isomorphic to the quotient space $A/J + J^*$.

Compact quantum hypersystem

Definition (Amini-Kh.)

A triple (S, Δ_S, R_S) is called a hypersystem structure if

1. S is an operator system;
2. $\Delta : S \to S \otimes_{\min} S$ is a linear, unital, completely positive map which is co-associative, i.e., $(\mathrm{id} \otimes \Delta_S) \circ \Delta_S = (\Delta_S \otimes \mathrm{id}) \circ \Delta_S$;
3. $R_S : S \to S$ is a unital, anti-linear, completely positive map such that $\boldsymbol{\sigma}(R_S \otimes R_S) \circ \Delta_S = \Delta_S \circ R_S$ and $R_S^2 = \mathrm{id}$.

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Let S^* denotes the set of all (completely) bounded linear functionals on S, Then S^* is a Banach $*$-algebra.

Theorem (Amini-Kh.)

Let (S, Δ_S, R_S) be a hypersystem structure. If the linear space spanned by the positive definite elements is dense in S, then there exists a self-adjoint functional on S, which is both left and right invariant.

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Definition (Amini-Kh.)

A hypersystem structure (A, Δ, R) is called a compact quantum hypersystem if it admits a faithful Haar state.

Definition (Amini-Kh.)

Let $\mathbb{G}$ be a locally compact quantum group and $(S, \Delta_S, R_S, \boldsymbol{h}_S)$ be a compact quantum hypersystem. Then $(S, \Delta_S, R_S, \boldsymbol{h}_S)$ is called a compact quantum subhypersystem of $\mathbb{G}$, if there exists a surjective completely positive map $\pi_S : \mathrm{C}_0^u(\mathbb{G}) \to S$ such that

$$\Delta_S \circ \pi_S = (\pi_S \otimes \pi_S) \circ \Delta_{\mathbb{G}}.$$

Example

Let $(S, \Delta_S, R_S, \boldsymbol{h}_S)$ be a compact quantum subhypersystem of $\mathbb{G}$, with the associated map π_S, then $\boldsymbol{h}_S \circ \pi_S$ is an idempotent state on $\mathbb{G}$.

Definition (Amini-Kh.)

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Question.

Does every idempotent state on $\mathbb{G}$ arise as a Haar idemptent state on a compact quantum subhypersystem (in a canonical way)?

Theorem (Amini-Kh.)

Let ω be an idempotent state on $C_0^u(\mathbb{G})$. Then there exists a compact quantum subhypersystem $(S, \Delta_S, R_S, \boldsymbol{h}_S)$ such that $\omega = \boldsymbol{h}_S \circ \pi_S$.

Theorem (Amini-Kh.)

Let ω be an idempotent state on $C_0^u(\mathbb{G})$. Then there exists a compact quantum subhypersystem $(S, \Delta_S, R_S, \boldsymbol{h}_S)$ such that $\omega = \boldsymbol{h}_S \circ \pi_S$. Moreover let $(S', \Delta_{S'}, R_{S'}, \boldsymbol{h}_{S'})$ be another compact quantum subhypersystem of $\mathbb{G}$ such that $\omega = \boldsymbol{h}_{S'} \circ \pi_{S'}$, then there exists a unique map

$$\pi : S \to S',$$

such that $\pi_{S'} = \pi \circ \pi_S$.

Sketch of the proof

Consider the left ideal $J_\omega = \{x \in C_0^u(\mathbb{G}) \,|\, \omega(x^*x) = 0\}$,

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$$J_\omega = C_0^u(\mathbb{G})^{**}p \cap C_0^u(\mathbb{G}) = C_0^u(\mathbb{G})p \cap C_0^u(\mathbb{G}).$$

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- $q = 1 - p$ is a closed projection in $\mathrm{C}_0^u(\mathbb{G})^{**}$ and

$$\frac{\mathrm{C}_0^u(\mathbb{G})}{J_\omega + J_\omega^*} \cong q\,\mathrm{C}_0^u(\mathbb{G})q.$$

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- $q \in q\, C_0^u(\mathbb{G})q$ and $q\, C_0^u(\mathbb{G})q$ is an operator system.

Sketch of the proof

- The map

$$\begin{aligned}\Delta_\omega : q\, C_0^u(\mathbb{G})q &\to q\, C_0^u(\mathbb{G})q \otimes_{\min} q\, C_0^u(\mathbb{G})q,\\ qaq &\mapsto (q \otimes q)\Delta_{\mathbb{G}}^u(a)(q \otimes q),\end{aligned}$$

  is a well-defined, unital, coassociative, completely positive map.

Sketch of the proof

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$$qaq \mapsto (q \otimes q)\Delta_{\mathbb{G}}^u(a)(q \otimes q),$$

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- The map

$$R_\omega : q\,\mathrm{C}_0^u(\mathbb{G})q \to q\,\mathrm{C}_0^u(\mathbb{G})q,$$
$$qaq \mapsto qR_{\mathbb{G}}^u(a)q,$$

  is a well-defined, unital, anti-linear, completely positive map such that $\boldsymbol{\sigma}(R_\omega \otimes R_\omega) \circ \Delta_\omega = \Delta_\omega \circ R_\omega$ and $R_\omega^2 = \mathrm{id}$.

Sketch of the proof

- The following map is a faithfull Haar state on $q\,C_0^u(\mathbb{G})q$,

$$\begin{aligned} \boldsymbol{h}_\omega &\to \mathbb{C}, \\ qaq &\mapsto \omega(a). \end{aligned}$$

Sketch of the proof

- The following map is a faithfull Haar state on $q\,\mathrm{C}_0^u(\mathbb{G})q$,

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- ω arises as a Haar state of $q\,\mathrm{C}_0^u(\mathbb{G})q$ in a canonical way.

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- ω arises as a Haar state of $q\,\mathrm{C}_0^u(\mathbb{G})q$ in a canonical way.

Corollary

Let ω be an idempotent state on a locally compact quantum group. Then the following conditions are equivalent:

- J_ω is an ideal,
- q is a central closed projection in $\mathrm{C}_0^u(\mathbb{G})^{**}$,
- ω arises as the Haar state of the compact quantum group $(\mathrm{C}_0^u(\mathbb{G})q, \Delta_\omega)$

Example

Let $\mathbb{G} = (A, \Delta)$ be the 8-dimentional Kac-Paljutkin finite quantum group, where $A = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$. Let

$$e_k = \delta_{1k} \oplus \delta_{2k} \oplus \delta_{3k} \oplus \delta_{4k} \oplus \begin{bmatrix} \delta_{5k} & \delta_{8k} \\ \delta_{7k} & \delta_{6k} \end{bmatrix} \quad , \quad k = 1, \cdots, 8.$$

Then the idempotent state $\omega = \frac{1}{4}(\rho_1 + \rho_4) + \frac{1}{2}\rho_6$ is not the haar state of any quantum subgroup $\mathbb{G}$. The null space of ω is not an ideal ($*$-preserving).

$$\begin{aligned} I &= \{x \in A \,|\, \omega(x^*x) = 0\} = \langle e_2, e_3, e_5, e_7 \rangle, \\ I^* &= \{x \in A \,|\, \omega(xx^*) = 0\} = \langle e_2, e_3, e_5, e_8 \rangle. \end{aligned}$$

Example

Define the projection $q = e_1 \oplus e_4 \oplus e_6 \in A$. It can be observed that as Banach spaces

$$\frac{A}{I + I^*} \cong qAq.$$

In this case qAq is a unital C^*-algebra. Since q is not central, the co-multiplication

$$\tilde{\Delta} : qAq \to qAq \otimes_{\min} qAq,$$
$$\tilde{\Delta}(qaq) = (q \otimes q)\Delta(a)(q \otimes q)$$

is not a homomorphism but it is completely positive. Now q is the support projection of ω and ω arises as the Haar state of qAq in a canonical way.

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thank you

Amenable Actions of Discrete Quantum Groups on von Neumann Algebras

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January 7, 2019

Motivation

The concept of amenability for a locally compact group G can be described in many different equivalent ways. Two of the most well-known characterizations are the following:

- there is a left invariant mean on $L^\infty(G)$;
- any affine G-action has a fixed point.

Motivation

The concept of amenability for a locally compact group G can be described in many different equivalent ways. Two of the most well-known characterizations are the following:

- there is a left invariant mean on $L^\infty(G)$;
- any affine G-action has a fixed point.

In 1978, **Zimmer** introduced the notion of amenable actions as a natural generalization of fixed point property.

He gave a criterion that an ergodic action be amenable in terms of the von Neumann algebra associated to it by the Murray–von Neumann construction.

Motivation

Delaroche transported the notion of amenable action into operator algebra terms:

- von Neumann algebra setting, 1979 and 1982;
- C*-algebra setting, 1987.

Definition (Delaroche, 1979)

The action $\alpha : G \curvearrowright N$ of a locally compact group G on a von Neumann algebra N is called amenable if there exists an equivariant conditional expectation

$$P : (\ell^\infty(G) \overline{\otimes} M, \tau \otimes \alpha) \to (\mathbf{1} \overline{\otimes} M, \alpha),$$

where τ denotes the left translation action of G on $\ell^\infty(G)$.

Delaroche extended Zimmer's criterion to the general setting:

Theorem (Delaroche, 1979)

Let $\alpha : G \curvearrowright N$ be an action of a locally compact group G on a von Neumann algebra N. TFAE:

- α is amenable;
- there is a conditional expectation from $B(L_2(G)) \overline{\otimes} N$ onto the crossed product $G \ltimes_\alpha N$.

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- α is amenable;
- there is a conditional expectation from $B(L_2(G)) \overline{\otimes} N$ onto the crossed product $G \ltimes_\alpha N$.

Using the automorphism

$$T_\alpha : \sum_{g \in G} (\delta_g \otimes x_g) \in \ell^\infty(G) \overline{\otimes} M \mapsto \sum_{g \in G} (\delta_g \otimes \alpha_g^{-1}(x_g)) \in \ell^\infty(G) \overline{\otimes} M$$

to get the following equivalent definition of amenable actions:
The action $\alpha : G \curvearrowright N$ is called amenable if there exists an equivariant conditional expectation

$$P : (\ell^\infty(G) \overline{\otimes} M, \tau \otimes \mathrm{id}) \to (\alpha(M), \tau \otimes \mathrm{id}),$$

Definition (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N. Then α is called amenable if there exists a conditional expectation $E_\alpha : \ell^\infty(\mathbb{G}) \overline{\otimes} N \to \alpha(N)$ such that

$$(\mathrm{id} \otimes E_\alpha)(\Delta \otimes \mathrm{id}) = (\Delta \otimes \mathrm{id})E_\alpha.$$

Facts:

- Amenability of trivial action $tr : \mathbb{G} \curvearrowright \mathbb{C}$ is equivalent to amenability of quantum group $\mathbb{G}$;
- every discrete quantum group acts amenably on itself.

von Neumann algebra Braided Tensor Products

For any discrete quantum group $\mathbb{G}$, there is an action $\gamma : \hat{\mathbb{G}} \curvearrowright \ell^\infty(\mathbb{G})$ of dual quantum group $\hat{\mathbb{G}}$ on $\ell^\infty(\mathbb{G})$ given by

$$\gamma(x) = \hat{W}^*(1 \otimes x)\hat{W}, \quad x \in \ell^\infty(\mathbb{G}),$$

where $\hat{W}$ is the left fundamental unitary of $\hat{\mathbb{G}}$.

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where $\hat{W}$ is the left fundamental unitary of $\hat{\mathbb{G}}$. Observe that

$$(\mathrm{ad}(W) \otimes \mathrm{id})(\mathrm{id} \otimes \gamma)\Delta(x) = (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\gamma(x), \; x \in \ell^\infty(\mathbb{G}),$$

where W is the left fundamental unitary of $\mathbb{G}$ and σ is the flip map.

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where W is the left fundamental unitary of $\mathbb{G}$ and σ is the flip map.
It imlies that

$$\overline{\mathrm{span}\{\gamma(\ell^\infty(\mathbb{G}))_{12}\alpha(N)_{13}\}}^{\text{weak*}} = \overline{\mathrm{span}\{\alpha(N)_{13}\gamma(\ell^\infty(\mathbb{G}))_{12}\}}^{\text{weak*}},$$

and it is a von Neumann sub-algebra of $B(\ell^2(\mathbb{G}))\,\overline{\otimes}\,\ell^\infty(\mathbb{G})\,\overline{\otimes}\,N$, which is called (von Neumann algebra) braided tensor product and is denoted by $\ell^\infty(\mathbb{G})\,\overline{\boxtimes}\,N$.

von Neumann algebra Braided Tensor Products

There is an action $\Delta \boxtimes \alpha : \mathbb{G} \curvearrowright \ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ of $\mathbb{G}$ on the von Neumann algebra $\ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ such that

$$(\Delta \boxtimes \alpha)(\gamma(a)_{12}\alpha(b)_{13}) = \big((\mathrm{id} \otimes \gamma)\Delta(a)\big)_{123}\big((\mathrm{id} \otimes \alpha)\alpha(b)\big)_{124}.$$

von Neumann algebra Braided Tensor Products

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$$(\Delta \boxtimes \alpha)(\gamma(a)_{12}\alpha(b)_{13}) = ((\mathrm{id} \otimes \gamma)\Delta(a))_{123}((\mathrm{id} \otimes \alpha)\alpha(b))_{124}.$$

Lemma (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N. There exists an equivarinat $*$-isomorphism

$$T_\alpha : (\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \Delta \boxtimes \alpha) \to (\ell^\infty(\mathbb{G}) \overline{\otimes} N, \Delta \otimes \mathrm{id})$$

such that $T_\alpha(1 \boxtimes a) = \alpha(a)$ for all $a \in N$ and $T_\alpha(x \boxtimes 1) = x \otimes 1$ for all $x \in \ell^\infty(\mathbb{G})$.

Theorem (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N. Then there is an equivariant isomorphism Φ from $((\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}, \widehat{\Delta \boxtimes \alpha})$ onto $(B(\ell^2(\mathbb{G})) \overline{\otimes} N, \widehat{\Delta \boxtimes \alpha})$ such that Φ maps $(\mathbf{1} \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}$ onto $N \ltimes_\alpha \mathbb{G}$.

A Characterization of Amenability

Theorem (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N. TFAE:

- The quantum group $\mathbb{G}$ is amenable.
- The action α is amenable and there exists an invariant state on N.

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Izumi Proved that the tracial Haar state $\hat{\varphi}$ of the dual quantum group $\hat{\mathbb{G}}$ is invariant with respect to the canonical action $\Delta^{\ell}_{|_{L^\infty(\hat{\mathbb{G}})}}$. It follows that:

Corollary (M., 2018)

Let $\mathbb{G}$ be a discrete Kac algebra. Then $\mathbb{G}$ is amenable if and only if the canonical action $\Delta^{\ell}_{|_{L^\infty(\hat{\mathbb{G}})}} : \mathbb{G} \curvearrowright L^\infty(\hat{\mathbb{G}})$ is amenable.

Non-commutative Poisson Boundaries

Let $\mathbb{G}$ be a discrete quantum group and $\mu \in \ell^1(\mathbb{G})$ be a state. In this case $\Phi_\mu(x) = (\mu \otimes \mathrm{id})\Delta(x)$ is a Markov operator, i.e. unital, normal and completely positive map, on $\ell^\infty(\mathbb{G})$. We can consider

$$\mathcal{H}_\mu = \{x \in \ell^\infty(\mathbb{G}) : \Phi_\mu(x) = x\}$$

of all μ-harmonic operators.

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of all μ-harmonic operators. There is a conditional expectation from $\ell^\infty(\mathbb{G})$ onto $\mathcal{H}_\mu$. Then the corresponding Choi–Effros product induces the von Neumann algebraic structure on $\mathcal{H}_\mu$. This von Neumann algebra is called non-commutative Poisson boundary with respect to μ.
Kalantar, Neufang and **Ruan** proved that the restriction of Δ to $\mathcal{H}_\mu$ induces a left action Δ_μ of $\mathbb{G}$ on the von Neumann algebra $\mathcal{H}_\mu$.

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Theorem (M., 2018)

Let $\mathbb{G}$ be a discrete quantum group and let $\mu \in \ell^1(\mathbb{G})$ be a state. The left action Δ_μ of $\mathbb{G}$ on the Poisson boundary $\mathcal{H}_\mu$ is amenable.

Non-commutative Analogue of Zimmer's criterion

Theorem (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N. TFAE:

1. The action α is amenable.
2. There is an equivariant conditional expectation

$$E : ((\ell^\infty(\mathbb{G}) \,\overline{\boxtimes}\, N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}, \widehat{\Delta \boxtimes \alpha}) \to ((\mathbf{1} \,\overline{\boxtimes}\, N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}, \widehat{\Delta \boxtimes \alpha}).$$

3. There is an equivariant conditional expectation

$$E : (B(\ell^2(\mathbb{G})) \,\overline{\otimes}\, N, \hat{\Delta}^{\mathrm{op}} \otimes \mathrm{id}) \to (N \ltimes_\alpha \mathbb{G}, \hat{\alpha}).$$

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In the case of the trivial action, the equivalence between (1) and (3) provides the characterization of amenability of a quantum group $\mathbb{G}$ in terms of injectivity of $L^\infty(\hat{\mathbb{G}})$ in the category of $\mathcal{T}(\ell^2(\mathbb{G}))$-modules investigated by **Crann** and **Neufang**.

Let $\beta : \mathbb{G} \curvearrowright K$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra K. We say that K is $\mathbb{G}$-injective if for every unital completely isometric equivariant map $\iota : (M, \alpha_1) \to (N, \alpha_2)$ and every unital completely positive equivariant map $\Psi : (M, \alpha_1) \to (K, \beta)$ there is a unital completely positive equivariant map $\overline{\Psi} : (N, \alpha_2) \to (K, \beta)$ such that $\overline{\Psi} \circ \iota = \Psi$.

Corollary (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N. TFAE:

1. N is injective and α is amenable.
2. The crossed product $N \ltimes_\alpha \mathbb{G}$ is $\hat{\mathbb{G}}$-injective.

Discrete Kac Algebra Actions

Theorem (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete Kac algebra $\mathbb{G}$ on a von Neumann algebra N. The following are equivalent:

1. The action α is amenable.
2. There is a conditional expectation from $B(\ell^2(\mathbb{G})) \overline{\otimes} N$ onto $N \ltimes_\alpha \mathbb{G}$.

Corollary (M., 2018)

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete Kac algebra $\mathbb{G}$ on a von Neumann algebra N. TFAE:

1. N is injective and α is amenable.
2. The crossed product $N \ltimes_\alpha \mathbb{G}$ is injective.

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Thank you for your attention!