Rotation theory III

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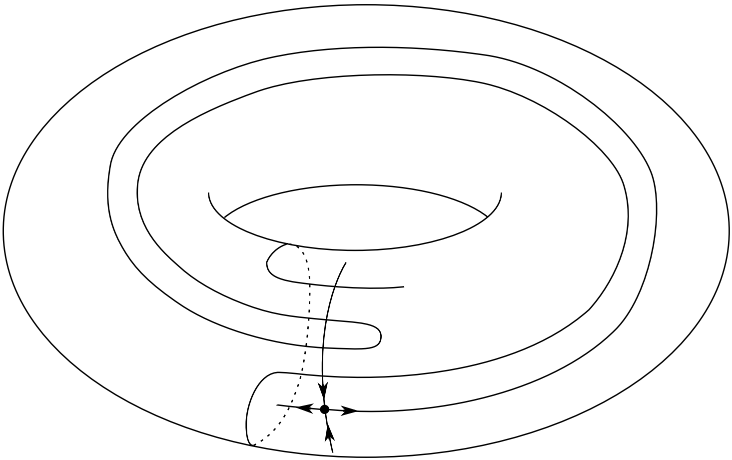
Rotation set

$f \in \mathrm{Homeo}_{\mathrm{Id}}(\mathbb{T}^2)$, $\quad \widetilde{f} \in \mathrm{Homeo}(\mathbb{R}^2)$ lift.

$$\rho(\widetilde{f}) = \left\{ \lim_{k\to\infty} \frac{\widetilde{f}^{n_k}(\widetilde{z}_k) - \widetilde{z}_k}{n_k} \right\} = \left\{ \lim_{k\to\infty} \frac{1}{n_k} \Delta^{n_k}(z_k) \right\}$$

- Compact, convex;
- equal to $\rho_m(\widetilde{f}) = \{\int_{\mathbb{T}^2} \Delta \, d\mu : \mu \in \mathcal{M}(f)\}$;
- Extremal elements always realized by ergodic measures;
- Rational elements realized by periodic points if <u>extremal</u> or <u>interior</u>;
- All interior elements realized by compact invariant sets;
- Interior $\implies$ entropy.

Example



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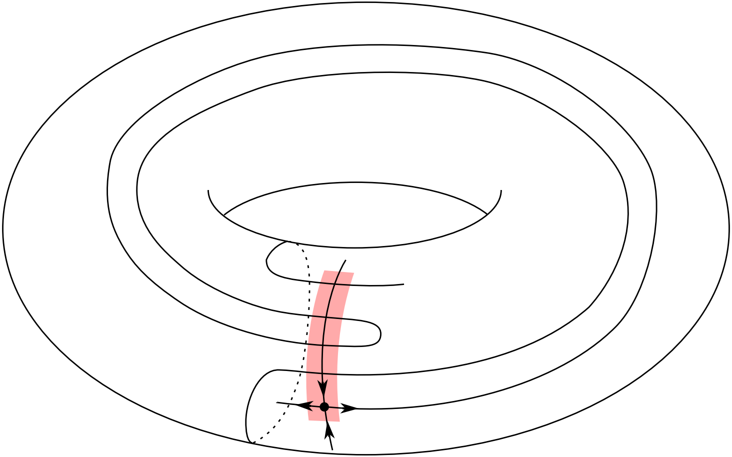

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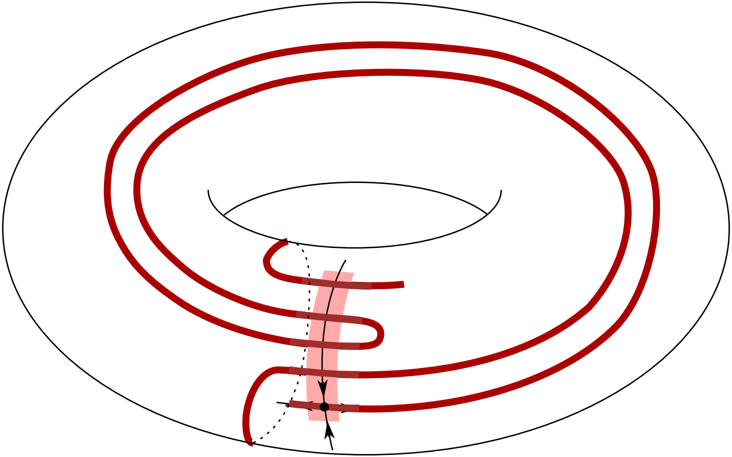

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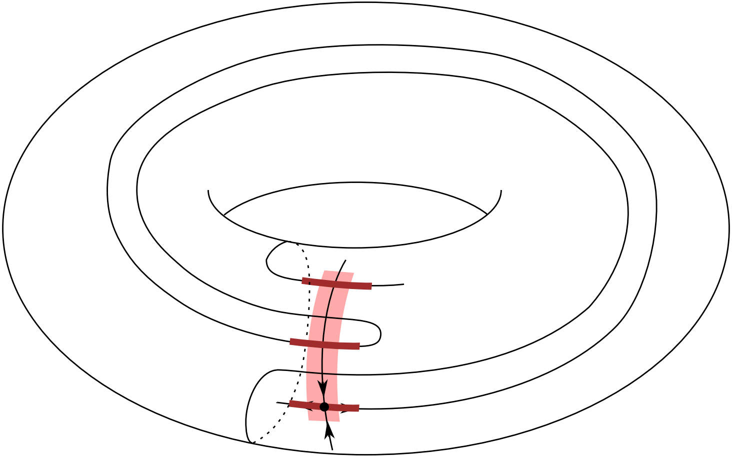

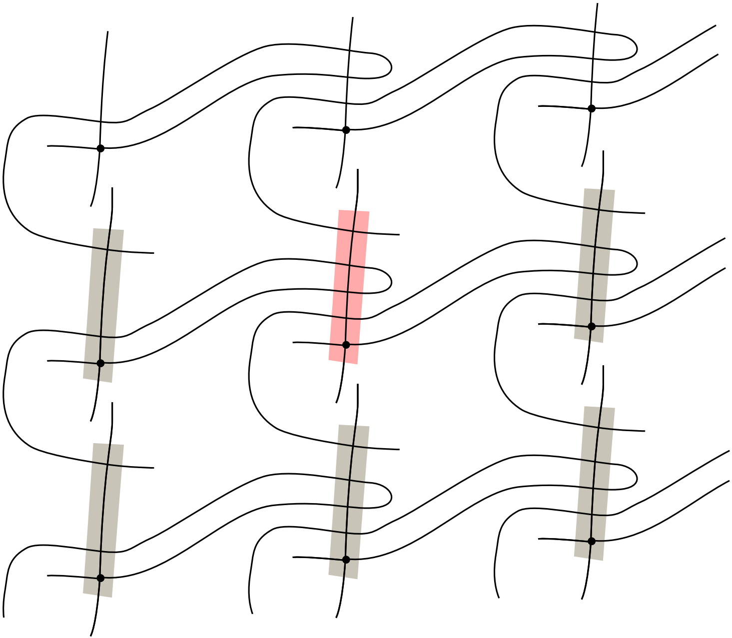

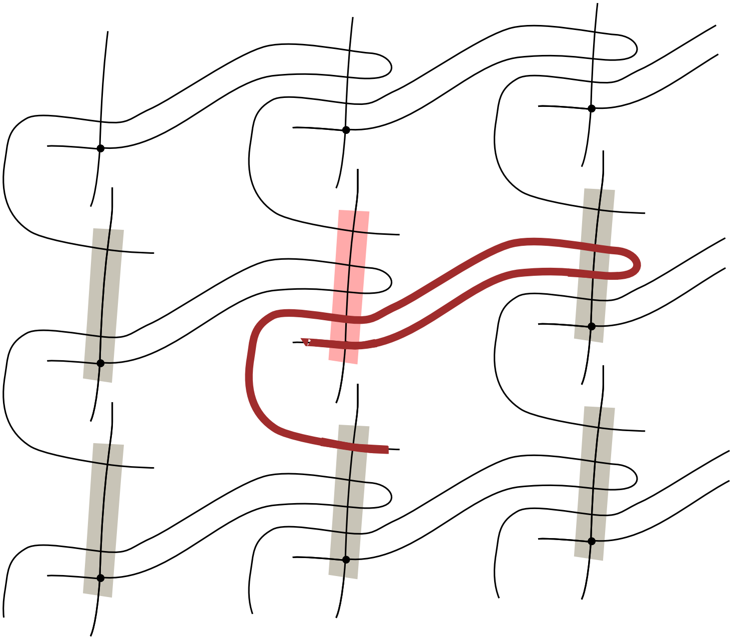

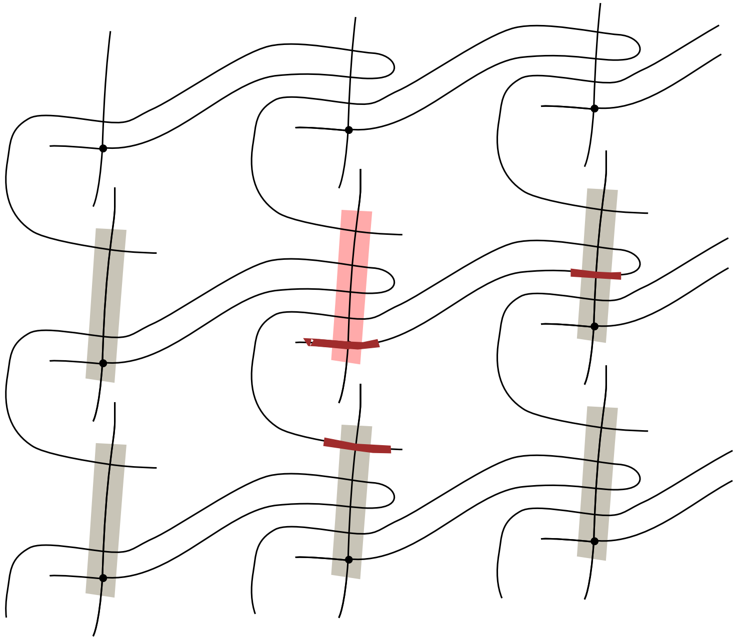

Rotation sets with nonempty interior

Theorem (Addas-Zanata 2013, 2015)

If f is $C^{1+\alpha}$ and $(0,0) \in \operatorname{int} \rho(\widetilde{f})$, then there exists a hyperbolic periodic point p for $\widetilde{f}$ such that for all $v \in \mathbb{Z}^2$, the stable manifold of p has a topologically transverse intersection with the unstable manifold of $p + v$ (and vice-versa).

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Theorem (de Carvalho, K., Tal)

If f is $C^{1+\alpha}$ and the rotation set has nonempty interior, then f is monotonely semiconjugate to a "model map" which is: transitive, with dense periodic points, continuum-wise expansive , and more.

Shapes of rotation sets with empty interior

Question

Which compact convex sets are realizable as rotation sets?

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Which compact convex sets are realizable as rotation sets?

With empty interior (intervals)

- Single points (rotations);
- intervals with rational slope containing rational points;
- intervals with irrational slope and one rational endpoint (Katok);

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Can an interval of rational slope without periodic points be realized?

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Some advances [K., Passeggi, Sambarino 2016], [Kocsard 2016].

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The set of extremal points is totally disconnected.

The only compact convex sets known to be non-realizable are those whose boundary contain an interval of irrational slope with a rational non-extremal point (Le Calvez, Tal 2016)

Stability of the rotation set

Continuity

$\widetilde{f} \mapsto \rho(\widetilde{f})$ is upper-semicontinuous (continuous when $\rho(\widetilde{f})$ has interior)

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The rotation set is stable if it does not change under small perturbations of the dynamics.

- Addas-Zanata 2004: C^0-stable $\implies$ rational extremal points;
- Guihéneuf 2016: Also C^1.
- Passeggi 2014: C^0-generically, stable + polygonal.
- Guihéneuf, K. 2016: Same thing area-preserving + estimates.

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We do not know any result about C^r-stability, or how to control the result even for C^0 perturbations. Related: C^r-enerically $\exists$ periodic point?

Mean rotation vectors in the area-preserving case

Let $\mu =$ Lebesgue measure on $\mathbb{T}^2$.

Lemma

If f, g are area-preserving, then $\rho(\widetilde{f}\widetilde{g}, \mu) = \rho(\widetilde{f}, \mu) + \rho(\widetilde{g}, \mu)$

Note: provides a group homomorphism $\mathrm{Diff}^r_\mu(\mathbb{T}^2) \to \mathbb{T}^2$.

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Theorem (Conley-Zehnder 83, Franks 88, Le Calvez 98)

If $\rho(\tilde{f}, \mu) = (0, 0)$ then $\tilde{f}$ has a fixed point (actually, 3, ess. different)

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Theorem

For area-preserving diffeomorphisms, C^r-generically the rotation set has nonempty interior (any r).

Proof: Use perturbations of the form $R_v \circ f$ with v small.

Rotational deviations: the case with nonempty interior

The rotation set measures "average" speed of rotation, but average rotation 0 does not mean "no rotation at all"; e. g.

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Theorem

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$$\forall n \in \mathbb{Z}, \quad \{\widetilde{f}^n(z) - z : z \in [0,1]^2\} = \Delta^n(\mathbb{T}^2) \subset B_M(n\rho(\widetilde{f})).$$

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- Dávalos 2014: rational polygons (BLC foliations, "forcing")
- Addas-Zanata 2015: $C^{1+\alpha}$ (Pesin theory, homoclinic intersections)
- Tal-Le Calvez 2016: general (BLC foliations, forcing theory)

Rotational deviations: the case with empty interior

Assume $\rho(\widetilde{f})$ has empty interior. If $\rho(\widetilde{f}) = \{v\}$ we call f a **pseudo-rotation**.

- $v \in \mathbb{R}^2 \backslash \mathbb{Q}^2$ (irrational pseudo-rotation).
  - Dynamics is aperiodc.
  - May be topologically weak-mixing, or even mixing (Kochergin)
  - May have unbounded rotational deviations (Kocsard, K., Jäger)
  - May have positive entropy (but not if f is smooth) (Rees, Katok)
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  - All of this may happen for area-preserving maps, minimal.
- $v = (p_1/q, p_2/q) \in \mathbb{Q}^2$ (rational pseudo-rotation)
  - Must have periodic points;
  - Interesting case: $v = (0, 0)$ (take $\widetilde{g} = \widetilde{f}^q - (p_1, p_2)$).
  - If $\rho(\widetilde{f}) = (0, 0)$ we say f is **irrotational**.
  - May have unbounded rotational deviations.
  - Katok's example.
  - Interesting case: f area-preserving.

Irrotational area-preserving homeomorphisms

Theorem (Lifted Poincaré recurrence) [K., Tal 2015]

If f is area-preserving and irrotational, then a.e. $z \in \mathbb{R}^2$ is $\widetilde{f}$-recurrent.

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If f is area-preserving and irrotational, then either $\mathrm{Fix}(f)$ is essential or the displacement is uniformly bounded: $\sup_{z \in \mathbb{T}^2, n \in \mathbb{Z}} \|\Delta^n(z)\| < \infty$.

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Question: Does the lifted Poincaré recurrence hold for irrotational area-preserving homeomorphisms of arbitrary surfaces?

Bounded deviations when $\rho(\widetilde{f})$ is an interval

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If $\rho(\widetilde{f})$ is an interval, are the rotational deviations bounded? In the direction perpendicular to $\rho(\widetilde{f})$?

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Theorem (Dávalos 2015)

Yes if $\rho(\widetilde{f})$ is an interval with rational slope intersecting $\mathbb{Q}^2$. (Model: Vertical interval through the origin.)

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Theorem (Kocsard, 2016)

Yes if f is minimal.

General case?

More on area-preserving homeomorphisms

From now on f is area-preserving.

Definition

f is irrotational if $\rho(\widetilde{f}) = \{(0,0)\}$ for some lift

Recall:

Theorem (Le Calvez, Tal 2016)

f is irrotational $\implies \mathrm{Fix}(f)$ essential or uniformly bounded displacement.

That is, if f is irrotational then either the fixed point set is <u>very large</u> or there is no rotation at all.

Irrotational example

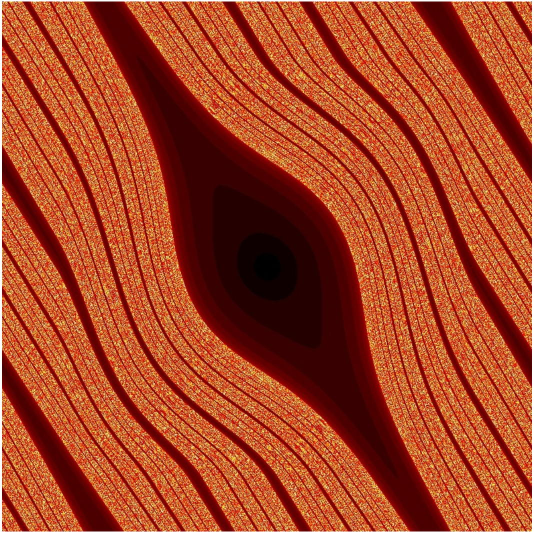
Irrotational diffeomorphisms with unbounded deviations (K, Tal 2013)

There exists a C^∞ Bernoulli ($\implies$ ergodic) diffeomorphism f with a lift $\widetilde{f}$ such that $\rho(\widetilde{f}) = \{(0,0)\}$ and the displacement is unbounded in all directions. More specifically, the orbit of almost every point intersects every fundamental domain in $\mathbb{R}^2$.

- Find an open topological disk U in $\mathbb{T}^2$ in a way that its lift to $\mathbb{R}^2$ intersects every fundamental domain.
- Choose a smooth ergodic diffeomorphism ϕ of the unit disk $\mathbb{D}$ which is the identity on $\partial\mathbb{D}$ and $\phi - \mathrm{Id}$ goes to 0 sufficiently fast near $\partial\mathbb{D}$ (Katok 1979).
- Extend as the identity on $\mathbb{T}^2 \backslash U$.
- Simpler example: blow up an orbit of a minimal flow on $\mathbb{T}^2$.

Note: $\mathrm{Fix}(f)$ **is huge!**.

Unbounded disk (with direction)



Invariant disks

General philosophy

If an open connected set U is invariant by an area-preserving homeomorphism, there are strong restrictions on the topology of ∂U (unless f has a "huge" set of fixed points).

In the area-preserving setting, connected open invariant (periodic) sets appear frequently: if U is open, the connected component of U in $\mathcal{O}_f(U) = \bigcup_{n\in\mathbb{Z}} f^n(U)$ is periodic. Also: KAM.

Bounded disks lemma

Recall: U inessential $\iff$ every loop in U is trivial in $\mathbb{T}^2$. An arbitrary set is inessential if it has an inessential neighborhood.

Covering diameter

For U open connected and inessential, $\mathcal{D}(U) = \operatorname{diam}(\widehat{U})$ where $\widehat{U}$ is a lift of U (= connected component of $\pi^{-1}(U)$).

Bounded disks lemma (K. Tal, 2014/17)

Suppose that f is area-preserving and $\operatorname{Fix}(f)$ is inessential. There exists $M > 0$ such that for any inessential open invariant connected set U one has $\mathcal{D}(U) \leqslant M$.

It holds on any surface. There is a version for non-simply connected sets.

Application: dynamically essential and inessential points

An open set $U \subset \mathbb{T}^2$ is **fully essential** in $\mathbb{T}^2$ if $\mathbb{T}^2 \backslash U$ is inessential.

Dynamically essential/inessential points

- $x \in \mathrm{Ine}(f)$ = **dynamically inessential** points if there is a neighborhood U of x such that $\mathcal{O}_f(U)$ is inessential in $\mathbb{T}^2$.
- $x \in \mathrm{Ess}(f)$ = **dynamically essential** points if $\mathcal{O}_f(U)$ is essential for every neighborhood U of x.
- $x \in \mathcal{C}(f)$ = **dynamically fully essential** points if $\mathcal{O}_f(U)$ is fully essential for every neighborhood U of x.

Area preserving $\implies$ every $x \in \mathrm{Ine}(f)$ belongs to a periodic open topdisk.

- $\mathrm{Ine}(f)$ is open invariant;
- $\mathrm{Ess}(f) = \mathbb{T}^2 \backslash \mathrm{Ine}(f)$ and $\mathcal{C}(f)$ are compact invariant.

Note: $\mathrm{Ine}(f)$ may be essential as a set, $\mathrm{Ess}(f)$ may be inessential.

Strictly toral dynamics

Theorem (K., Tal 2014)

If $\rho(\widetilde{f})$ has nonempty interior then:

- $\mathrm{Ine}(f)$ is a disjoint union of periodic homotopically bounded topological disks;
- $\mathrm{Ess}(f)$ is a fully essential continuum and $\mathcal{C}(f) = \mathrm{Ess}(f)$, and
  - $\mathcal{C}(f)$ is weakly syndetically transitive;
  - For any lift $\widetilde{f}$ of f and U neighborhood of $x \in \mathcal{C}(f)$, $\rho(\widetilde{f}, U) = \rho(\widetilde{f})$.
  - Every rotation vector realized by a periodic point or ergodic measure can be realized in $\mathcal{C}(f)$.

Moreover, $\mathcal{C}(f)$ is indecomposable.

There is a version for higher genus surfaces (K, Tal 2017).

Strictly toral dynamics