Pseudofiniteness and measurability of the everywhere infinite forest

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Pseudofinite structures

Pseudofinite structures

Definition

An $\mathcal{L}$-structure M is said to be pseudofinite if any of the following equivalent properties holds:

- *Every $\mathcal{L}$-sentence σ that is true in M, is also satisfied in some finite $\mathcal{L}$-structure M_0^σ.*
- $M \models \mathrm{FIN}_{\mathcal{L}}$.
- *M is elementarily equivalent to an ultraproduct $\prod_{\mathcal{U}} M_i$ of finite $\mathcal{L}$-structures.*

Observation: An ultraproduct of finite structures can only be finite or of size $2^{\aleph_0}$. Thus, the last condition allows us to describe structures that are "similar" to ultraproducts of finite structures, but have different cardinalities (for example, can be countable).

Examples of structures that are not pseudofinite

- The linear orders $(\mathbb{Q}, <), (\mathbb{Z}, <)$ are not pseudofinite.
- The field $(\mathbb{C}, +, \cdot)$ is not pseudofinite: the function $f(x) = x^2$ is definable and surjective, but not injective. Hence $(\mathbb{C}, +, \cdot) \models \forall y \exists x (x^2 = y) \wedge \exists x, y (x \neq y \wedge x^2 = y^2)$, but this cannot be true in any finite field.
- $(\mathbb{Z}, +)$ is not pseudofinite: the function $x \mapsto x + x$ is injective, but not surjective.

Examples of structures that are pseudofinite

- Every ultraproduct of finite $\mathcal{L}$-structures is pseudofinite.
- Pseudofinite fields:

Theorem (James Ax, 1968)

An infinite field K is pseudofinite if and only if it satisfies the following conditions:

1. K is perfect.
2. K has a unique extension of degree n for each $n \in \mathbb{N}$.
3. K is *pseudo-algebraically closed* every absolutely irreducible variety over K has a K-rational point.

- Vector spaces over $\mathbb{F}_p$ are pseudofinite: we can simply take $\prod_{\mathcal{U}} \mathbb{F}_p^n$.
- The group $(\mathbb{R}, +)$ is isomorphic to $\prod_{\mathcal{U}}(\mathbb{Z}/p\mathbb{Z}, +)$: both are torsion-free divisible abelian groups of cardinality $2^{\aleph_0}$.
- Vector spaces over $\mathbb{Q}$ are pseudofinite in the language $\mathcal{L}_{vs}$.

The random graph

Theorem (Erdős, Rényi - 1963)

Given a fix number $r \geq 1$, $lim_{n\to\infty}\mathbb{P}(\mathbb{G}(n,p) \models \mathcal{A}_r) = 1$. In particular, for each finite graph H, $lim_{n\to\infty}\mathbb{P}(H \text{ is a subgraph } \mathbb{G}(n,p)) = 1$.

Theorem (Bollobás, Thomason - 1985)

Let U, W be disjoint subsets of $\mathbb{F}_q$ ($q \equiv 1 \pmod 4$), such that $|U \cup W| = m$, and let S be the set of non-zero squares in $\mathbb{F}_q$. Let $v(U,W)$ be the set of elements $x \in \mathbb{F}_q$ such that $x - U \subseteq S$ and $x - W \subseteq \mathbb{F}_q \setminus S$.
Then, $||v(U,W)| - \frac{q}{2^m}| \leq \frac{1}{2}(m - 2 + 2^{-m+1})q^{\frac{1}{2}} + \frac{m}{2}$.

Corollary

Every infinite ultraproduct of the Paley graphs $\{P_q : q \equiv 1 (\mathrm{mod}\, 4)\}$ satisfies $\prod_{\mathcal{U}} P_q \models \mathcal{A}_r$ for every r, hence $\prod_{\mathcal{U}} P_q \models$ RG .

Theories of tree-like graphs

Definition

A tree is a (simple) graph without cycles. This property can be axiomatized in the language of graphs $\mathcal{L} = \{R\}$ by the theory:

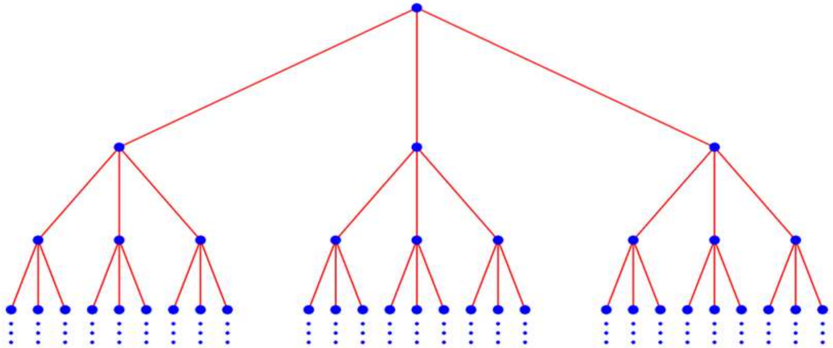
$$\mathrm{Tree} = \{\forall x(\neg xRx), \forall x, y(xRy \to yRx)\}$$
$$\cup \left\{ \neg\exists x_1, \ldots, x_n \left(\bigwedge_{1\leq i<j\leq n} (x_i \neq x_j) \wedge \bigwedge_{i=1}^{n-1} (x_i R x_{i+1}) \wedge x_n R x_1 \right) : n \geq 2 \right\}.$$

Question (may be too wide)

- *(May be too wide) Which kind of infinite trees are pseudofinite?*
- *(perhaps less wide)Is every infinite tree of bounded diameter pseudofinite?*

Pseudofiniteness in countable trees

Example of a countable tree that is not pseudofinite.



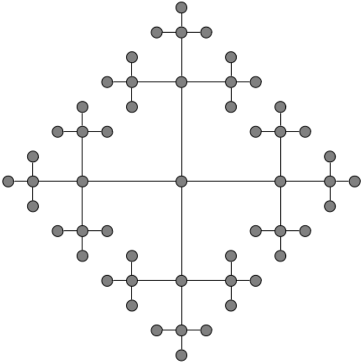
$$\sigma_{(1;3,4)} := \exists x\, [\deg(x) = 3 \wedge \forall y (y \neq x \rightarrow \deg(y) = 4)]$$

This sentence does not have finite models, due to the *Handshaking lemma*:

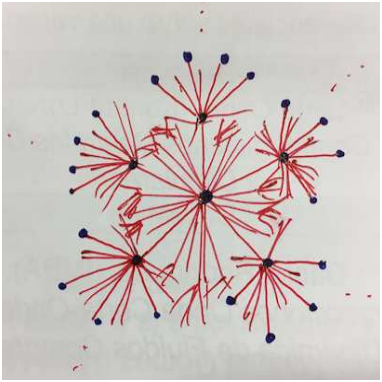
$$\sum_{v \in V} \deg(v) = 2|E(G)|.$$

The r-regular and the everywhere infinite forest

The theory T_r is the theory of an infinite tree such that every vertex has degree r. The theory T_∞ (also known as the theory of the everywhere infinite forest) is the theory of an infinite tree in which every vertex has infinite degree.



T_4



T_∞ (artistic representation)

Basic properties of T_r and T_∞

- Both $T_r = \mathrm{Tree} \cup \{\forall x\, \exists^{=r} y(xRy)\}$ and $T_\infty = \mathrm{Tree} \cup \{\forall x\, \exists^{\geq n} : n \geq 1\}$ are complete theories, and have quantifier elimination in the language $\mathcal{L}' = \{D_n : n \geq 0\}$, where $D_n(x,y) \Leftrightarrow \mathrm{dist}(x,y) = n$.
- The theory T_r is strongly minimal. Moreover, for every $M \models T_r$ and $A \subseteq M$,

$$\begin{aligned}\mathrm{acl}_M(A) &= \bigcup_{a \in A} \mathrm{acl}_M(a) \\ &= \text{connected components containing vertices from A}\end{aligned}$$

- The theory T_∞ is ω-stable of Morley rank ω. In fact,

$$\mathrm{RM}(D_n(x,a)) = n$$

Proposition

Let $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ be a class of finite graphs such that:

(a) Each graph G_n is r-regular (resp. d_n-regular)

(b) $\mathrm{girth}(G_n) \to \infty$

Then, every infinite ultraproduct M of graphs in $\mathcal{C}$ is a model of T_r (resp. a model of T_∞ if $d_n \to \infty$.

Theorem (G., Robles)

Such classes of finite graphs exists, therefore the theories T_r and T_∞ are both pseudofinite.

Why study pseudofinite structures?

- The idea is that the counting measure on a class of finite structures can be lifted using Łoś' theorem to give notions of dimension and measure on their ultraproduct.
- This kind of finite/infinite connection can sometimes be used to prove qualitative properties of large finite structures, and in the other direction, quantitative properties in the finite structures often induce desirable qualitative properties in their ultraproducts.
  - Szemerédi's Regularity (Goldbring, Towsner)
  - Freiman conjecture for non-abelian groups (Hrushovski)
  - Expanders maps in finite fields (Tao)
  - Stable graphs and Erdős-Hajnal conjecture (Malliaris, Shelah / Chernikov, Starchenko)

Counting pairs and counting measures

- L will denote a first-order language.
- $\mathcal{C} = \{M_i : i \in I\}$ will denote a class of finite L-structures, and $\mathcal{U}$ will denote an ultrafilter on I.
- We will assume that for every $n \in \mathbb{N}$, $\{i \in I : |M_i| \geq n\} \in \mathcal{U}$. That is, $\lim_{i \to \mathcal{U}} |M_i| = \infty$.
- We enrich L to a 2-sorted language L^+ with sorts $\mathbb{D}$ (carrying the language L) and $\mathbb{OF}$ (carrying the language of ordered fields). Also, for every L-formula $\phi(\overline{x}, \overline{y})$, a function symbol $f_\phi : \mathbb{D}^{|\overline{y}|} \to \mathbb{OF}$.
- Every finite L-structure $M_i \in \mathcal{C}$ gives rise to an L^+-structure $K_i = \langle M_i; (\mathbb{R}, +, \cdot, 0, 1, <)\rangle$ with the functions f_ϕ interpreted as:

$$\begin{aligned} f_\phi^{K_i} :\ M_i^{|\overline{y}|} &\longrightarrow \mathbb{R} \\ \overline{b} &\longmapsto \left|\phi(M_i^{|x|}, \overline{b})\right| \end{aligned}$$

Counting pairs and counting measures

- Consider now the ultraproduct of the structures K_i with respect to $\mathcal{U}$,

$$K := \prod_{i \in \mathcal{U}} K_i = \left(\prod_{i \in \mathcal{U}} M_i, \mathbb{R}^* \right).$$

- We put $M := \prod_{i \in \mathcal{U}} M_i$, and $T = \mathrm{Th}(M)$.
- If $X = \phi(M^r; \overline{b})$ is an L-definable set in M, $|X| := f_\phi(\overline{b})$ will denote its (non-standard) cardinality.
- *Counting measures:* For a non-empty definable subset D of M, there is a finitely-additive real valued probability measure μ_D defined as:

$$\mu_D(X) := \mathrm{st}\left(\frac{|X \cap D|}{|D|} \right) = \mathrm{st}\left(\lim_{i \to \mathcal{U}} \frac{|X(M_i) \cap D(M_i)|}{|D(M_i)|} \right)$$

Strongly minimal ultraproducts of finite structures

Theorem (Pillay, 2015)

Let $M = \prod_{\mathcal{U}} M_i$ be a **strongly minimal** ultraproduct of finite structures, and let $\alpha \in \mathbb{N}^*$ be the pseudofinite cardinality of M ($\alpha = |M|$). Then,

1. For any definable (with parameters) set $X \subseteq M^n$, there is a polynomial $P_X(x) \in \mathbb{Z}[x]$ with positive leading coefficient such that $|X| = P_X(\alpha)$. Moreover, $RM(X) = \text{degree}(P_X)$.
2. For any L-formula $\phi(\overline{x}, \overline{y})$ there is a finite number of polynomials $P_1, \ldots, P_k \in \mathbb{Z}[x]$ and L-formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that:
   (a) $\{\psi_i(\overline{y}) : i \leq k\}$ is a partition of the $\overline{y}$-space.
   (b) For any $\overline{a}$, $|\phi(M^{|x|}; \overline{a})| = P_i(\alpha)$ if and only if $M \models \psi_i(\overline{a})$.

For instance, if $M = (\mathbb{R}, +) = \prod_{\mathcal{U}} (\mathbb{Z}/p\mathbb{Z}, +)$, and we consider the formula $\phi(x_1; y_1, y_2) :\ x = y_1 \vee x \neq y_2$, we have

$$|\phi(M; a_1, a_2)| = \begin{cases} \alpha & \text{if } a_1 = a_2, \\ \alpha - 1 & \text{if } a_1 \neq a_2. \end{cases}$$

Consider the theory T_r and an ultraproduct of finite graphs $M \models T_r$,

- For the formula $\phi(x, y) := \neg D_2(x, y)$,

$$|\phi(M, a)| = \alpha - r(r-1).$$

- For the formula $\phi(x; y_1, y_2) := D_2(x, y_1) \wedge D_3(x, y_2)$

$$|\phi(M, a_1, a_2)| = \begin{cases} r(r-1) & \text{if } M \models D_1(a_1, a_2) \\ r & \text{if } M \models D_5(y_1, y_2) \\ 0 & \text{if } M \models \neg D_1(a_1, a_2) \wedge D_5(a_1, a_2) \end{cases}$$

- For the formula $\phi(x; y_1, y_2) := \neg D_2(x, y_1) \wedge \neg D_3(x, y_2)$

Asymptotic classes

Asymptotic classes of finite structures

Theorem (Chatzidakis, van den Dries, Macintyre, 1992)

Let $\varphi(x_1,\dots,x_n;y_1,\dots,y_m)$ be a formula in the language of rings. Then there is a positive constant C and finitely many pairs (d_i,μ_i) $(1\leq i\leq k)$ with $d_i\in\{0,1,\dots,n\}$ and $\mu_i\in\mathbb{Q}^{>0}$ a positive rational number such that for each finite field $\mathbb{F}_q$, and each $\overline{a}\in\mathbb{F}_q^m$, if the set $\varphi(\mathbb{F}_q^n;\overline{a})$ is non-empty then

$$\left||\varphi(\mathbb{F}_q^n;\overline{a})|-\mu_i q^{d_i}\right|<Cq^{d_i-1/2}$$

for some $i\leq k$. Moreover, for each pair (d_i,μ_i) there is a formula $\psi_i(y_1,\dots,y_m)$ in the language of rings such that $\psi_i(\mathbb{F}_q^m)$ consists precisely of those $\overline{a}\in\mathbb{F}_q^m$ for which the corresponding inequality (μ_i,d_i) holds.

Asymptotic classes of finite structures

Definition (Macpherson, Steinhorn)

Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. We say that $\mathcal{C}$ is a 1-dimensional asymptotic class if for every $\mathcal{L}$-formula $\varphi(x;\overline{y})$ there is a positive constant $C_\varphi > 0$ and a finite set $E_\varphi \subseteq \mathbb{R}^{>0}$ such that the following hold:

(a) *For every $M \in \mathcal{C}$ and $\overline{a} \in M^{\overline{y}}$, either $|\varphi(M;\overline{a})| \leq C$ or there is $\mu \in E$ such that*

$$||\varphi(M,\overline{a})| - \mu|M|| \leq C|M|^{1/2}. \qquad (*)$$

(b) *For every $\mu \in E$ there is a formula $\psi_\mu(\overline{y})$ such that for every $M \in \mathcal{C}$, and $\overline{a} \in M^{|y|}$, $M \models \psi_\mu(\overline{a}) \Leftrightarrow (*)$ holds.*

Elwes (2007): notion of N-dimensional asymptotic classes, with dimensions $0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}$ for formulas in one variable.

Examples of asymptotic classes

- Finite fields. (Chatzidakis, van den dries, Macyntire)
- Finite cyclic groups. (Macpherson, Steinhorn, based on Szmielew)
- Finite simple groups of fixed Lie type. (Elwes)
- Finite fields with a Frobenius automorphism. (Ryten)
- **Paley graphs** (Macpherson, Steinhorn - using Bollobás-Thomason to get the random graph, and later using QE for conditions (a), (b))
- Some classes of residue rings, e.g. $\{(\mathbb{Z}/p^d\mathbb{Z}, +, \cdot, 0, 1) : p \text{ prime}\}$. (Bello-Aguirre)

Ultraproducts of asymptotic classes

Theorem (Macpherson, Steinhorn)

1. If every ultraproduct of a class $\mathcal{C}$ is strongly minimal, then $\mathcal{C}$ is a 1-dimensional asymptotic class.
2. Every infinite ultraproduct of structures in a 1-dimensional asymptotic class is supersimple of SU-rank 1.

Similarly, every ultraproduct of an N-dimensional asymptotic class is supersimple of finite rank ($\leq N$).

Idea: Each instance of dividing for formulas in one variable is witnessed by a drop of dimension. In ultraproducts of asymptotic classes, there are only finitely many possible dimensions.

Definition (Macpherson, Steinhorn)

A structure M is measurable if there is a function $h = (\dim, \text{meas}) : \text{Def}(M) \to \mathbb{N} \times \mathbb{R}^{>0}$ satisfying the following conditions:

(i) *If $X \in \text{Def}(M)$ is finite and non-empty, then $h(X) = (0, |X|)$*

(ii) *(Definability and finiteness) For every formula $\phi(\overline{x}, \overline{y})$, there is a finite set $D_\phi \subseteq \mathbb{N} \times \mathbb{R}^{>0}$ so that*

  (a) *For every $\overline{a} \in M^{|\overline{y}|}$, $h(\phi(M^{|\overline{x}|}, \overline{a})) \in D_\phi$*

  (b) *For every $(d, \mu) \in D_\phi$ there is a formula $\psi_{(d,\mu)}(\overline{y})$ such that for every $\overline{a} \in M^{|\overline{y}|}$,*

$$M \models \psi(\overline{a}) \Leftrightarrow h(\phi(M^{|\overline{x}|}, \overline{a}) = (d, \mu)\}$$

(iii) *(Fubini property) Suppose $f : X \to Y$ is a definable surjection, and $Y = Y_1 \cup \cdots \cup Y_r$ is a definable partition such that $\dim(Y_i) = e_i$, $\text{meas}(Y_i) = \nu_i$ and $h(f^{-1}(\overline{y})) = (d_i, \mu_i)$ for every $\overline{y} \in Y$. Suppose also that the maximum $c := \text{máx}\{d_1 + e_1, \ldots, d_r + e_r\}$ is attained by the values $d_1 + e_1, \ldots, d_s + e_s$. Then, $h(X) = (c, \mu_1 \cdot \nu_1 + \cdots + \mu_s \cdot \nu_2)$*

Multidimensional asymptotic classes

Definition (Anscombe, Macpherson, Steinhorn, Wolf)

*Let $\mathcal{C}$ be a class of finite structures and let R be any set of functions $\mathcal{C} \to \mathbb{R}^{\geq 0}$. We say that $\mathcal{C}$ is an R-**multidimensional asymptotic class** (or an R-**m.a.c.** if for every formula $\phi(\overline{x}, \overline{y})$ there is a finite -definable partition Φ of $(\mathcal{C}, \overline{y})$ and a set $H_\Phi := \{h_P \in R : P \in \Phi\}$ of functions such that*

$$||\phi(M^{|\overline{x}|}, \overline{a}) - h_P(M)| = o(h_P(M))$$

for $(M, \overline{a}) \in P$, as $|M| \to \infty$.
*In addition, we say that $\mathcal{C}$ is an R-m.e.c (multidimensional **exact** class) if in the condition above we have $|\phi(M^{|\overline{x}|}, \overline{a}) = h_P(M)$.*

There is a corresponding notion for **generalized measurable structures**, and it turns out that every ultraproduct of structures in an R-mac is a generalized measurable structure.

Examples of macs

1. (G. Macpherson, Steinhorn) The class of 2-sorted structures $(V, \mathbb{F}_q)$ with V a finite-dimensional vector space over $\mathbb{F}_q$. Given a formula $\phi(\overline{x}, \overline{y})$ there is a finite set E_ϕ of polynomials $g(\boldsymbol{V}, \boldsymbol{F})$ with coefficients in $\mathbb{Q}$ such that if $M = (V, F)$, then $h_P(M)$ has the form $g(|V|, |F|)$ for some $g \in E_\phi$.
   The ultraproducts of structures in this class are supersimple, but the V-sort may have rank ω.
2. (Bello Aguirre) In the language of rings, for a fixed $d \in \mathbb{N}$ we can consider the class C_d of all residue rings $\mathbb{Z}/n\mathbb{Z}$ where n is the product of powers of at most d primes, each with exponent at most d. Then $\mathcal{C}_d$ is a m.a.c. (after an appropriate expansion by unary predicates). Note here that the class $\{(\mathbb{Z}/p^d\mathbb{Z} : p \text{ prime}\}$ is a d-dimensional asymptotic class.

Theorem (G., Robles)

Let $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ be a class of finite graphs such that each graph G_n is d_n-regular and $d_n, \mathrm{girth}(G_n) \to \infty$.

Let M be an infinite ultraproduct of graphs in $\mathcal{C}$ (a model of T_∞) and fix the non-standard integers $\alpha = |M|$ and $\beta = [d_n]_{\mathcal{U}}$. Then for every formula $\phi(\overline{x}, \overline{y})$ in the language of graphs there is a finite number of polynomials $p_1(X, Y), \ldots, p_k(X, Y) \in \mathbb{Z}[X, Y]$ such that:

1. For every $\overline{a} \in M^{|\overline{y}|}$, $|\phi(M^{|\overline{x}|}, \overline{a})| = p_i(\alpha, \beta)$ for some $i \leq k$.
2. Moreover, there are formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that for every $\overline{a} \in M^{|\overline{y}|}$,
$$M \models \psi_i(\overline{a}) \Leftrightarrow |\phi(M^{|\overline{x}|}, \overline{a})| = p_i(\alpha, \beta).$$

This is enough to show that any class of graphs with the properties above is a multidimensional exact class.

Final remarks/questions

- Which nice classes of graphs satisfy the conditions described to obtain models of T_r, T_∞ as ultraproducts? (Ramanujan graphs, expanders, etc.) [work in progress with Melissa Robles]
- In famous examples of pseudofinite structures, what do we know about the classes of finite structures approximating them?
  For instance, is $\mathcal{M}_\alpha$ (the generic limit of the class of graphs with predimension $\delta_\alpha(X) = |X| - \alpha|R(X)|$) a generalized measurable structure?
- What kind of measurability properties are preserved when we apply different constructions (H-structures, lovely pairs) to ultraproducts of finite structures? [some progress here with A. Berenstein and T. Zou]

Example (Anscombe)

If M is the Fraïssé limit of a free amalgamation class then M is generalized measurable. (note for example that the generic triangle-free graph is an example, that has a TP1 and TP2 theory)

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Thank you