Variational Principles for Nonlinear Eigenvalue Problems

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Variational principles are very powerful tools when studying self-adjoint linear operators on a Hilbert space $\mathcal{H}$. Bounds for eigenvalues, comparison theorems, interlacing results and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few.

The purpose of this talk is to survey generalizations of Rayleigh's principle, of the minmax characterization of Poincaré and the maxmin characterization of Courant, Fischer and Weyl to the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{1}$$

and to trace the history of these generalizations. Here $T(\lambda)$, $\lambda \in J$, is a family of linear self-adjoint and bounded operator on a Hilbert space $\mathcal{H}$, and J is a real open interval which may be unbounded. As in the linear case $T(\lambda) := \lambda I - A$ we call $\lambda \in J$ an eigenvalue of $T(\cdot)$ if equation (1) has a nontrivial solution $x \neq 0$ and the solution x is called a corresponding eigenelement.

We demonstrate the benefits of these characterization in several applications

- Sylvesters law of inertia for nonlinear problems
- Nonlinear low rank modification of a symmetric eigenvalue problem
- Safeguarded iteration for computing particular eigenvalues
- Detecting hyperbolic matrix polynomials
- Free vibrations of fluid-solid structures
- Regularization of total least squares problems

References

[1] R.J. Duffin: A minimax theory for overdamped networks. J. Rat. Mech. Anal. 4, 221-233 (1955)

[2] K.P. Hadeler: Variationsprinzipien bei nichtlinearen Eigenwertaufgaben. Arch. Ration. Mech. Anal. 30, 297-307 (1068)

[3] A.S. Markus: Introduction to the Spectral Theory of Polynomial Operator Pencils. Translations of Mathematical Monographs 71. AMS, Providence 1988

[4] E.H. Rogers: A minimax theory for overdamped systems. Arch. Ration. Mech. Anal. 16, 89-96 (1964)

[5] H. Voss: A maxmin principle for nonlinear eigenvalue problems with application to a rational spectral problem in fluid–solid vibration. Appl. Math. 48, 607-622 (2003)

[6] H. Voss: A minmax principle for nonlinear eigenproblems depending continuously on the eigenparameter. Numer. Lin. Algebra Appl. 16, 899-913 (2009)

[7] H. Voss &B. Werner: A minimax principle for nonlinear eigenvalue problems with applications to nonoverdamped systems. Math. Meth. Appl. Sci. 4, 415-424 (1982)