Abstracts of the Seminar Talks

(In Alphabetical Order)

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Linearity Defect of the Residue Field of Local Rings

Rasoul Ahangari Maleki
IPM, Iran

Let R be a Noetherian local ring with the residue field k. The linearity defect of a finitely generated R-module M which is denoted by $\mathrm{ld}_R(M)$, is a numerical measure of how far M is from having linear resolution. In this talk we introduce the notion of linearity defect and we will deal with the question raised by Herzog and Iyengar of whether $\mathrm{ld}_R(k) < \infty$ implies $\mathrm{ld}_R(k) = 0$.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

An Invitation to Teissier's Program for Resolution of Singularities

Razieh Ahmadian
IPM, Iran

The problem of resolution of singularities of a reduced scheme X asks if there exists a proper birational morphism $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is a non-singular scheme. One of the leading approaches to the problem has been introduced by Zariski consisting of local uniformization[1] and gluing up local data. Using this strategy, resolution of singularities for surfaces and 3–folds have been proven by Zariski himself over an algebraically closed field of characteristic zero [Z1, Z2, Z3], and by Abhyankar in positive characteristic in 1966 (see [C1]). Another major strategy has been outstandingly applied by Hironaka in his complete proof in characteristic zero [H1]. He has also a program [H2] and an unpublished article available on his personal homepage in positive characteristic.

Recently, significant attempts are being made to prove the resolution problem in positive characteristic and generalized resolution problems. Of interest to us is Teissier's program (outlined in [T1, T2]), in which the skillful use of techniques in toric geometry and valuation theory helps him to propose a completely different method for proving local uniformization which is blind to the characteristic. In this talk, we will give an overview of this program, and the results obtained so far.

References

[C1] Cutkosky, S.D., *Resolution of singularities for 3-folds in positive characteristic*, Amer. J. Math. **131** (2009), no. 1, 59 – 127.

[H1] Hironaka, H., *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math **79** (1964), 109 – 326.

[H2] Hironaka, H., *A program for resolution of singularities, in all characteristics* $p > 0$ *and in all dimensions*, Lecture notes of the school and conference on Resolution of singularities, Trieste, 2006.

[1]**Local Uniformization.** Suppose that X is an algebraic variety over a field k, with function field K, and let V be a valuation ring of K containing k. The problem of local uniformization is to find a regular local ring R, essentially of finite type over k and with quotient field K such that the valuation ring V dominates R, i.e., $R \subset V$ and the maximal ideal of R is the intersection of the maximal ideal of V with R. This would imply local resolution of singularities.

[T1] Teissier, B., *Valuations, deformations and toric geometry*, Valuation theory and its applications II, F.V. Kuhlmann, S. Kuhlmann and M. Marshall, editors, Fields Institute Communications **33**, Amer. Math. Soc., Providence, RI, (2003), 361 – 459.

[T2] Teissier, B., *Monomial ideals, binomial ideals, polynomial ideals*, Trends in Commutative Algebra, MSRI publications, Cambridge University Press (2004), 211–246.

[T3] Teissier, B., *Overweight Deformations of Affine Toric Varieties and Local Uniformation*, arXive: 1401.5204v4 (2016).

[Z1] Zariski, O., *The reduction of singularities of an algebraic surface*, Ann. of Math. **40** (1939), 639–689.

[Z2] Zariski, O., *Local uniformization of algebraic varieties*, Annals of Math. **41** (1940), 852 – 896.

[Z3] Zariski, O., *Reduction of singularities of algebraic three–dimensional varieties*, Ann. of Math. **45** (1944), 472–542.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Cohen-Macaulay Auslander Algebras

Javad Asadollahi
University of Isfahan and IPM, Iran

In this talk, we study Cohen-Macaulay Auslander algebras using certain intermediate extension functors. In particular, it will be shown that two Gorenstein algebras of G-dimension one that are of finite Cohen-Macaulay type are Morita equivalent if and only if their Cohen-Macaulay Auslander algebras are Morita equivalent.

The talk is based on a joint work with Rasool Hafezi.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Cotorsion Pairs in the Category of N-Complexes

Payam Bahiraei
University of Guilan, Iran

The concept of cotorsion pairs (or cotorsion theory) was invented by Salce in the category of abelian groups and was rediscovered by Enochs and coauthors in the 1990's. In short, a cotorsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{F}, \mathcal{C})$ of classes of objects of $\mathcal{A}$ each of which is the orthogonal complement of the other with respect to the Ext functor. In this talk, we show some ways of getting complete cotorsion pairs in the category of N-complexes. As an application, we introduce the existence of adjoint functors between particular homotopy categories of N-complexes.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Quadratic Monomial Ideals with Large Number of Linear Steps

Mina Bigdeli
IPM, Iran

Let I be a quadratic monomial ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, $\mathbb{K}$ a field, with the property that the minimal free resolution of I is linear up to the homological degree t. It is well-known that the following are equivalent:

(i) $t = \infty$, i.e. I has a linear resolution;

(ii) the polarization of I is the edge ideal of a graph with a chordal complement;

(iii) all powers of I have a linear resolution.

The equivalence of (i) and (ii) is due to R. Fröberg, and (ii)$\Leftrightarrow$(iii) is given by J. Herzog et al.

In this talk, we discuss the possibility of extension of this result to the more general case $t \geq \text{pd}(I) - 2$, where $\text{pd}(I)$ denotes the projective dimension of I. As an outcome, we give a complete combinatorial classification of quadratic monomial ideals whose resolution is linear up to the homological degree $\text{pd}(I) - 2$. This classification in particular implies that, unlike the general case, the numerical data of the resolution of such ideals does not depend on the choice of the base field. We compute some of these data. Moreover, we investigate the linearity of the resolution of powers of I.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

On a Family of Cohomological Degrees

Doan Trung Cuong
Institute of Mathematics, Hanoi, Vietnam

Cohomological degrees (or extended degrees) were introduced by Doering-Gunston-Vasconcelos twenty years ago as a measure for the complexity of algebraic structures of finitely generated modules over a Noetherian ring. These degrees plays the role of multiplicities in the case of Cohen-Macaulay modules. In particular, Doering-Gunston-Vasconcelos obtained several upper bounds for number of generators, Castelnuovo-Mumford regularity, Betti numbers, etc in terms of a cohomological degree. On the other hand, Vasconcelos and Gunston introduced respectively the homological degree hdeg and the extremal degree bdeg as two examples of cohomological degrees. Recently, NT Cuong-PH Quy has introduced the notion of unmixed degree and showed that it is also a cohomological degree. This is the third example of cohomological degree. In this talk, we will discuss a construction that gives rise to an infinite family of cohomological degrees. The construction relies on special properties of annihilators of local cohomology modules.

This is a joint work with Pham Hong Nam.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

On the Index of Reducibility of Noetherian Modules

Nguyen Tu Cuong
Institute of Mathematics, Hanoi, Vietnam

One of the fundamental results in commutative algebra is the irreducible decomposition theorem [1, Satz II and Satz IV] proved by Emmy Noether in 1921. In this paper she had showed that any ideal I of a Noetherian ring R can be expressed as a finite intersection of irreducible ideals, and the number of irreducible ideals in such an irredundant irreducible decomposition is independent of the choice of the decomposition. This number is then called the index of reducibility of I and denoted by $\mathrm{ir}_R(I)$. Although irreducible ideals belong to basic objects of commutative algebra, there are not so much papers on the study of irreducible ideals and the index of reducibility. Then the purpose of this talk is to investigate the index of reducibility of submodules of a finitely generated R-module M as well as the behaviour of the function of indices of reducibility $\mathrm{ir}_M(I^nM)$, where I is an ideal of R, and to present applications of the index of reducibility for the studying the structure of the module M. We show that this function is in fact a polynomial for sufficiently large n. Moreover, we can prove that the big height $\mathrm{bight}_M(I) - 1$ is a lower bound and the analytic spread $\ell_M(I) - 1$ is an upper bound for the degree of this polynomial. However, the degree of this polynomial is still mysterious to us. We can only give examples to show that these bounds are optimal. A classical result of Northcott [2] says that the index of reducibility of a parameter ideal in a Cohen-Macaulay local ring is dependent only on the ring and not on the choice of the parameter ideal. We will generalize Northcott's result and get a characterization for Cohen-Macaulayness of a Noetherian module in terms of the index of reducibility of parameter ideals.

References

[1] E. Noether, Idealtheorie in Ringbereichen, *Math. Ann.* **83** (1921), 24–66.

[2] D.G. Northcott, On irreducible ideals in local rings, *J. London Math. Soc.* **32** (1957), 82–88.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Coxeter Diagrams and the Köthe's Problem

Ziba Fazelpour
IPM, Iran

In this talk we give results on the solution of the old problem stated by Köthe in 1935, developed later in 1960-1980 by a group of researchers studying classification problems of representation theory of finite groups, finite-dimensional algebras, and artinian rings. A ring R is called right Köthe if each of the right R-modules is a direct sum of cyclic modules. Köthe posed the problem, known as Köthe's problem, to classify the right (resp., left) Köthe rings. The Köthe's problem is still open. We give a complete quiver solution of the problem in two cases basic hereditary rings and rings with radical square zero by using of the powerful species and Coxeter diagram technique introduced and developed by Dlab-Ringel in 1975-1976.

This talk is based on a joint work with Alireza Nasr-Isfahani.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Brauer-Thrall Theory for Lattices

Fahimeh Sadat Fotouhi
University of Isfahan, Iran

Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay local ring and let Λ be an R-order. In this talk, we will study Brauer-Thrall theorems in the category of Λ-lattices. We assign a numerical invariant for any lattice, called h-length. It will turn out that an R-order Λ is bounded lattice type if it is of finite lattice type and if there are infinitely many non-isomorphic indecomposable lattices of the same h-length, then Λ has strongly unbounded lattice type, meaning that there is an infinite sequence $n_1 < n_2 < \cdots$ of positive integers such that Λ has, for each i, infinitely many non-isomorphic indecomposable lattices of the same h-length n_i.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Gorenstein Singularity Categories

Rasoul Hafezi
IPM, Iran

In my talk, I will introduce the notion of Gorenstein singularity category of an Artin algebra which is inspired of the work " Kalck, Martin and Yang, Dong, Relative singularity categories I: Auslander resolutions. Adv. Math. 301 (2016), 9731021". Then, I will explain that how this Gorenstein notion can be related to the classical singularity category defined as the Verdier localization of the bounded derived category by the full triangulated subcategory of perfect complexes.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

The Structure of Residual Intersections

Hamid Hassanzadeh
Federal University of Rio de Janeiro, Brazil

Residual intersection is a generalization of linkage. It provides geometric and algebraic classifications of ideals. The definition is the following: In a (Cohen-Macaulay) local ring R, an ideal J is an s-residual intersections of an ideal I if there is an s-generated ideal a such that J=a: I and codim(J) is at least s. Although this is a fundamental construction in Algebra and Geometry, very few are known about the structure of J (generators and other Betti numbers), except for some particular cases. In this talk, we show how one can determine the generators of J by using the DG-algebra structure of the Koszul complex of I. The results cover most of the classical works in this direction which include some works of Peskine-Szpiro, Huneke-Ulrich, Kustin, Bruns among others.

The talk bases on joint work with Vinicius Bouca.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Cyclic Covers and the F-regularity of Hankel Determinantal Rings

Maral Mostafazadehfard
Federal University of Rio de Janeiro, Brazil

Hankel determinantal rings, i.e., rings defined by minors of Hankel matrices of indeterminates, arise as homogeneous coordinate rings of higher order secant varieties of rational normal curves. We study cyclic covers of Hankel determinantal rings in broad generality, and use these to prove that certain Hankel determinantal rings are F-regular.

Based on a joint work with Aldo Conca, Anurag K. Singh and Matteo Varbaro

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Normality and Normally Torsion-Freeness of Monomial Ideals Under Monomial Operations

Mehrdad Nasernejad
Khayyam University, Mashhad, Iran

This talk has two parts. In the first part, we explore the normality of monomial ideals, and produce a procedure for constructing new normal monomial ideals from other ideals that are assumed to be normal. This enables us to prove that if the cover ideal of a graph G is normal, then the cover ideal of the graph H is normal as well, where the graph H is obtained by connecting all vertices in G with a new vertex. We use these ideas to explore the normality of the cover ideals of some imperfect graphs. Furthermore, we investigate the normality under some monomial operations such as expansion, weighting, monomial multiple, monomial localization, contraction, and deletion. In the second part, our aim is to investigate the normally torsion-freeness of monomial ideals by using some monomial operations such as expansion, weighting, monomial multiple, monomial localization, contraction, and deletion. Especially, we introduce several methods for constructing new normally torsion-free monomial ideals based on the monomial ideals which have normally torsion-freeness.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Hypersurfaces with Linear Type Singular Subscheme

Abbas Nasrollah Nejad
Institute for Advanced Studies in Basic Sciences, Zanjan, Iran

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field k of characteristic zero. Let $X = V(f) \subset \mathbb{A}_k^n$ be a reduced singular hypersurfaces which defined by a reduced polynomial $f \in R$. The singular locus of X is defined by the Jacobian ideal $I_f = (f, J_f)$, where $J_f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is so called the gradient ideal. The reduced hypersurface X is said to be of *Jacobian linear type* if the Jacobian ideal $I_f \subseteq R$ is of linear type, i.e., the symmetric algebra of I_f (naive blowup) is isomorphic with the corresponding Rees algebra (blowup).

In this talk, we give necessary and sufficient criterion for reduced hypersurface only with isolated singularities to be of Jacobian linear type. We prove that a projective hypersurface is of *gradient linear type* if and only if the corresponding affine hypersurface in the affine chart associated to singular point is locally Eulerian. We show that any projective plane curve with simple singularities(ADE) is of gradient linear type. In particular, any reduced projective quartic curve is of gradient linear type.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

On the Depth of Binomial Edge Ideals

Mohammad Rouzbahani Malayeri
Amirkabir University of Technology, Iran

Binomial edge ideals associated to graphs were introduced in 2010 by Herzog et al. in [6] and independently by Ohtani in [9]. Let G be a simple graph on the vertex set $[n] = \{1, \dots, n\}$ and the edge set $E(G)$. Let $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring over a field $\mathbb{K}$. Then the binomial edge ideal of G denoted by J_G is an ideal in S whose generators are all quadrics of the form $f_{ij} = x_i y_j - x_j y_i$, where $\{i, j\} \in E(G)$ and $1 \le i < j \le n$.

One of the interesting homological invariants in commutative algebra, is depth. Let $H^i_{\mathfrak{m}}(S/J_G)$ denote the i^{th} local cohomology module of S/J_G supported on the irrelevant maximal ideal $\mathfrak{m} = (x_1, \dots, x_n, y_1, \dots, y_n)$. Then we have

$$\operatorname{depth} S/J_G = \min\{i : H^i_{\mathfrak{m}}(S/J_G) \neq 0\}.$$

Let C_n denote the cycle on n vertices. In [10] it was shown that $\operatorname{depth} S/J_{C_n} = n$. Also, in [4] the authors showed that $\operatorname{depth} S/J_G = n + 1$, for a connected *block* graph G. Later in [7] the authors computed the depth of a wider class of graphs which are called *generalized block* graphs. In [8], a nice formula was given for the depth of the *join product* of two graphs G_1 and G_2.

In [1] the authors gave an upper bound for the depth of the binomial edge ideal of a graph in terms of some graphical invariants. Indeed, they showed that for a non-complete connected graph G, $\operatorname{depth} S/J_G \le n - \kappa(G) + 2$, where $\kappa(G)$ denotes the vertex connectivity of G.

There is also a lower bound for $\operatorname{depth} S/J_G$. Indeed, let $\operatorname{cd}(J_G, S)$ denote the cohomological dimension of J_G in S. Now a result in [5] due to Faltings, together with [2] and [3] implies that

$$\operatorname{depth} S/J_G \ge \left\lfloor \frac{2n-1}{\operatorname{bigheight} J_G} \right\rfloor.$$

In this talk, among other results about the depth of S/J_G, we supply an explicit characterization of graphs G for which $\operatorname{depth} S/J_G = 4$. Moreover, we show that this is the smallest possible value for $\operatorname{depth} S/J_G$.

This talk is based on a joint work with Dariush Kiani and Sara Saeedi Madani.

References

[1] A. Banerjee, L. L. Núñez-Betancourt, *Graph connectivity and binomial edge ideals*, Proc. Amer. Math. Soc. 145 (2017), 487-499.

[2] A. Conca, M. Varbaro, *Square-free Gröbner degenerations*, (2018) arXiv:1805.11923v2.

[3] H. Dao, A. De Stefani, L. Ma, *Cohomologically full rings*, Int. Math. Res. Not. (2019), https://doi.org/10.1093/imrn/rnz203.

[4] V. Ene, J. Herzog, T. Hibi, *Cohen-Macaulay binomial edge ideals*, Nagoya Math. J. 204 (2011), 57-68.

[5] G. Faltings, *Über lokale kohomologiegruppen hoher ordnung*, J. Reine Angew . Math. 313 (1980), 43-51.

[6] J. Herzog, T. Hibi, F. Hreinsdóttir, T. Kahle, J. Rauh, *Binomial edge ideals and conditional independence statements*, Adv. Appl. Math. 45 (2010), 317-333.

[7] D. Kiani, S. Saeedi Madani, *Some Cohen-Macaulay and unmixed binomial edge ideals*, Comm. Algebra. 43 (2015), 5434-5453.

[8] A. Kumar, R. Sarkar, *Depth and extremal Betti number of binomial edge ideals*, Math. Nachr. (2019), to appear.

[9] M. Ohtani, *Graphs and ideals generated by some 2-minors*, Comm. Algebra. 39 (2011), 905-917.

[10] Z. Zahid, S. Zafar, *On the Betti numbers of some classes of binomial edge ideals*, The Electronic Journal of Combinatorics. 20(4) (2013), ♯ P37.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Squarefree Powers of Edge Ideals

Sara Saeedi Madani
IPM and Amirkabir University of Technology, Iran

Let G be a finite simple graph on $[n] = \{1, \ldots, n\}$ with the edge set $E(G)$. Let K be a field, and let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K. Associated to each edge $e = \{i, j\}$ of G is the monomial x_ix_j of S. The so-called *edge ideal* of G is the monomial ideal $I(G)$ of S which is generated by those monomials $e = x_ix_j$ with $e \in E(G)$.

The study of the minimal graded free resolution of edge ideals and of their powers has been and still is one of the current trends of commutative algebra. In this talk, instead of the ordinary powers of edge ideals, we discuss their *squarefree powers*. Let I be a squarefree monomial ideal in S. For any $k \geq 1$ we denote by the $I^{[k]}$ the kth *squarefree power* of I which is defined to be the squarefree monomial ideal generated by the squarefree monomials in I^k. Note that $I^{[k]} = (0)$ for $k \gg 0$. Our study on squarefree powers of edge ideals is closely connected with the classical theory of matchings in graph theory.

In this talk, we discuss some bounds for the Castelnuovo-Mumford regularity of the squarefree powers $I(G)^{[k]}$. We also investigate about those squarefree powers which have linear resolution. We also consider the so-called *squarefree Ratliff property*.

This talk is based on a joint work with N. Erey, J. Herzog and T. Hibi.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

Involutory Quandles and Persistent Homology in Machine Learning

Amir Saki
Amirkabir University of Technology, Iran

Let Δ be a simplicial complex and the function $\phi : \Delta \to \mathbb{R}$ be order preserving (i.e. $\phi(\sigma) \leq \phi(\delta)$ for any $\sigma, \delta \in \Delta$ with $\sigma \subseteq \delta$). It follows that $\Delta_t = \phi^{-1}((-\infty, t])$ is a subcomplex of Δ for any $t \in \mathbb{R}$. Let k be an integer and $s, t \in \mathbb{R}$ with $s \leq t$. Then, the inclusion $i : \Delta_s \hookrightarrow \Delta_t$ induces a homomorphism $i_k : \tilde{H}_k(\Delta_s) \to \tilde{H}_k(\Delta_t)$. The k^{th} *persistent homology group of* Δ with respect to ϕ at (s,t) is denoted by $PH_k^{\phi}(s,t)$ and is defined by $PH_k^{\phi}(s,t) = i_k\left(\tilde{H}_k(\Delta_s)\right)$. Also, the k^{th} *persistent Betti number of* Δ with respect to ϕ at (s,t) is denoted by $r_k(\phi)(s,t)$ and is defined to be the rank of $PH_k^{\phi}(s,t) = i_k\left(\tilde{H}_k(\Delta_s)\right)$. Note that r_k^{ϕ} is a function called the k^{th} *Betti function* with respect to ϕ. Let $\mathcal{A}$ be a set of order preserving functions from Δ to $\mathbb{R}$. Then, a pseudo-metric d_{match} is defined on the set of all k^{th} Betti functions of $\mathcal{A}$ (see [2]).

Let X be a set and Φ be a set of bounded functions $\phi : X \to \mathbb{R}$, which we call each of them a *signal*. Then, we denote the metric arising from the L^{∞}-norm on Φ by D_Φ. Signals induce a pseudo-metric on X as follows: $D_X(x,y) = \sup_{\phi\in\Phi} |\phi(x) - \phi(y)|$ for any $x, y \in X$. Hence, D_X induces a topology on X. The set of Φ-*preserving homeomorphisms* of X is denoted by $\text{Homeo}_\Phi(X)$ and is defined to be the set of all homeomorphisms $g : X \to X$ such that $\phi \circ g, \phi \circ g^{-1} \in \Phi$ for any $\phi \in \Phi$. Let G be a subgroup of $\text{Homeo}_\Phi(X)$. To see the real difference between two signals the following pseudo-metric called the *natural pseudo-metric* on Φ is defined: $d_G(\phi, \psi) = \inf_{g\in G} D_\Phi(\phi, \psi \circ g)$ for any $\phi, \psi \in \Phi$. Note that based on computations, the natural pseudo-metric on Φ is not reasonable, and hence a suitable way to approximate d_G is needed. In the following, we explain a method based on using persistent homology on the so-called *group equivariant non-expansive operators*. The pair (Φ, G) is called a *perception pair*. Assume that (Ψ, H) is another perception pair and $T : G \to H$ is a homomorphism. A map $F : \Phi \to \Psi$ is called a *group equivariant operator* (or simply a *GEO*) if $F(\phi \circ g) = F(\phi) \circ T(g)$ for any $\phi \in \Phi$ and $g \in G$. A group equivariant non-expansive operator (or simply a *GENEO*) is a GEO satisfying the condition $D_\Psi(F(\phi_1), F(\phi_2)) \leq D_\Phi(\phi, \psi)$ for any $\phi_1, \phi_2 \in \Phi$. A suitable approximation for $d_G(\phi_1, \phi_2)$ is the pseudo-metric $D^k_{\text{match}}(\phi_1, \phi_2) = \sup_{F\in\mathcal{F}} d_{\text{match}}(r_k(F(\phi_1)), r_k(F(\phi_2)))$ for any $\phi_1, \phi_2 \in \Phi$, where $\mathcal{F}$ is the set of all GENEOs from Φ to Ψ. Finally, in the case that $\mathcal{F}$ is compact, a nice approximation of d_G could be reached.

A quandle is a set Q together with a binary operation $\triangleright$ which satisfies the following conditions: For any $x, y, z \in Q$, we have that $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (x \triangleright z)$, there exists a unique element $w \in Q$ with $w \triangleright x = y$, and $x \triangleright x = x$. The inner group of a quandle Q, denoted by $\text{Inn}(Q)$, is the subgroup of the permutation group on Q generated by all maps

f_x. An involutory quandle is a quandle $(Q, \triangleright)$ such that $(x \triangleright y) \triangleright y = x$ for any $x, y \in Q$. For instance, one could define $x \triangleright y$ to be the reflection of x through y for any $x, y \in \mathbb{R}^2 \backslash \{0\}$.

This talk is based on proposing an algebraic model to be attached to the theory of GENEOs, which is a topological theory in neural networks. A *symmetric space* is an involutory quandle from the Algebraic perspective. Let $(X, \triangleright)$ be an involutory quandle. Then, we consider a set Φ of signals $\phi : X \to \mathbb{R}$. Also for any $x \in X$, we consider the map $f_x : X \to X$ sending each $y \in X$ to $y \triangleright x$ and call it a *symmetry* of X. If we want to get practically suitable results, the set of signals Φ should be rich enough with the aim of learning process. Hence, we assume that $\phi \circ f_x, \phi \circ f_x^{-1} \in \Phi$ for any $x \in X$ and any $\phi \in \Phi$. For instance, in the case that our signals are photos, we expect that different rotations of the patterns (that we would like to recognize) belong to Φ. We show that all symmetries of X are isometries with respect to D_X. Then, we work with $G = \mathrm{Inn}(X)$ which is a subgroup of $\mathrm{Homeo}_\Phi(X)$. This approach shows that our algebraic model and the topological model considered in [1] are compatible. Note that the example discussed in [1], could bee seen as an example of our model.

This talk is based on a joint work with Dariush Kiani.

References

[1] M. G. Bergomi, P. Frosini, D. Giorgi, and N. Quercioli, Towards a topological-geometrical theory of group equivariant non-expansive operators for data analysis and machine learning, Nature Machine Intelligence, vol. 2019 (2019).

[2] A. Cerri, B. D. Fabio, M. Ferri, P. Frosini, and C. Landi, Betti numbers in multidimensional persistent homology are stable functions, Mathematical Methods in the Applied Sciences, 36(12)(2013), 1543-1557.

[3] H. Edelsbrunner and J. Harer, Computational Topology, an Introduction, AMS (2010).

[4] D.E. Joyce, An algebraic approach to symmetry with applications to knot theory, PhD thesis, University of Pennsylvania (1979).

[5] D. Kiani and A. Saki, The lattice of subracks is atomic, J. Combin. Theory Ser. A, vol. 162 (2019), 55-64.

[6] A. Verri, C. Uras, P. Frosini, and M. Ferri, On the use of size functions for shape analysis. Biological Cybernetics 70 (1993), 99-107.

[7] A. Zomorodian and G. Carlsson. Computing Persistent Homology. Discrete and Computational Geometry, 33(2)(2005), 249-274.

The 16th Seminar on
Commutative Algebra and Related Topics, January 15 and 16, 2020
School of Mathematics, IPM, Tehran

The Importance of Counterexamples

Sverre Olaf Smalo
Norwegian University of Science and Technology, Norway

I will present a couple of counterexamples I have made to some homological questions raised by H. Bass and M. Auslander.