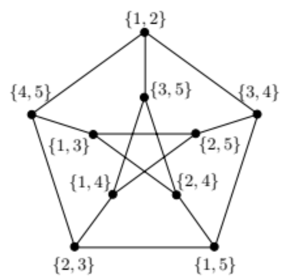
Differential Topology

Project
The Chromatic Number of Kneser Graphs

Let $k \leq n$ be two positive integers. We define the *Kneser Graph* associated to these parameters, which is denoted by $K(n,k)$ as follows. Consider all the k-element subsets of $\{1,2,\ldots,n\}$ as the vertices and connect two vertices corresponding to two subsets A and B with an edge if and only if $A \cap B = \emptyset$. Verify that there are exactly $\binom{n}{k}$ vertices. For example, the graph below is the Kneser graph for $n = 5$ and $k = 2$.



Obviously if $2k > n$ the graph has no edges, and so we assume $n \geq 2k$.

By a (vertex) coloring of a graph G, we mean associating colors to its vertices so that any two adjacent vertices have different colors and *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors so that such coloring exists.

a) Show that vertices of $K(n,k)$ can not be colored with less than $\frac{n}{k}$ number of colors ($n \geq 2k$).

b) Show that if $n = 2k + d$ for some integer $d \geq 0$, $K(n,k)$ has a coloring with $d + 2$ colors and so $\chi\big(K(n,k)\big) \leq d + 2$.

Now our goal is to show the reverse inequality $\chi\big(K(n,k)\big) \geq d+2$ (where $n = 2k + d \geq 2k$). For this purpose, suppose that the vertices of $K(n,k)$ are colored in $d + 1$ colors $c_1, c_2, \ldots, c_{d+1}$ and we want to show that there are two adjacect vertices of the same color (two disjoint k-element subsets of the same color).

Take $n = 2k+d$ points on the unit sphere $\mathbb{S}^{d+1} \subseteq \mathbb{R}^{d+2}$ in general position (i.e. no $d+2$ points of them lie on a proper linear subspace of $\mathbb{R}^{d+2}$). Label these points with $1, 2, \ldots, n$ arbitrarily. For any point $x \in \mathbb{S}^{d+1}$ let H_x be

the open hemishpere with pole x.
For any color c_i $(1 \leq i \leq d+1)$ define O_i to be the set of all points $x \in \mathbb{S}^{d+1}$ such that H_x contains k points with labels $t_1, \dots, t_k$ so that $\{t_1, \dots, t_k\}$ is colored by c_i.

c) Show that each O_i $(1 \leq i \leq d+1)$ is an open set of $\mathbb{S}^{d+1}$.

d) Show that $\mathbb{S}^{d+1} \setminus (O_1 \cup \cdots \cup O_{d+1})$ can not contain two antipodal points.

e) Use Borsuk-Ulam theorem to show that at least one of O_i's contains two antipodal points and show that this is in contradiction with the assumpition of coloring with $d+1$ colors. This finishes the proof $\chi\big(K(n,k)\big) = n-2k+2$.

References

[AZ] Martin Aigner, Günter M Ziegler, and Alfio Quarteroni. Proofs from the Book, volume 274. Springer, 2010.

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