TOPOLOGY OF SMOOTH MANIFOLDS
EXERCISES; WEEK 1

Problem-1. A. Let G be a topological group with unity element e. For loops
$$\gamma_1, \gamma_2 : (S^1, *) \longrightarrow (G, e)$$
define the loop
$$\gamma_1 \bullet \gamma_2 : (S^1, *) \longrightarrow (G, e)$$
by $\gamma_1 \bullet \gamma_2(t) := \gamma_1(t)\gamma_2(t)$, i.e. by pointwise multiplication in G. Show that $\gamma_1 \bullet \gamma_2$ is homotopic to $\gamma_1 \star \gamma_2$.
B. If G is a topological group with unity element e, use part A to show that $\pi_1(G, e)$ is abelian.
Problem-2. Compute the fundamental group of the n-dimensional torus
$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}.$$
Problem-3. Show that any map of the projective plane to itself which is non-trivial on the fundamental group can be lifted to a map $f : S^2 \to S^2$ such that $f(-x) = -f(x)$ for all $x \in S^2$. You may use the fact that S^n is simply connected for $n \geq 2$.
Problem-4. Either prove, or give a counterexample:
Let $p : \tilde{X} \to X$ be a covering map and $f : \tilde{X} \to \tilde{X}$ be a continuous map such that $p \circ f = p$. Then f is a deck transformation.
Problem-5. Let G be a finite group which freely acts on the Hausdorff topological space X (i.e. if $gx = x$ for some $g \in G$ and $x \in X$ then g is the identity element of G). Show that the action of G on X is properly discontinuous.
Problem-6. A. Construct at least three coverings of the figure 8 space with three sheets.
B. Use the covering spaces constructed in part A to show that the fundamental group of the figure 8 space is not abelian.

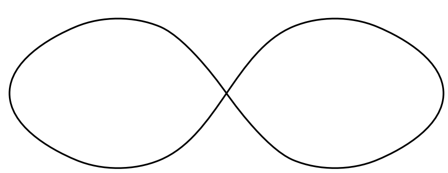

FIGURE 1. Figure 8 Space