Midterm Exam of
Topology of Smooth Manifolds

Azar 15th, 1397

8:30 – 12:00

Definitions and Basic Concepts

(A1) Give a definition of a regular point, a regular value, a submersion and an immersion, and give simple examples in each case.

(A2) State Sard's theorem and provide an example.

(A3) What is a *stable class* of maps? Show that diffeomorphisms form a stable class of maps on compact manifolds. List three other stable classes of maps and informally explain what makes them stable. Does the same hold on non-compact manifolds? Why?

(A4) Define the degree of a smooth map (both mod 2 and oriented degree). Define the Euler characteristic of a manifold. State all required conditions for the definitions and explain why these concepts are well-defined.

Problems and Theorems.

(B1) Show that $O(n)$, the group of linear transformations of $\mathbb{R}^n$ that preserve distance, is a smooth manifold and thus a Lie group. Moreover, compute the dimension of $O(n)$. (Hint: Use the fact that $A \in O(n)$ if and only if $A^t A = I$.)

(B2) Prove that every k-dimensional manifold admits a one-to-one immersion in $\mathbb{R}^{2k+1}$.

(B3) If a compact submanifold X of a manifold Y may be deformed to another compact submanifold Z of Y, then X and Z are cobordant. Give an example to show that the converse is false. If X and Z are cobordant in Y, then for every compact submanifold C of Y with dimension complementary to X and Z, show that $I_2(X, C) = I_2(Z, C)$.

Due date for the following projects: Azar 18th, 1397

Project 1

(C1) Given a positive integer n, consider a polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

with $a_n \neq 0$. Show that there is a connected submanifold L of $\mathbb{C}$ of dimension 1 which contains 0 and $P(0) = a_n$ such that $P^{-1}(L) \subset \mathbb{C}$ is a one-dimensional submanifold of $\mathbb{C}$.

(C2) Use part (C1) to prove the fundamental theorem of algebra, that P has a root in $\mathbb{C}$.

Project 2

Theorem. *The Euler characteristic of a compact oriented manifold is zero if and only if it admits a diffeomorphism isotopic to identity without any fixed point.*

Prove this theorem by taking the following steps:

(D1) Read section 5 of chapter 3 (pages 132-138) of [GP].

(D2) Solve problems 5, 11, 12, 13, 14, 15, 16 and 17 on pages 139-141 of [GP]. (You may omit the details, to a reasonable extent, when you write your answer.)

(D3) Show that if x and y are zeros of a vector field v on a manifold M with opposite index and γ is a path in M connecting x to y which is disjoint from other zeros of v, then there is another vector field w on M which is identical with v outside an open neighborhood U of γ and is everywhere non-zero on U.

(D4) Combine the Lefschetz Fixed-Point Theorem and your findings from parts (D2) and (D3) to prove the theorem.

References

[GP] Guillemin, V., Pollack, A., *Differential Topology*, Prentice Hall.

E-mail address: gt@ipm.ir