Auslander-Reiten principle in derived categories.

Yuji Yoshino

Let R be a Cohen-Macaulay local ring of Krull dimension d with canonical module ω and let M, N be maximal Cohen-Macaulay modules over R. If R has only an isolated singularity, then we have an isomorphism

$$\underline{\mathrm{Hom}}_R(M, N)^\vee \cong \mathrm{Ext}^1_R(N, \tau M), \tag{1}$$

where $\tau M = \mathrm{Hom}_R(\Omega^d(\mathrm{Tr}M), \omega)$. This isomorphism is known as Auslander-Reiten duality, or simply AR duality, which plays a crucial role in the theory of maximal Cohen-Macaulay modules. In fact, one can derive from it the existence of Auslander-Reiten sequence in the category of maximal Cohen-Macaulay modules. Further assuming that R is Gorenstein, it assures us that the stable category of the category of maximal Cohen-Macaulay modules has $(d-1)$-Calabi-Yau property.

We propose a general principle behind the AR duality, which is a general theorem for chain complexes of modules in a kind of general form that encompasses the AR duality and its generalization.

Theorem[AR Principle] *Let R be a commutative Noetherian ring and let W be specialization-closed subset of* $\mathrm{Spec}(R)$*. Given a bounded complex I of injective R-modules with $I^i = 0$ for all $i > n$ and a complex X such that the support of $H^i(X)$ is contained in W for all $i < 0$, the natural map $\Gamma_W I \to I$ induces isomorphisms*

$$\mathrm{Ext}^i_R(X, \Gamma_W I) \xrightarrow{\cong} \mathrm{Ext}^i_R(X, I) \qquad \textit{for } i > n.$$

This theorem is very similar to a version of the local duality theorem. If we consider the case where $(R, \mathfrak{m})$ is a local ring, $W = \{\mathfrak{m}\}$, and I is a dualizing complex of R, then it naturally induces a generalization of the original AR duality (1). If $W = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} \leqq 1\}$, then we can deduce from AR principle the generalization of AR duality due to Iyama and Wemyss. It also can be applied to the Auslander-Reiten conjecture for modules over Gorenstein rings.

Theorem Let R be a Gorenstein local ring of dimension d that is larger than 2. Assume that M is a maximal Cohen-Macaulay R-module whose non-free locus has dimension $\leqq 1$. Furthermore we assume that $\mathrm{Ext}^{d-1}_R(M, M) = 0 = \mathrm{Ext}^{d-2}_R(M, M)$. Then M is a free R-module.

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Degeneration for Cohen-Macaulay modules.

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One of the interesting problems for Cohen-Macaulay modules is a degeneration problem. Let R be an associative k-algebra where k is any field. We take a discrete valuation ring (V, tV, k) which is a k-algebra and t is a prime element. We denote by K the quotient field of V. Then we have the natural functors

$$\mathrm{mod}(R) \xleftarrow{\ r\ } \mathrm{mod}(R \otimes_k V) \xrightarrow{\ \ell\ } \mathrm{mod}(R \otimes_k K),$$

where $r = - \otimes_V V/tV$ and $\ell = - \otimes_V K$.

Definition For modules $M, N \in \mathrm{mod}(R)$, we say that M **degenerates to** N if there exist a discrete valuation ring (V, tV, k) which is a k-algebra and a module $Q \in \mathrm{mod}(R \otimes_k V)$ that is V-flat such that $\ell(Q) \cong M \otimes_k K$ and $r(Q) \cong N$.

The module Q, regarded as a bimodule ${}_RQ_V$, is a flat family of R-modules with parameter in V.

The following theorem is a key for the study of degenerations.

Theorem [2] Let $M, N \in \mathrm{mod}(R)$. Then M degenerates to N if and only if there is a short exact sequence in $\mathrm{mod}(R)$

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

We will show some recent computational results on degeneration of Cohen-Macaulay modules.

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Auslander-Bridger theory for unbounded projective complexes.

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Let R be a commutative Noetherian ring and let $\mathscr{K}(R)$ be the homotopy category of all complexes of finitely generated projective modules over R. For any $X \in \mathscr{K}(R)$ the R-dual complex $X^* = \mathrm{Hom}_R(X, R)$ is defined and the operation $(-)^*$ gives the duality on $\mathscr{K}(R)$. The main theorem of this talk is the following:

Theorem [2] *Let $X \in \mathscr{K}(R)$ and assume that R is a generically Gorenstein ring. Then, X is acyclic if and only if X^* is acyclic.*

Recall that R is called a generically Gorenstein ring if the total ring of quotients is Gorenstein. This theorem includes the Tachikawa conjecture and the dependence of totally reflexivity conditions for modules over a generically Gorenstein ring.

To prove this theorem we need to develop and establish the Auslander-Bridger type theory for $\mathscr{K}(R)$. Precisely speaking, we have a natural mapping $\rho^i_{X,R} : H^{-i}(X^*) \to H^i(X)^*$ for $X \in \mathscr{K}(R)$ and $i \in \mathbb{Z}$. We say that a complex $X \in \mathscr{K}(R)$ is ***torsion-free** (resp. ***reflexive**) if $\rho^i_{X,R}$ are injective (resp. bijective) mappings for all $i \in \mathbb{Z}$. Let $\mathrm{Add}(R)$ be the additive full subcategory of $\mathscr{K}(R)$ consisting of all split complexes. We can show that $\mathrm{Add}(R)$ is functorially finite in $\mathscr{K}(R)$ and hence every complex in $\mathscr{K}(R)$ is resolved by complexes in $\mathrm{Add}(R)$. Define $\underline{\mathscr{K}(R)}$ to be the factor category $\mathscr{K}(R)/\mathrm{Add}(R)$. Then we are able to define the syzygy functor Ω and the cosyzygy functor Ω^{-1} on $\underline{\mathscr{K}(R)}$, and as a result we have an adjoint pair (Ω^{-1}, Ω) of functors. Then we can show that X is *torsion-free iff $X \cong \Omega^{-1}\Omega X$ in $\underline{\mathscr{K}(R)}$. And under the assumption that R is generically Gorenstein, X is *reflexive iff $X \cong \Omega^{-2}\Omega^2 X$ in $\underline{\mathscr{K}(R)}$.

There is a triangles of the form

$$\Delta^{(n,0)}(X) \to \Omega^{-n}\Omega^n(X) \to X \to \Delta^{(n,0)}(X)[1],$$

for $X \in \mathscr{K}(R)$ and $n > 0$, where $\Delta^{(n,0)}(X)$ has a finite $\mathrm{Add}(R)$-resolution of length at most $n-1$. This is one of the key theorems in order to prove Main Theorem. The second key observation is that any syzygy complex $\Omega^r X$ ($\forall r > 0$) is *torsion-free if $H(X^*) = 0$.

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