Introduction to Group Actions

3rd Problem Set
Due Aban 16th, 1398

- Exercise 2.2.7, 2.2.8 and 2.2.9, pages 36-37 from [N].

- Exercises 7.1.4 and 7.1.5, page 159 from [BS].

- Consider $f, g \in \mathrm{Homeo}^+(\mathbb{S}^1)$ so that g is a factor of f. Show that $\rho(f) = \rho(g)$.

- For any real number θ and any complex number a with $|a| < 1$, define the following map on the complex plane.

$$g_{\theta,a}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}.$$

  a) Show that restriction of $g_{\theta,a}$ to the unit circle ($\{z : |z| = 1\}$) defines a homeomorphism of this circle.
  b) Compute the rotation number of this map in terms of θ and a.

- Let $f, g \in \mathrm{Homeo}^+(\mathbb{S}^1)$ be two commuting homeomorphisms (i.e. $f \circ g = g \circ f$). Prove that $\rho(f \circ g) = \rho(f) + \rho(g)$.

- Let f be an element of $\mathrm{Homeo}^+(\mathbb{S}^1)$ so that for any $g \in \mathrm{Homeo}^+(\mathbb{S})^1$, $\rho(f \circ g) = \rho(f) + \rho(g)$. Show that f is the identity map.

- a) Let $\lambda \neq 0$ be a real number. Consider the following homeomorphisms of $\mathbb{R}$.

$$f(x) = x + 1, \quad g(x) = \lambda x.$$

  Prove that the group generated by f and g is not free.

  b) Let $\mathrm{Aff}(\mathbb{R})$ be the group of affine transformations of the real line.

$$\mathrm{Aff}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f(x) = ax + b, \text{ for some } a, b \in \mathbb{R}, a \neq 0\}.$$

  Show that this group does not contain any subgroup isomorphic to $\mathbb{F}_2$ (free group on two generators).

References

[BS] Brin, M., Stuck, G., *Introduction to Dynamical Systems*, Cambridge University Press.

[N] Navas, A., *Groups of Circle Diffeomorphisms*, The University of Chicago Press.

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