Introduction to Group Actions

4th Problem Set
Due Aban 23rd, 1398

- Let G be a finitely generated group and S a finite generating subset of G. For any $g \in G$, define
$$l_S(g) := \min\{n : g = s_1^{\pm 1} \cdots s_n^{\pm 1}, s_i \in S\}. \quad (l_S(e) = 0)$$
a) Show that $d_S(g_1, g_2) := l_S(g_1^{-1} g_2)$ defines a metric on G (This is called word-metric on G relative to S).
b) Let S' be another finite generating subset of G (possibly $|S| \neq |S'|$). Show that there is a constant $C > 1$ so that
$$\frac{1}{C} d_{S'}(g_1, g_2) \leq d_S(g_1, g_2) \leq C d_{S'}(g_1, g_2).$$
c) Deduce that the growth rate of a finitely generated group is independent of choice of finite generating subsets.

- Consider the following subgroup of $\mathrm{SL}(3, \mathbb{Z})$ known as Heisenberg group.
$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}.$$
a) Prove that H is finitely generated and has polynomial growth.
b) Show that H is nilpotent.

- Let G be a finitely generated group and H a finite index subgroup of G.
a) Show that H is also finitely generated.
b) Show G has exponential growth (polynomial growth) if and only if H has exponential growth (polynomial growth).

- Let $\lambda \neq 1$ be a positive real number. Consider the following homeomorphisms of $\mathbb{R}$.
$$f(x) = x + 1, \quad g(x) = \lambda x.$$
Prove that the group generated by f and g is of exponential growth, but it does not contain any free subgroup.

- a) Prove that two martrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is a generating set for $\mathrm{SL}(2, \mathbb{Z})$.
b) Show that $\mathrm{SL}(2, \mathbb{Z})$ contains a free subgroup of finite index.

- Prove that every finitely generated nilpotent group has polynomial growth. (Hint. Use induction on the nilpotency class of the group which the minimum positive integer n so that the n-th subgroup in the lower central series of the group is trivial.)

- Prove that any finitely generated subgroup of $\mathrm{SL}(2, \mathbb{R})$ is either of polynomial or exponential growth.

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