RADIUS OF COMPARISON OF FIXED POINT ALGEBRAS AND CROSSED PRODUCTS OF ACTIONS ON FINITE GROUPS.

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These notes are written for a talk which is given at the School of Mathematics of Institute for Research in Fundamental Sciences (IPM) on January 6-9, 2020. Parts of this talk concern my joint work with N. Christopher Phillips and Nasser Golestani.(See [1] for more details.)

1. INTRODUCTION

The radius of comparison, based on the Cuntz semigroup, was introduced to distinguish examples of nonisomorphic simple separable unital AH algebras with the same Elliott invariant. The comparison theory of projections is fundamental to the theory of von Neumann algebras, and is the basis for the type classification of factors. A C*-algebra might have few or no projections, in which case their comparison theory tells us little about the structure of the C*-algebra. The appropriate replacement for projections is positive elements. This idea was first introduced by Cuntz in [4] with a view to studying dimension functions on simple C*-algebras. Then the appropriate definition of the radius of comparison of C*-algebras was introduced by Andrew S. Toms in Section 6 of [12]. Significant progress has been made on the radius of comparison of a C*-algebra A and a large subalgebra of A by N. Christopher Phillips. (See [8] and [9].) Also, we refer to the work of Bruce Blackadar, Leonel Robert, Aaron Tikuisis, Andrew S. Toms, and Wilhelm Winter [3] and the work of Zhuang Niu and George A. Elliott [5] for more details about the radius of comparison of commutative C*-algebras. Since the radius of camparison of C*-algebras helps us to complete Elliott classification program for simple separable nuclear unital C*-algebras it is so important in classification of C*-algebras.

Strict comparison of positive elements is a property of the Cuntz semigroup. The Cuntz semigroup is generally large and complicated; roughly speaking, among simple nuclear C*-algebras, the classifiable ones are those whose Cuntz semigroups are easily accessible. With the near completion of the Elliott program, attention is turning to nonclassifiable C*-algebras, and the Cuntz semigroup is the main additional available invariant.

2. Preliminaries

Conjecture 2.1 (Toms-Winter 2008)**.** If A is a simple stably finite separable amenable C*-algebra, then the following are equivalent:

(1) A has finite nuclear dimension;
(2) A is $\mathcal{Z}$-stable (where $\mathcal{Z}$ is the Jiang–Su algebra);
(3) A has strict comparison of positive elements (i.e., $\mathrm{rc}(A) = 0$).

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Notation 2.2. We use the following standard notation. We write $\mathcal{K} = K(\ell^2)$. We write A_+ for the set of positive elemnts of A. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. We denote by A^α the fixed point algebra, given by

$$A^\alpha = \{a \in A \colon \alpha_g(a) = a \text{ for all } g \in G\}.$$

Definition 2.3. Let A be a C*-algebra. Let $m, n \in \mathbb{Z}_{>0}$, let $a \in M_n(A)_+$, and let $b \in M_m(A)_+$.

(1) We say that a is *Cuntz subequivalent* to b in A, written $a \precsim_A b$, if there exists a sequence $(x_k)_{k=1}^\infty$ in $M_{n,m}(A)$ such that $\lim_{k\to\infty} x_k b x_k^* = a$.
(2) We say that a is *Cuntz equivalent* to b in A, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$.

Example 2.4. Let $n \in \mathbb{Z}_{>0}$, let $A = M_n(\mathbb{C})$, and let $a, b \in A_+$. Then

$$a \precsim_A b \Longleftrightarrow \mathrm{rank}(a) \leq \mathrm{rank}(b).$$

Example 2.5. Let $n \in \mathbb{Z}_{>0}$, let $A = M_n\big(C([0,1])\big)$, and let $f, g \in A_+$. Then

$$f \precsim_A g \Longleftrightarrow \mathrm{rank}(f)(t) \leq \mathrm{rank}(g)(t) \quad \text{for all } t \in [0,1].$$

Example 2.6. Let X be a compact metric space, and let $f, g \in C(X)_+$. Then

$$f \precsim_{C(X)} g \Longleftrightarrow \{x \in X \colon f(x) \neq 0\} \subseteq \{x \in X \colon g(x) \neq 0\}.$$

Example 2.7. Let $A = M_n(C(X))$ with X is a CW-complex of $\dim(X) \geq 3$ and $n \geq 2$. Then there exist $f, g \in M_n(C(X))_+$ such that $\mathrm{rank}(a)(t) = \mathrm{rank}(b)(t)$ for all $t \in [0,1]$ and $f \nsim g$.

Definition 2.8. Let A be a C*-algebra. We denote by $M_\infty(A)$ the algebraic limit of the direct system $(M_n(A), \varphi_n)$ where $\varphi_n \colon M_n(A) \to M_{n+1}(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Define $\mathrm{W}(A) = M_\infty(A)_+/\sim_A$ and $\mathrm{Cu}(A) = \mathrm{W}(A \otimes \mathcal{K})$. We further define an addition $\langle a\rangle_A + \langle b\rangle_A = \left\langle \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\rangle_A$ and a partial order $\langle a\rangle_A \leq \langle b\rangle_A$ if $a \precsim_A b$. Then we get an ordered abelian semigroup.

Example 2.9. Let $n \in \mathbb{Z}_{>0}$ and let $A = M_n(\mathbb{C})$. Then $\mathrm{Cu}(A) = \mathbb{Z}_{>0} \cup \{\infty\}$ with $x + \infty = \infty$, $\infty + \infty = \infty$ and $\langle 1_A\rangle_A = n \in \mathbb{Z}_{>0}$.

Definition 2.10. Let A be a C*-algebra. A function $\tau\colon A \to \mathbb{C}$ is a 2-*quasitrace* if the following hold:

(1) $\tau(x^*x) = \tau(xx^*) \geq 0$ for all $x \in A$.
(2) $\tau(a + ib) = \tau(a) + i\tau(b)$ for $a, b \in A_{\mathrm{sa}}$.
(3) $\tau|_B$ is linear for every commutative C*-subalgebra $B \subseteq A$.
(4) There is a function $\tau_2\colon M_2(A) \to \mathbb{C}$ satisfying (1), (2), and (3) with $M_2(A)$ in place of A, and such that, with $(e_{j,k})_{j,k=1}^2$ denoting the standard system of matrix units in $M_2(\mathbb{C})$, for all $x \in A$ we have

$$\tau(x) = \tau_2(x \otimes e_{1,1}).$$

A 2-quasitrace τ on a unital C*-algebra is *normalized* if $\tau(1) = 1$. The set of normalized quasitraces on A is denoted by $\mathrm{QT}(A)$. The set of normalized traces on a unital C*-algebra A is denoted by $\mathrm{T}(A)$.

All quasitraces on a unital exact C*-algebra are traces, by Theorem 5.11 of [6]. Also, $\mathrm{QT}(A) \neq \varnothing$ for a stably finite unital C*-algebra A. (See [2].)

Definition 2.11. Let A be a unital stably finite C*-algebra. For every $\tau \in \mathrm{QT}(A)$ define $d_\tau \colon M_\infty(A)_+ \to \mathbb{R}^+$ by

$$d_\tau(a) = \lim_{n \to \infty} \tau(a^{\frac{1}{n}}).$$

The following is Definition 6.1 of [12], except that we allow $r = 0$ in (2.12). This change makes no difference.

Definition 2.12. Let A be a unital stably finite C*-algebra.

(1) Let $r \in [0, \infty)$. We say that A has *r-comparison* if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.

(2) The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\big(\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\}\big).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

Example 2.13. $\mathrm{rc}\big(M_n(\mathbb{C})\big) = 0$ for $n \in \mathbb{Z}_{>0}$.

Example 2.14. Let A be a stably finite exact simple $\mathcal{Z}$-stable unital C*-algebra. Then $\mathrm{rc}(A) = 0$. (See corollary 4.6 of [10].)

Let X be a compact metric space. The covering dimension of X is denoted by $\dim(X)$ and the cohomological dimension with rational coefficients is denoted by $\dim_{\mathbb{Q}}(X)$. (See [5].)

Example 2.15. Let X be a compact metric space with $\dim_{\mathbb{Q}}(X) = \dim(X)$.

(1) If $\dim X$ is even, then $\frac{\dim X}{2} - 2 \leq \mathrm{rc}\big(C(X)\big) \leq \frac{\dim X}{2} - 1$.

(2) If $\dim X$ is odd, then $\mathrm{rc}\big(C(X)\big) = \max\left(0, \frac{(\dim X - 1)}{2} - 1\right)$.

Proposition 2.16. Let A, B be unital and stably finite C*-algebras. Then:

(1) $\mathrm{rc}(A \oplus B) = \max\big(\mathrm{rc}(A), \mathrm{rc}(B)\big)$.

(2) If $k \in \mathbb{Z}_{>0}$, then $\mathrm{rc}\big(M_k(A)\big) = \frac{1}{k} \cdot \mathrm{rc}(A)$.

Definition 2.17. Let A be a unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. The action α has the *Rokhlin property* if, for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $p_g \in A$ for $g \in G$ such that

(1) $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$,

(2) $\|p_g a - a p_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,

(3) $\sum_{g \in G} p_g = 1$.

We call $(p_g)_{g \in G}$ a family of Rokhlin projections for α, F, and ε.

We can generalize Definition 2.17 under the name *weak tracial Rokhlin property*. We replace Rokhlin projections by positive contractions and replace condition 3 by $1 - \sum_{g \in G} p_g$ is small in Cuntz semigroup.

3. Radius of comparison of the fixed point algebra and crossed product

Lemma 3.1. Let A be a simple unital C*-algebra and let $\alpha\colon G \to A$ be an action of a finite group G on A with the weak tracial Rokhlin property. Let $a, b \in (A^{\alpha})_{+}$ and suppose that 0 is a limit point of $\mathrm{sp}(b)$. Then $a \precsim_{A} b$ if and only if $a \precsim_{A^{\alpha}} b$.

Proof. For simplicity, we may assume that α has the Rokhlin property. By some technical difficulties, the proof does work if α has the weak tracial Rokhlin property. Let $\varepsilon > 0$. We show that $(a-\varepsilon)_{+} \precsim_{A^{\alpha}} b$. Since $a \precsim_{A} b$, there is $w \in A$ such that

$$\big\| wbw^{*} - a \big\| < \frac{\varepsilon}{10\,\mathrm{card}(G)}.$$

Since $b, a \in A^{\alpha}$, it follows that

$$\big\| \alpha_{g}(w) b \alpha_{g}(w^{*}) - a \big\| < \frac{\varepsilon}{10\,\mathrm{card}(G)} \tag{3.1}$$

for all $g \in G$. Now set $F = \{a, b, w, w^{*}\}$ and

$$\varepsilon' = \frac{\varepsilon}{10\big(\|w\|+1\big)^{2}\big(\|b\|+1\big)\big(\mathrm{card}(G)^{2}+1\big)}.$$

Applying Definition 2.17 with F as given and with ε' in place of ε, we get orthogonal projections $p_{g} \in A$ for $g \in G$ such that

(1) $\|\alpha_{g}(p_{h}) - p_{gh}\| < \varepsilon'$ for all $g, h \in G$,
(2) $\|p_{g}a - ap_{g}\| < \varepsilon'$ for all $g \in G$ and all $a \in F$,
(3) $\sum_{g\in G} p_{g} = 1$.

Now define $v = \sum_{g} \alpha_{g}(p_{1}w)$. Clearly $v \in A^{\alpha}$. We can show

$$\|vbv^{*} - a\| < \varepsilon.$$

This relation implies that $(a-\varepsilon)_{+} \precsim_{A^{\alpha}} vbv^{*} \precsim_{A^{\alpha}} b$. □

Lemma 3.1 fails if 0 is not a limit point of $\mathrm{sp}(b)$. But if α has the Rokhlin property, the assumption "0 is a limit point of $\mathrm{sp}(b)$" can be omitted.

Theorem 3.2. Let G be a finite group, let A be a simple unital stably finite, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the weak tracial Rokhlin property. Then $\mathrm{rc}(A^{\alpha}) \leq \mathrm{rc}(A)$.

Proof. Let $a, b \in (A^{\alpha} \otimes M_{l})_{+}$ for $l \in \mathbb{Z}_{>0}$. We may assume $l = 1$. Suppose

$$d_{\tau}(A) + \mathrm{rc}(A) < d_{\tau}(b)$$

for all $\tau \in \mathrm{QT}(A^{\alpha})$. For every $\rho \in \mathrm{T}(A)$, we have $\rho|_{A^{\alpha}} \in \mathrm{QT}(A^{\alpha})$. Therefore

$$d_{\tau}(A) + \mathrm{rc}(A) < d_{\tau}(b)$$

for all $\tau \in \mathrm{QT}(A)$. Applying the definition of radius of comparison for A, we get $a \precsim_{A} b$. By Lemma 3.1, we have $a \precsim_{A^{\alpha}} b$. Therefore $\mathrm{rc}(A^{\alpha}) \leq \mathrm{rc}(A)$. □

By some technical difficulties , Theorem 3.2 can be proved if α has the weak tracial Rokhlin property.

Lemma 3.3. Let G be a finite group, let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Let $p = \frac{1}{\mathrm{card}(G)} \sum_{g\in G} 1 \cdot u_{g}$. Then $A^{\alpha} \cong pC^{*}(G, A, \alpha)p$.

Lemma 3.3 is true, even G be a locally compact group. It was prove by Rosenberg in 1979. (See [11].)

Lemma 3.4. [Rc of a corner] Let A be a stably finite simple unital C*-algebra, let p be a full projection in A, and define $\beta = \sup\{\tau(p)\colon \tau \in \mathrm{T}(A)\}$ Then:

(1) $\beta \in (0\,, 1]$.
(2) $\mathrm{rc}(A) \leq \beta \mathrm{rc}(pAp)$.

Theorem 3.5. Let G be a finite group, let A be a stably finite simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the weak tracial Rokhlin property. Then

$$\mathrm{rc}\big(C^*(G, A, \alpha)\big) \leq \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A).$$

Proof. It follows from 3.3 that $A^\alpha \cong pC^*(G, A, \alpha)p$. Since α has the Rokhlin property, for all $\mathrm{T}(A)$ we have

$$\tau(p) = \tau\left(\frac{1}{\mathrm{card}(G)} \sum_{g \in G} 1 \cdot u_g\right) = \frac{1}{\mathrm{card}(G)}.$$

Applyiny Lemma 3.4, we get $\beta = \frac{1}{\mathrm{card}(G)}$. Therefore

$$\mathrm{rc}\big(C^*(G, A, \alpha)\big) \leq \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}\big(pC^*(G, A, \alpha)p\big) = \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A^\alpha) \leq \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A).$$

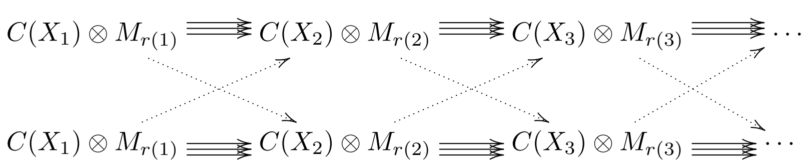
□

By some technical difficulties, Theorem 3.2 can be proved if α has the weak tracial Rokhlin property and A is not exact.

Remark 3.6. Applying the definition of the Rokhlin property finite set in $C^*(G, A, \alpha)$, we can approximate $C^*(G, A, \alpha)$ by $M_{\mathrm{card}(G)}(eAe)$ with a suitable Rokhlin projection e. It seems that this idea doesn't work here because we don't know enough about the radius of comparison of eAe. Let me explain more. $\mathrm{rc}(eAe) = \beta \mathrm{rc}(A)$ if $\tau(e) = \beta$ for all $\tau \in \mathrm{T}(A)$, but that if not all the traces take the same value, things are more complicated.

4. An example

Now we give an example of a simple AH algebra A with $\mathrm{rc}(A) > 0$ and an action $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$ which has the Rokhlin property. We use two copies of the same system. Writing the direct system sideways, our combined system looks like the following diagram, in which the solid arrows represent many partial maps and the dotted arrows represent a small number of point evaluations:



The order two automorphism exchanges the two rows.

Since we don't care about contractibility, we can use products of copies of S^2 instead of cones over such spaces as in [7].

Example 4.1. We construct a nice example by the following steps:

(1) Let $s(n)$ and $r(n)$ be sequences in $\mathbb{Z}_{>0}$ with $\lim_{n\to\infty} \frac{s(n)}{r(n)} \neq 0$.
(2) For $n \in \mathbb{Z}_{\geq 0}$, set $X_n = (S^2)^{s(n)}$.
(3) For $n \in \mathbb{Z}_{\geq 0}$ and $\nu = 1, 2, \ldots, d(n+1)$, let $P_\nu^{(n)} \colon X_{n+1} \to X_n$ be the ν coordinate projection.
(4) For $n \in \mathbb{Z}_{\geq 0}$, set $A_n = [C(X_n) \oplus C(X_n)] \otimes M_{r(n)}$.
(5) Define

$$\Lambda_{n+1,n} \colon [C(X_n) \oplus C(X_n)] \otimes M_{r(n)} \to [C(X_{n+1}) \oplus C(X_{n+1})] \otimes M_{r(n+1)},$$

by

$$(f, g) \otimes c \mapsto \begin{pmatrix} (f \circ P_1^{(n)}, g \circ P_1^{(n)}) & & 0 & \\ & \ddots & & \\ & & (f \circ P_{d(n+1)}^{(n)}, g \circ P_{d(n+1)}^{(n)}) & \\ 0 & & & \big(g(x_n), f(x_n)\big) \end{pmatrix} \otimes c$$

for $f, g \in C(X_n)$ and $c \in M_{r(n)}$.
(6) Set

$$A = \varinjlim (A_n, \, \Lambda_{n+1, \, n}).$$

(7) For $n \in \mathbb{Z}_{\geq 0}$, define $\alpha^{(n)} \colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A_n)$ by

$$\alpha_1^{(n)}\big((f, \, g) \otimes c\big) = (g, \, f) \otimes c$$

for $f, g \in C(X_n)$ and $c \in M_{r(n)}$.
(8) A is a simple unital stable rank one AH-algebra with

$$\mathrm{rc}(A) = \mathrm{rc}(A^\alpha) = \frac{1}{2}\mathrm{rc}\big(C^*(\mathbb{Z}/2\mathbb{Z}, \, A, \, \alpha)\big).$$

5. AN OPEN PROBLEM

Question 5.1. Let G be a finite group, let A be an infinite-dimensional stably finite simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the weak tracial Rokhlin property. Does it follow that

$$\mathrm{rc}(A^\alpha) = \mathrm{rc}(A) \qquad \text{and} \qquad \mathrm{rc}\big(C^*(G, A, \alpha)\big) = \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A)?$$

One might even hope that the reverse inequalities

$$\mathrm{rc}(A^\alpha) \geq \mathrm{rc}(A) \qquad \text{and} \qquad \mathrm{rc}\big(C^*(G, A, \alpha)\big) \geq \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A).$$

hold without restrictions on the action. Quite different methods seem to be needed for this question.

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