Mathematical Notes, vol. 79, no. 4, 2006, pp. 561–571.
Translated from Matematicheskie Zametki, vol. 79, no. 4, 2006, pp. 607–618.

Irrationality of the Sums of Zeta Values

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Received April 13, 2005

Abstract—In this paper, we establish a lower bound for the dimension of the vector spaces spanned over $\mathbb{Q}$ by 1 and the sums of the values of the Riemann zeta function at even and odd points. As a consequence, we obtain numerical results on the irrationality and linear independence of the sums of zeta values at even and odd points from a given interval of the positive integers.

Key words: *Riemann zeta function, sum of zeta values, irrationality, polylogarithm, Dirichlet beta function, Nesterenko test for linear independence, hypergeometric series.*

1. INTRODUCTION

Consider the polylogarithmic function defined for all integers $k \geq 1$ by the series

$$\operatorname{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \tag{1}$$

where $|z| < 1$ for $k = 1$ and $|z| \leq 1$ for $k \geq 2$. The linear independence of the values of the polylogarithms at rational points close to zero was studied in many works (see, for example, [1–4]). The problem of the linear independence of the values of the functions (1) at points close to the boundary of the disk of convergence, in particular, at the point $z = 1$ still remains open. The transcendence of π implies the transcendence and linear independence at the point $z = 1$ of the values of the polylogarithms with even numbers

$$\operatorname{Li}_{2k}(1) = \zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2 \cdot (2k)!} B_{2k};$$

here $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function and the $B_{2k} \in \mathbb{Q}$ are the Bernoulli numbers. The irrationality $\operatorname{Li}_3(1) = \zeta(3)$ was proved by Apéry in [5].

Using the Nesterenko test for linear independence [6] and Nikishin's construction from [1], Rivoal proved in [7] that, for any rational α, $|\alpha| < 1$, there exists infinitely many integers j such that $\operatorname{Li}_j(\alpha)$ is irrational. More recently, this result was generalized to the set of numbers [8]

$$\left\{ \lambda \operatorname{Li}_k(\alpha) + \mu \frac{\log^k(\alpha)}{(k-1)!} \;\middle|\; k \in \mathbb{N},\ \lambda, \mu \in \mathbb{R},\ (\lambda, \mu) \neq (0,0),\ \alpha \in \mathbb{Q},\ |\alpha| \leq 1 \right\},$$

which also contains infinitely many irrational numbers. Moreover, the appropriate symmetry of the rational function from works of Ball and Rivoal was used to prove [9] that, among the values of the Riemann zeta function at odd points, $\zeta(2k+1) = \operatorname{Li}_{2k+1}(1)$, $k \geq 1$, there are infinitely many irrational values and also to obtain the best quantitative result [10] concerning the irrationality of at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$. A similar approach was also used in [11] to study the values of the Dirichlet beta function

$$\beta(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k}$$

at even points $k \ge 2$.

In the present paper, we modify Rivoal's construction [9] and prove the following results.

Theorem 1. *Suppose that* λ_1, λ_2 *are arbitrary real numbers not all zero. Then, in each numerical set*

$$\{\lambda_1\zeta(2k) + 2k\lambda_2\zeta(2k+1),\ k \in \mathbb{N}\}, \qquad \{\lambda_1\zeta(2k+1) + (2k+1)\lambda_2\zeta(2k+2),\ k \in \mathbb{N}\},$$

there are infinitely many irrational numbers. More exactly, for the dimensions $\delta_1(a)$, $\delta_2(a)$ *of the vector spaces spanned over* $\mathbb{Q}$ *by the numbers*

$$\lambda_1,\ \lambda_2,\ \lambda_1\zeta(2k) + 2k\lambda_2\zeta(2k+1), \qquad k = 1, 2, \dots, \frac{a-1}{2},$$

and

$$\lambda_1,\ \lambda_2,\ \lambda_1\zeta(2k+1) + (2k+1)\lambda_2\zeta(2k+2), \qquad k = 1, 2, \dots, \frac{a-1}{2},$$

respectively, where a *is odd, the following estimates hold*:

$$\delta_1(a), \delta_2(a) \ge \frac{\log a}{1+\log 2}(1+o(1)) \qquad \text{as} \quad a \to \infty.$$

If, in Theorem 1, we set $\lambda_1 = 4\lambda\pi$, $\lambda_2 = 1$, for the first of the sets and $\lambda_1 = 1$, $\lambda_2 = \lambda/(2\pi)$, where $\lambda \in \mathbb{Q}$, for the second set, then we obtain the following assertion.

Corollary 1. *For any rational* λ, *in each of the numerical sets*

$$\left\{\zeta(2k+1) - \frac{\lambda(-1)^k 2^{2k} B_{2k}}{k(2k)!} \cdot \pi^{2k+1},\ k \in \mathbb{N}\right\},$$
$$\left\{\zeta(2k+1) - \frac{\lambda(-1)^k (2k+1) 2^{2k} B_{2k+2}}{(2k+2)!} \cdot \pi^{2k+1},\ k \in \mathbb{N}\right\},$$

there are infinitely many irrational numbers.

Theorem 2. *Suppose that* λ_1, λ_2 *are arbitrary rational numbers not all zero. Then, in each numerical collection*

$$\{\lambda_1\zeta(2k) + 2k\lambda_2\zeta(2k+1),\ k = 1, 2, \dots, 6\},$$
$$\{\lambda_1\zeta(2k+1) + (2k+1)\lambda_2\zeta(2k+2),\ k = 1, 2, \dots, 6\},$$

there is, at least, one irrational number.

Theorem 3. *For any irrational* λ, *there exists an even* a *and an odd number* b, $2 \le a, b \le 339$, *such that each of the triples of numbers*

$$1,\ \lambda,\ \zeta(a) + a\lambda\zeta(a+1) \quad \text{and} \quad 1,\ \lambda,\ \zeta(b) + b\lambda\zeta(b+1)$$

is linearly independent over $\mathbb{Q}$.

Immediately from Theorem 3 we obtain

Corollary 2. *For any irrational* λ, *in each numerical collection*

$$\{\lambda\zeta(a)+a\zeta(a+1),\ a=2,4,\dots,338\},\qquad \{\zeta(a)+a\lambda\zeta(a+1),\ a=2,4,\dots,338\}$$
$$\{\lambda\zeta(b)+b\zeta(b+1),\ b=3,5,\dots,339\},\qquad \{\zeta(b)+b\lambda\zeta(b+1),\ b=3,5,\dots,339\},$$

there is at least one irrational number.

Theorem 4. *For any rational* μ, *there exist natural numbers* c *and* d *of identical parity, where* $2\le c<d\le 339$, *such that the numbers*

$$1,\ \zeta(c)+c\mu\zeta(c+1),\ \zeta(d)+d\mu\zeta(d+1)$$

are linearly independent over $\mathbb{Q}$.

Concerning earlier results on the sums of zeta values, we note Gutnik's work [12], where it was proved that, for any rational λ, $\lambda\ne 0$, at least one of the numbers $3\zeta(3)+\lambda\zeta(2)$, $\zeta(2)+2\lambda\ln 2$ is irrational. The linear independence and irrationality of the values of the zeta function was also studied in [13], where, in particular, it was proved that there exist odd numbers $a_1\le 145$ and $a_2\le 1971$ such that the numbers $1,\zeta(3),\zeta(a_1),\zeta(a_2)$ are linearly independent over $\mathbb{Q}$.

2. ANALYTIC CONSTRUCTION

Suppose that $r\ge 1$, $a>4r$ are integers and n is a natural parameter. Let

$$R_n(t)=(n!)^{a-4r}\frac{(t-rn)_{rn}^2(t+n+1)_{rn}^2}{(t)_{n+1}^a}, \tag{2}$$

where $(\alpha)_k$ is the Pochhammer symbol:

$$(\alpha)_0=1 \qquad\text{and}\qquad (\alpha)_k=\alpha(\alpha+1)\cdots(\alpha+k-1)\quad\text{for}\ \ k\ge 1.$$

For complex z, $|z|\ge 1$, we define two series

$$E_{n,1}(z)=\sum_{t=1}^{\infty}R_n(t)z^{-t}, \tag{3}$$
$$E_{n,2}(z)=-\sum_{t=1}^{\infty}R_n'(t)z^{-t}, \tag{4}$$

which are absolutely convergent for $|z|\ge 1$, because

$$R_n(t)=O(t^{(4r-a)n-a}). \tag{5}$$

Let us expand the rational function $R_n(t)$ into the sum of partial fractions:

$$R_n(t)=\sum_{k=1}^{a}\sum_{j=0}^{n}\frac{A_{k,j,n}}{(t+j)^k}; \tag{6}$$

then we have

$$A_{k,j,n}=\frac{1}{(a-k)!}\left(\frac{d}{dt}\right)^{a-k}\big(R_n(t)(t+j)^a\big)\big|_{t=-j}.$$

Then we obtain the following representations for the series (3) and (4):

$$E_{n,1}(z) = \sum_{k=1}^{a} P_{k,n}(z)\,\mathrm{Li}_k(z^{-1}) - P_{0,n}(z), \tag{7}$$

$$E_{n,2}(z) = \sum_{k=1}^{a} kP_{k,n}(z)\,\mathrm{Li}_{k+1}(z^{-1}) - P_{-1,n}(z), \tag{8}$$

where

$$P_{k,n}(z) = \begin{cases} \displaystyle\sum_{j=0}^{n} A_{k,j,n} z^j & \text{if } 1 \le k \le a, \\ \displaystyle\sum_{m=1}^{a}\sum_{j=0}^{n}\sum_{l=1}^{j} \frac{A_{m,j,n} z^{j-l}}{m^k l^{m-k}} & \text{if } -1 \le k \le 0. \end{cases}$$

Note that $E_{n,1}(z)$ is the fully balanced hypergeometric series (see [14, p. 188 of the Russian translation]),

$$E_{n,1}(z) = z^{-rn-1}(n!)^{a-4r}\Gamma^2((2r+1)n+2)\left(\frac{\Gamma(rn+1)}{\Gamma((r+1)n+2)}\right)^{a+2}$$
$$\times\ {}_{a+4}F_{a+3}\left(\begin{matrix}(2r+1)n+2,\ (2r+1)n+2,\ rn+1,\ \dots,\ rn+1 \\ 1,\ (r+1)n+2,\ \dots,\ (r+1)n+2\end{matrix}\ \middle|\ z^{-1}\right),$$

which allows us to obtain the expansions (9) and (10) in the values of the zeta function of identical parity.

Lemma 1. *Suppose that a is odd. Then the following relations hold*:

$$\begin{cases} E_{n,1}(1) = \displaystyle\sum_{k=1}^{(a-1)/2} P_{2k,n}(1)\zeta(2k) - P_{0,n}(1), \\ E_{n,2}(1) = \displaystyle\sum_{k=1}^{(a-1)/2} 2kP_{2k,n}(1)\zeta(2k+1) - P_{-1,n}(1) \end{cases} \tag{9}$$

if n is odd and

$$\begin{cases} E_{n,1}(1) = \displaystyle\sum_{k=1}^{(a-1)/2} P_{2k+1,n}(1)\zeta(2k+1) - P_{0,n}(1), \\ E_{n,2}(1) = \displaystyle\sum_{k=1}^{(a-1)/2} (2k+1)P_{2k+1,n}(1)\zeta(2k+2) - P_{-1,n}(1) \end{cases} \tag{10}$$

if n is even.

Proof. Note that

$$P_{1,n}(1) = \sum_{j=0}^{n} A_{1,j,n} = \sum_{j=0}^{n} \operatorname*{Res}_{t=-j} R_n(t) = -\operatorname*{Res}_{t=\infty} R_n(t) = 0$$

by virtue of (5) and

$$\lim_{\substack{z\to 1\\ |z|>1}} P_{1,n}(z)\,\mathrm{Li}_1(z^{-1}) = 0.$$

Therefore, letting $z \to 1$ in (7) and (8), we obtain

$$E_{n,1}(1) = \sum_{k=2}^{a} P_{k,n}(1)\zeta(k) - P_{0,n}(1), \tag{11}$$

$$E_{n,2}(1) = \sum_{k=2}^{a} kP_{k,n}(1)\zeta(k+1) - P_{-1,n}(1). \tag{12}$$

It follows from Definition (2) that $R_n(-t-n) = (-1)^{a(n+1)}R_n(t)$, whence, taking the uniqueness of expansion (6) into account, we obtain

$$A_{k,n-j,n} = (-1)^{a(n+1)+k}A_{k,j,n}, \qquad k=1,\dots,a, \quad j=0,\dots,n.$$

Thus, the polynomials $P_{k,n}(z)$ at the point $z=1$ satisfy the equality

$$P_{k,n}(1) = (-1)^{a(n+1)+k}P_{k,n}(1), \qquad k=1,\dots,a,$$

and, therefore, $P_{k,n}(1) = 0$ if $a(n+1)+k$ is odd. The assertion of the lemma now follows from (11), (12). □

Lemma 2. *For all* $k=-1,0,\dots,a$, *the following inequality holds*:

$$\limsup_{n\to\infty} \frac{1}{n}\log|P_{k,n}(1)| \le (a-4r)\log 2 + (4r+2)\log(2r+1).$$

Proof. To prove the lemma, it suffices to find an upper bound for the coefficients $A_{k,j,n}$, for which, by Cauchy's formula, we have

$$A_{k,j,n} = \frac{1}{2\pi i}\int_{|u+j|=1/2} R_n(u)(u+j)^{k-1}\,du, \qquad k=1,\dots,a, \quad j=0,\dots,n.$$

Further, using estimates from [7, Lemma 3.4], we obtain the required inequality. □

Lemma 3. *For all* $k=-1,0,\dots,a$, *the following inclusions hold*:

$$D_n^{a-k}P_{k,n}(1) \in \mathbb{Z},$$

where $D_n = \mathrm{LCM}(1,2,\dots,n)$.

Proof. The assertion follows from the proof of Lemma 3.5 in [7] and the relation

$$R_n(t)(t+j)^a = \biggl(\prod_{m=1}^{r} F_m(t)\biggr)^2 \cdot \biggl(\prod_{m=1}^{r} G_m(t)\biggr)^2 \cdot H(t)^{a-4r},$$

where, in the notation of the lemma,

$$F_m(t) = \frac{(t-mn)_n}{(t)_{n+1}}(t+j), \qquad G_m(t) = \frac{(t+mn+1)_n}{(t)_{n+1}}(t+j), \qquad H(t) = \frac{n!}{(t)_{n+1}}(t+j). \quad □$$

3. ASYMPTOTICS OF THE LINEAR FORMS

Lemma 4. *For the sum* $E_{n,2}(1)$, *the following integral representation holds*:

$$E_{n,2}(1) = \frac{1}{2\pi i}\int_{X-i\infty}^{X+i\infty} R_n(t)\left(\frac{\pi}{\sin \pi t}\right)^2 dt$$
$$= \frac{(n!)^{a-4r}}{2\pi i}\int_{X-i\infty}^{X+i\infty} \frac{\Gamma^{a+2}(t)\Gamma^2(t+(r+1)n+1)\Gamma^2(rn+1-t)}{\Gamma^{a+2}(t+n+1)}\, dt, \qquad (13)$$

where X *is an arbitrary constant from the interval* $(0, rn+1)$.

Proof. The assertion of the lemma easily follows from the residue theorem, property (5) and the fact that $R_n(t)$ has zeros of second order at the points $1, 2, \dots, rn$ (see, for example, [15, Lemma 2]). $\square$

In what follows, we need the following auxiliary assertion.

Lemma 5. *Suppose that* $r \ge 1$ *and* $a > 4r$ *are integers and*

$$h(\tau) = \tau^{a+2}(\tau+r+1)^2 - (\tau+1)^{a+2}(\tau-r)^2.$$

Then the polynomial $h(\tau)$ *has exactly two positive roots* τ_1 *and* τ_2, *with* $0 < \tau_2 < r < \tau_1$.

Proof. Note that since $h(0) < 0$, $h(r) > 0$, and

$$h(\tau) = (4r-a)\tau^{a+3} + O(\tau^{a+2}) \qquad \text{as} \quad \tau \to \infty,$$

it follows that there exists at least one root on each of the intervals $(0, r)$ and $(r, +\infty)$; let us prove that that there is exactly one root on each of the intervals. First, consider the interval $(r, +\infty)$ and expand $h(\tau)$ in the product

$$h(\tau) = \big(\tau^{a/2+1}(\tau+r+1) + (\tau+1)^{a/2+1}(\tau-r)\big)\big(\tau^{a/2+1}(\tau+r+1) - (\tau+1)^{a/2+1}(\tau-r)\big);$$

hence all the roots $h(\tau)$ from $(r, +\infty)$ are solutions of the equation (and conversely)

$$\tau^{a/2+1}(\tau+r+1) - (\tau+1)^{a/2+1}(\tau-r) = 0.$$

Dividing both sides of this equation by $(\tau+1)^{a/2+2}$ and using the equality $1/(\tau+1) = 1-\tau/(\tau+1)$, we obtain

$$\left(\frac{\tau}{\tau+1}\right)^{a/2+1}\left(r+1-\frac{r\tau}{\tau+1}\right) - \left(\frac{(r+1)\tau}{\tau+1} - r\right) = 0. \qquad (14)$$

Consider the mapping $s = \tau/(\tau+1)$ which is one-to-one from the interval $(r, +\infty)$ to $(r/(r+1), 1)$ and from Eq. (14) to

$$H_1(s) = 0, \qquad \text{where} \quad H_1(s) = s^{a/2+1}(r+1-rs) - (r+1)s + r.$$

Note that

$$H_1(s) > 0 \qquad \text{on} \quad \left(0, \frac{r}{r+1}\right], \qquad (15)$$

$$H_1'(s) = -r\left(\frac{a}{2}+2\right)s^{a/2+1} + (r+1)\left(\frac{a}{2}+1\right)s^{a/2} - (r+1),$$

$$H_1''(s) = \left(\frac{a}{2}+1\right)s^{a/2-1}\left(\frac{a}{2}(r+1) - r\left(\frac{a}{2}+2\right)s\right) > 0 \qquad \text{on} \quad (0, 1).$$

Thus, $H_1'(s)$ is monotone increasing on the interval $(0,1)$ and $H_1'(0)=-r-1$, $H_1'(1)=(a-4r)/2$. Therefore, in view of (15) and the equality $H_1(1)=0$, we find that there exists a unique point $s_1\in(r/(r+1),1)$ in which $H_1(s_1)=0$. This point corresponds to the unique root

$$\tau_1=\frac{s_1}{1-s_1}\in(r,+\infty)$$

of the polynomial $h(\tau)$.

To study the interval $(0,r)$, we proceed in a similar way. In the same way as above, we find that it suffices to consider the equation

$$\tau^{a/2+1}(\tau+r+1)-(\tau+1)^{a/2+1}(r-\tau)=0,$$

which, by substituting $s=\tau/(\tau+1)$, can be reduced to the form

$$H_2(s)=0,\qquad s\in\left(0,\frac{r}{r+1}\right), \tag{16}$$

where $H_2(s)=s^{a/2+1}(r+1-rs)+(r+1)s-r$, $H_2(0)=-r$, $H_2(r/(r+1))>0$,

$$H_2'(s)=-r\left(\frac{a}{2}+2\right)s^{a/2+1}+(r+1)\left(\frac{a}{2}+1\right)s^{a/2}+(r+1),$$

and $H_2''(s)=H_1''(s)>0$ on $(0,1)$. Therefore, $H_2'(s)$ is monotone increasing on $(0,1)$, and because $H_2'(0)=r+1>0$, it follows that $H_2'(s)>0$ on $(0,1)$, and hence Eq. (16) has a unique root. Therefore, the polynomial $h(\tau)$ also has a unique root τ_2 in the interval $(0,r)$. The lemma is proved. □

Consider two functions

$$\begin{aligned} f_1(\tau)&=(a+2)\tau\log\tau+2(\tau+r+1)\log(\tau+r+1)\\ &\qquad-(a+2)(\tau+1)\log(\tau+1)-2(\tau-r)\log(\tau-r),\\ f_2(\tau)&=(a+2)\tau\log\tau+2(\tau+r+1)\log(\tau+r+1)\\ &\qquad-(a+2)(\tau+1)\log(\tau+1)-2(\tau-r)\log(r-\tau), \end{aligned} \tag{17}$$

defined in $\mathbb{C}\setminus(-\infty,r]$ and $\mathbb{C}\setminus\{(-\infty,0]\cup[r,+\infty)\}$, respectively, and fix the principal branches of the logarithms. Note that $f_1(\tau)$ and $f_2(\tau)$ regarded as functions of a real variable, are continuous on the corresponding intervals of the real axis $(r,+\infty)$ and $(0,r)$ and have a removable singularity at the point $\tau=r$, because

$$L=\lim_{\substack{\tau\to r\\ \tau>r}}f_1(\tau)=\lim_{\substack{\tau\to r\\ \tau<r}}f_2(\tau)=(a+2)r\log r+2(2r+1)\log(2r+1)-(a+2)(r+1)\log(r+1).$$

Therefore, their definition can be extended to include the point $\tau=r$ by setting

$$f_1(r)=f_2(r)=L.$$

Lemma 6. *The following asymptotic formulas hold as $n\to\infty$:*

$$\lim_{n\to\infty}\frac{\log|E_{n,i}(1)|}{n}=f_i(\tau_i),\qquad i=1,2,$$

where

$$f_2(\tau_2)<f_2(r)=f_1(r)<f_1(\tau_1)<2(r+1)\log2-(a-4r)\log(r+1),$$

and the numbers τ_1,τ_2 are defined in Lemma 5.

Proof. For the linear form $E_{n,1}(1)$, we have

$$E_{n,1}(1)=\sum_{t=rn+1}^{\infty}R_n(t)=(n!)^{a-4r}\sum_{t=rn+1}^{\infty}\frac{\Gamma^{a+2}(t)\Gamma^2(t+(r+1)n+1)}{\Gamma^{a+2}(t+n+1)\Gamma^2(t-rn)}.$$

Further, applying to the last sum the second method of the proof of Lemma 3 from [16], we obtain

$$\lim_{n\to\infty}\frac{\log|E_{n,1}(1)|}{n}=\max_{\tau\in(r,+\infty)}f_1(\tau).$$

With the derivative of the function $f_1(\tau)$,

$$f_1'(\tau)=(a+2)\log\tau+2\log(\tau+r+1)-(a+2)\log(\tau+1)-2\log(\tau-r),$$

we associate the polynomial $h(\tau)$ defined in Lemma 5. The polynomial $h(\tau)$ has exactly one root $\tau_1\in(r,+\infty)$; besides, $h(r)>0$, $h(+\infty)<0$, and hence $h(\tau)$ changes sign from "+" to "−" on $(r,+\infty)$ only once and, thus, τ_1 is the unique maximum point of the function $f_1(\tau)$ on $(r,+\infty)$. Finally, we obtain

$$\lim_{n\to\infty}\frac{\log|E_{n,1}(1)|}{n}=f_1(\tau_1)>f_1(r).$$

Further, for

$$f_1(\tau_1)=\tau_1\cdot f_1'(\tau_1)+\log\frac{(\tau_1+r+1)^{2r+2}(\tau_1-r)^{2r}}{(\tau_1+1)^{a+2}},$$

taking $\tau_1>r$ into account, we find that

$$\begin{aligned}\frac{(\tau_1+r+1)^{2r+2}(\tau_1-r)^{2r}}{(\tau_1+1)^{a+2}}&=\frac{\bigl(1+r/(\tau_1+1)\bigr)^{2r+2}\bigl(1-(r+1)/(\tau_1+1)\bigr)^{2r}}{(\tau_1+1)^{a-4r}}\\&<\frac{\bigl(1+r/(r+1)\bigr)^{2r+2}}{(r+1)^{a-4r}}<\frac{2^{2(r+1)}}{(r+1)^{a-4r}}\end{aligned}$$

and, therefore,

$$f_1(\tau_1)<2(r+1)\log2-(a-4r)\log(r+1).$$

Let us calculate the asymptotics of the linear form $E_{n,2}(1)$. To do this, in the integral (13), put $t=n\tau$, where $n\in\mathbb{N}$, $\tau=\tau_2+iy$, $y\in(-\infty,+\infty)$, the number $\tau_2\in(0,r)$ is defined in Lemma 5, and we use the asymptotics of the Γ-function:

$$\log\Gamma(u)=\left(u-\frac12\right)\log u-u+\log\sqrt{2\pi}+r(u),\qquad |r(u)|\le K|\operatorname{Re}u|^{-1},$$

where K is an absolute constant. Then we find that the integral (13) can be reduced to the form

$$E_{n,2}(1)=\frac{c}{n^{a/2+2r}}\int_{\tau_2-i\infty}^{\tau_2+i\infty}\frac{(r-\tau)(r+1+\tau)}{(\tau(\tau+1))^{a/2+1}}e^{nf_2(\tau)}\bigl(1+O(n^{-1})\bigr)\,d\tau, \tag{18}$$

where the function $f_2(\tau)$ is defined in (17), $c=c(a,r)$ is a complex constant distinct from zero and the constant in $O(\,\cdot\,)$ is absolute. Let us apply the saddle-point method to the integral (18). Note that $f_2'(\tau_2)=0$ by Lemma 5. Let us show that, for $v\in(-\infty,+\infty)$, the function $\operatorname{Re}f_2(\tau_2+iv)$ attains its maximum at the unique point $v=0$. For $\tau=\tau_2+iv$, $v\in(-\infty,+\infty)$, we obtain

$$\begin{aligned}\frac{d}{dv}\operatorname{Re}f_2(\tau_2+iv)&=-\operatorname{Im}\frac{d}{d\tau}f_2(\tau_2+iv)\\&=(a+2)\arg(\tau+1)-(a+2)\arg\tau+2\arg(r-\tau)-2\arg(\tau+r+1).\end{aligned} \tag{19}$$

On the contour of integration for $v > 0$, we have $\arg(\tau+1) < \arg\tau$, $\arg(\tau+r+1) > 0$, and since $\operatorname{Re}(r-\tau) > 0$, $\operatorname{Im}(r-\tau) < 0$, it follows that $\arg(r-\tau) < 0$. Hence

$$\frac{d}{dv}\operatorname{Re} f_2(\tau_2 + iv) < 0 \qquad \text{for} \quad v > 0.$$

Since (19) is odd in the variable v, we obtain

$$\frac{d}{dv}\operatorname{Re} f_2(\tau_2 + iv) > 0 \qquad \text{for} \quad v < 0$$

and, therefore, $v = 0$ is the unique maximum point of the function $\operatorname{Re} f_2(\tau_2 + iv)$ on the vertical line $(\tau_2 - i\infty, \tau_2 + i\infty)$. Also, note that

$$f_2''(\tau) = \frac{a+2}{\tau(\tau+1)} + \frac{2(2r+1)}{(\tau+r+1)(r-\tau)} > 0 \qquad \text{on} \quad (0, r).$$

Thus, as $n \to \infty$, using the saddle-point method, we obtain

$$E_{n,2}(1) = \frac{c}{n^{a/2+r}} \frac{(r-\tau_2)(r+1+\tau_2)}{(\tau_2(\tau_2+1))^{a/2+1}} e^{nf_2(\tau_2)} |f_2''(\tau_2)|^{-1/2}\big(1 + O(n^{-1})\big);$$

this implies the required asymptotics $\lim_{n\to\infty} \log|E_{n,2}(1)|/n = f_2(\tau_2)$. Let us now prove that $f_2(\tau_2) < f_2(r)$. By Lemma 5, τ_2 is the unique root of the polynomial $h(\tau)$ in the interval $(0, r)$, $h(0) < 0$ and $h(r) > 0$. Therefore, $h(\tau)$ only once changes sign from "$-$" to "$+$" on $(0, r)$. Taking the signs into account, we obtain $f_2(\tau_2) = \min_{\tau\in(0,r)} f_2(\tau) < f_2(r) = f_1(r)$, and the lemma is proved. □

4. PROOF OF THE THEOREMS

The proof of Theorem 1–4 is based on the Nesterenko test for linear independence [6] given below.

Test for linear independence. *Suppose that, for a given collection of real numbers $\theta_0, \dots, \theta_m$, $m \ge 1$, there exists a sequence of linear forms*

$$E_n = A_{0,n}\theta_0 + A_{1,n}\theta_1 + \dots + A_{m,n}\theta_m, \qquad n = 1, 2, \dots,$$

with integer coefficients and numbers $\alpha > 0$, $\beta > 0$ such that

$$\log|E_n| = -n\alpha + o(n), \quad \log\max_{0\le j\le m} |A_{j,n}| \le n\beta + o(n) \qquad \text{as} \quad n \to \infty.$$

Then

$$\dim_{\mathbb{Q}}(\mathbb{Q}\theta_0 + \mathbb{Q}\theta_1 + \dots + \mathbb{Q}\theta_m) \ge 1 + \frac{\alpha}{\beta}.$$

Proof of Theorem 1. Suppose that a is an odd integer, $a \ge 5$, and $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1^2 + \lambda_2^2 \ne 0$. Denote

$$\delta_1(a) = \dim_{\mathbb{Q}}\Bigg(\mathbb{Q}\lambda_1 + \mathbb{Q}\lambda_2 + \sum_{k=1}^{(a-1)/2} \mathbb{Q}\big(\lambda_1\zeta(2k) + 2k\lambda_2\zeta(2k+1)\big)\Bigg),$$

$$\delta_2(a) = \dim_{\mathbb{Q}}\Bigg(\mathbb{Q}\lambda_1 + \mathbb{Q}\lambda_2 + \sum_{k=1}^{(a-1)/2} \mathbb{Q}\big(\lambda_1\zeta(2k+1) + (2k+1)\lambda_2\zeta(2k+2)\big)\Bigg).$$

For $n \in \mathbb{N}$, define

$$l_n = D_{2n+1}^{a+1}\big(\lambda_1 E_{2n+1,1}(1) + \lambda_2 E_{2n+1,2}(1)\big),$$
$$\tilde{l}_n = D_{2n}^{a+1}\big(\lambda_1 E_{2n,1}(1) + \lambda_2 E_{2n,2}(1)\big),$$

as well as

$$p_{k,n} = D_{2n+1}^{a+1} P_{2k,2n+1}(1), \quad k = 0, 1, \dots, \frac{a-1}{2}, \qquad p_{-1,n} = D_{2n+1}^{a+1} P_{-1,2n+1}(1),$$
$$\tilde{p}_{k,n} = D_{2n}^{a+1} P_{2k+1,2n}(1), \quad k = -1, 1, \dots, \frac{a-1}{2}, \qquad \tilde{p}_{0,n} = D_{2n}^{a+1} P_{0,2n}(1).$$

Then, by Lemmas 1 and 3, it follows that l_n and $\tilde{l}_n$ are linear forms with integer coefficients in the sums of zeta values:

$$l_n = \sum_{k=1}^{(a-1)/2} p_{k,n}\big(\lambda_1\zeta(2k) + 2k\lambda_2\zeta(2k+1)\big) - p_{0,n}\lambda_1 - p_{-1,n}\lambda_2, \tag{20}$$

$$\tilde{l}_n = \sum_{k=1}^{(a-1)/2} \tilde{p}_{k,n}\big(\lambda_1\zeta(2k+1) + (2k+1)\lambda_2\zeta(2k+2)\big) - p_{0,n}\lambda_1 - p_{-1,n}\lambda_2. \tag{21}$$

Further, applying the test for linear independence with

$$m = (a+1)/2, \qquad \alpha = -2(a+1) - 2f_1(\tau_1), \qquad \beta = 2(a+1) + 2(a-4r)\log 2 + 2(4r+2)\log(2r+1)$$

to the forms (20), (21), we obtain the estimates

$$\delta_1(a), \delta_2(a) \ge 1 + \frac{\alpha}{\beta} = 1 + \frac{-a-1-f_1(\tau_1)}{a+1+(a-4r)\log 2 + (4r+2)\log(2r+1)}$$
$$\ge \frac{\log r + ((a-2r)/(a+2))\log 2}{1 + \log 2 + ((4r+2)/(a+2))\log(r+1)}. \tag{22}$$

Next, setting $r = [a/\log^2 a]$, as $a \to \infty$, we find

$$\log r + \frac{a-2r}{a+2}\log 2 = \big(1 + o(1)\big)\log a, \qquad 1 + \log 2 + \frac{4r+2}{a+2}\log(r+1) = 1 + \log 2 + o(1).$$

Hence we have

$$\delta_1(a), \delta_2(a) \ge \frac{\log a}{1 + \log 2}(1 + o(1)) \qquad \text{as} \quad a \to \infty,$$

and the theorem is proved. □

Proof of Theorem 2. Choosing $a = 13$, $r = 1$, we find that

$$\tau_1 \approx 1.0178067, \qquad f_1(\tau_1) \approx -14.16840167 < -a - 1;$$

therefore, by inequality (22), we have $\delta_1(13), \delta_2(13) > 1$, i.e., $\delta_1(13), \delta_2(13) \ge 2$, and the theorem is proved. □

Proof of Theorem 3. Setting $a = 339$, $r = 11$, we find that

$$\tau_1 \approx 11.00000829, \qquad f_1(\tau_1) \approx -1029.5000921,$$

and $\delta_1(339), \delta_2(339) > 2.001$ by inequality (22). Therefore, $\delta_1(399), \delta_2(399) \ge 3$, and the theorem now follows from (20), (21) if we set $\lambda_1 = 1$, $\lambda_2 = \lambda$. □

Proof of Theorem 4. The proof follows from that of Theorem 3 and (20), (21) if we set $\lambda_1 = 1$, $\lambda_2 = \mu$. □

ACKNOWLEDGMENTS

This research was supported in part by the Institute for Studies in Theoretical Physics and Mathematics of Iran under grants no. 84110026 (the first author) and no. 84110027 (the second author).

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