Dynamic Topological Logic
Day 2

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Recap: Dynamic Topological Logic

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- $[\![\square\varphi]\!] = \bigcap_{n<\omega} S^{-n}\,[\![\varphi]\!]$ (henceforth).

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- The formula $\Box\blacksquare p \to \blacksquare\Box p$ is Kripke-valid, but not topologically valid.
- A formula is satisfiable iff it is satisfiable on a non-deterministic quasimodel.

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2. Exhibit a decidable sub-language which remains expressive enough to reason about asymptotic behaviour

A completeness proof sketch for LTL

Fix a consistent formula φ. We construct a model for φ.

1. Assign a characteristic formula to each possible moment $\mathfrak{m}$.
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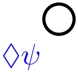

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4. Conclude that any possible moment can be included in a realizing path an hence an LTL model.

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4. Prove that $\mathcal{J}_\Sigma$ is a quasimodel.
5. Conclude that if φ is consistent then it is true on some possible moment, hence on the quasimodel $\mathcal{J}_\Sigma$, and hence on some dynamic topological model (by last week's results).

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Note: Labelled spaces generalize topological models since we may always define

$$\ell^+(x) = \{\varphi \in \Sigma : x \in [\![\varphi]\!]\}$$
$$\ell^-(x) = \{\varphi \in \Sigma : x \notin [\![\varphi]\!]\}$$

Simulations revisited

A simulation between labelled spaces (X, ℓ_X) and (Y, ℓ_Y) is a continuous relation $E \subset X \times Y$ which preserves labels:

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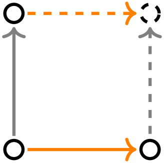
Simulations between labelled preorders are forward-confluent:



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3. $\mathfrak{v}\ R\ \mathfrak{w}$ if there is a sensible, root-preserving relation between $\mathfrak{v}$ and $\mathfrak{w}$

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Today: The relation $\trianglelefteq$ defines a well quasiorder on $\mathcal{I}_\Sigma$.

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2. If $U \subset A$ is upwards-closed under $\leq$, there are finitely many
$$u_1, \ldots, u_n \in A$$
such that for every $a \in A$ there is $i \leq n$ with $u_i \leq a$.

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Proof.

Every moment $\mathfrak{m}$ is bisimilar to a tree-like moment $\mathfrak{m}'$, and an embedding E from $\mathfrak{m}'$ to $\mathfrak{n}'$ yields a simulation between $\mathfrak{m}$ and $\mathfrak{n}$. So, we can apply Kruskal's tree theorem. $\square$

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First used in the setting of DTL in **Konev, Kontchakov, Wolter and Zakharyaschev 2006.**

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Proposition

The property $\mathfrak{m} \trianglelefteq (X, \ell, x)$ *is not definable in* $\mathcal{L}_{\blacksquare}$*, even when* X *is a Kripke model and* $\mathfrak{m}$ *is the* $\{\mathbf{p}, q\}$*-cluster.*

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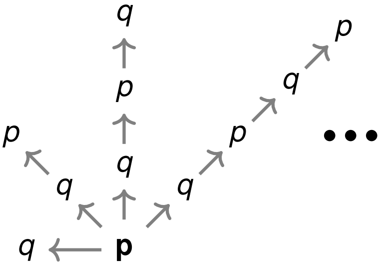

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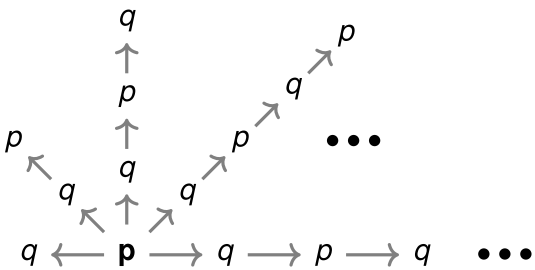

The tangled closure

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$\mathcal{L}^*_{\blacksquare(\circ\Box)}$: We consider an extension of L where $\blacklozenge$ is allowed to act on *sets* of formulas, and define

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Theorem (Dawar and Otto)

$\mathcal{L}^*_{\blacksquare}$ is equally expressive as the μ-calculus over the class of finite preorders.

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- In general, $w \in [\![\blacklozenge\{\varphi_0, \ldots, \varphi_n\}]\!]$ iff there is a path

$$w = w_0 \preccurlyeq w_1 \preccurlyeq w_2 \preccurlyeq \ldots$$

  such that for each $i \leq n$ there are infinitely many j such that $w_j \in [\![\varphi_i]\!]$.

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4. If $A = \mathbb{Q}$ and $B = \mathbb{Q} + \pi$ then $\{A, B\}^*$

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4. If $A = \mathbb{Q}$ and $B = \mathbb{Q} + \pi$ then $\{A, B\}^* = \mathbb{R}$

Simulation formulas

Theorem (DFD)

Given a locally finite labelled preorder $(W, \preccurlyeq, \ell)$*, there exist formulas* $\big(Sim(w)\big)_{w \in W} \in \mathcal{L}^*_{\blacksquare}$ *such that for any dynamic topological model* $\mathcal{M} = (X, S, [\![\cdot]\!])$*, tfae:*

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Example

If $\mathfrak{m}$ *is the* $\{\mathbf{p}, q\}$*-cluster then*

$$Sim(\mathfrak{m}) = p \wedge \blacklozenge\{p, q\}$$

An axiomatization for DTL*

Taut	All propositional tautologies.

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Rules:		MP, Subs, $\mathrm{N}_\blacksquare$, $\mathrm{N}_\Box$, $\mathrm{N}_\circ$

The set of possible moments

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Fix finite Σ closed under subformulas. A moment $\mathfrak{m}$ of $\mathcal{I}_\Sigma$ is possible if *Sim*$(\mathfrak{m})$ is consistent, and $\mathcal{J}_\Sigma$ is the substructure of possible moments.

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ω-sensibility

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Let w be a moment of $\mathcal{J}_\Sigma$ and $\Box\psi \in \ell^-(w)$, $R^*(w)$ be the set of worlds reachable from w in $\mathcal{J}_\Sigma$
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Completeness

Theorem (DFD)

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we obtain that w_* is possible for some w_* with $\varphi \in \ell^+(w_*)$, hence w_* is a world of $\mathcal{J}_\Sigma$.
Thus $\mathcal{J}_\Sigma$ is a quasimodel satisfying φ, and it follows that φ is satisfiable on some dynamical topological model. $\Box$

Kremer's intuitionistic temporal logic

Kremer 2004: Work over $\mathcal{L}_{\circ\Box}$ and use the topological semantics of intuitionistic logic to interpret $\to$

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However, the following standard validities fail

- $\Box p \to \circ\Box p$
- $\Box\circ p \to \circ\Box p$
- $\Box p \to \Box\Box p$

Topological semantics for intuitionistic logic

Models

$\mathcal{M} = (X, \mathcal{T}, V)$, where:

- $(X, \mathcal{T})$ is a topological space
- $V \colon \mathbb{PV} \to \mathcal{T}$

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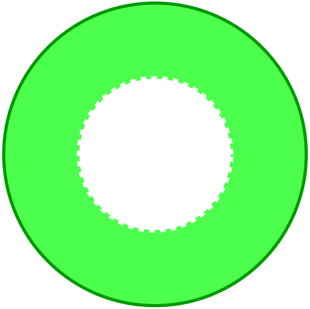
Truth sets

- $[\![\bot]\!] = \varnothing$
- $[\![p]\!] = V(p)$
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$
- $[\![\varphi \to \psi]\!]$ $= \left([\![\varphi]\!]^c \cup [\![\psi]\!] \right)^\circ$

Interior of $A \subseteq X$:

$$A^\circ = \bigcup \{ U \in \mathcal{T} : U \subseteq A \}$$

Classical regions



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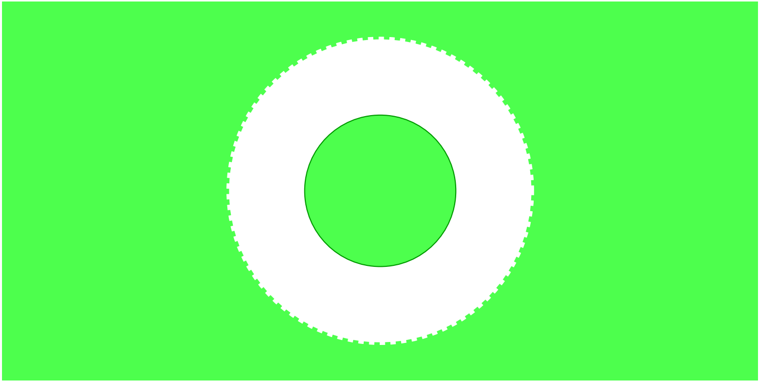

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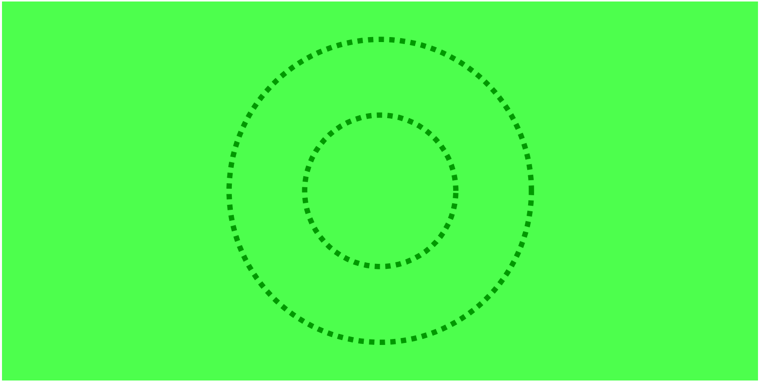

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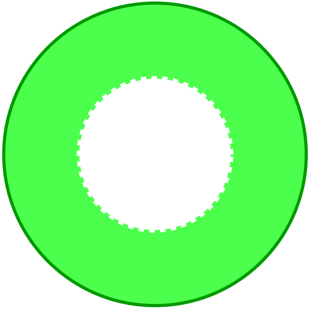

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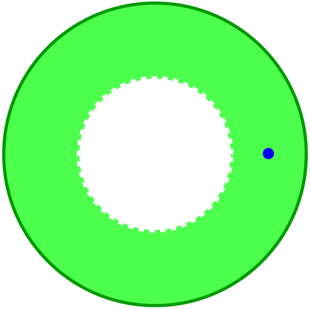

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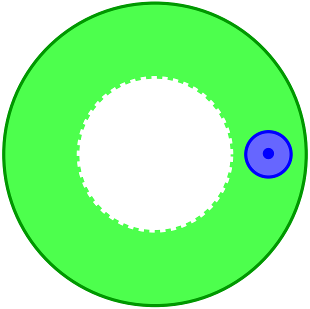

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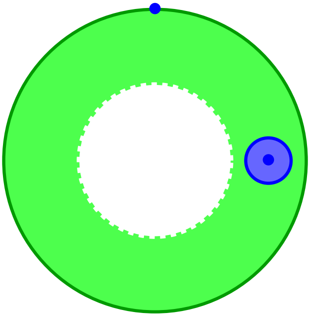

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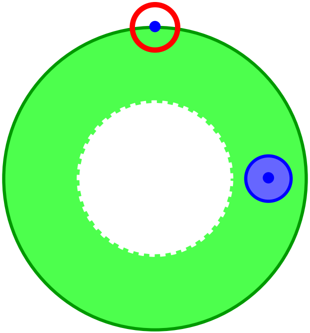

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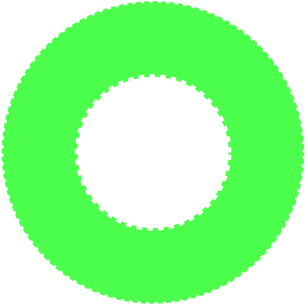

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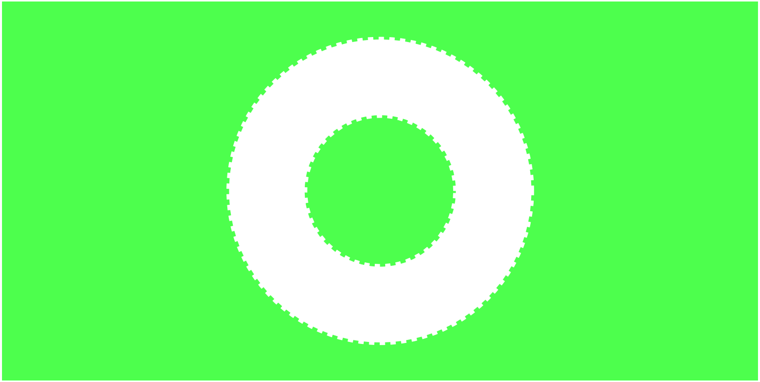

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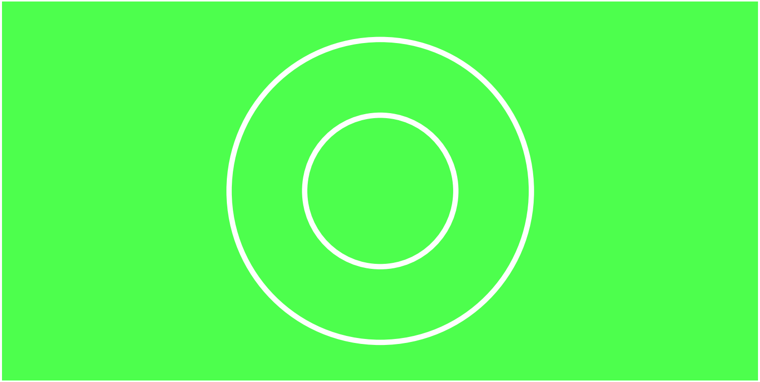

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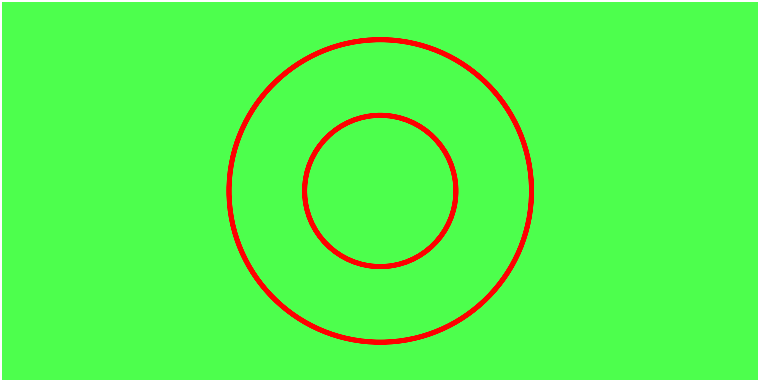

$[\![p \vee \neg p]\!]^{I}$ Fails!

Intuitionistic temporal logic

Language $\mathcal{L}_{\circ\Diamond\Box\forall}$: $\varphi, \psi :=$

$$p \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \circ\varphi \mid \Diamond\varphi \mid \Box\varphi \mid \forall\varphi$$

Models: (X, S, V), where $S \colon X \to X$ is continuous

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Truth of temporal operators

$$[\![\circ\varphi]\!] = S^{-1}\,[\![\varphi]\!]$$

$$[\![\Diamond\varphi]\!] = \bigcup_{n<\omega} S^{-n}\,[\![\varphi]\!]$$

$$[\![\Box\varphi]\!] = \left(\bigcap_{n<\omega} S^{-n}\,[\![\varphi]\!]\right)^{\circ}$$

$$[\![\forall\varphi]\!] = \begin{cases} X & \text{if } [\![\varphi]\!] = X \\ \varnothing & \text{otherwise} \end{cases}$$

Gödel-Tarski translation

The translation $\varphi \mapsto \varphi^{\blacksquare}$ embeds $\mathcal{L}_{\circ\Diamond\Box}$ into the classical $\mathcal{L}_{\blacksquare\circ\Box}$ by setting

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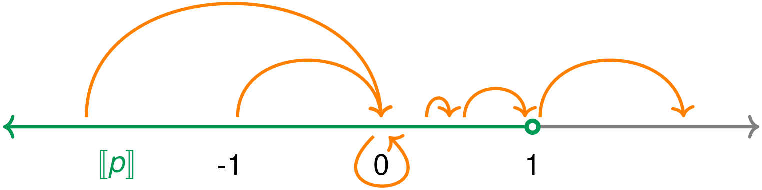
Corollary

The set of $\mathcal{L}_{\circ\Diamond\Box}$-formulas valid over the class of dynamical systems is computably enumerable.

Kremer's counterexample: $\Box p \to \circ\Box p$ fails!

- $X = \mathbb{R}$
- $V(p) = (-\infty, 1)$

- $S(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$



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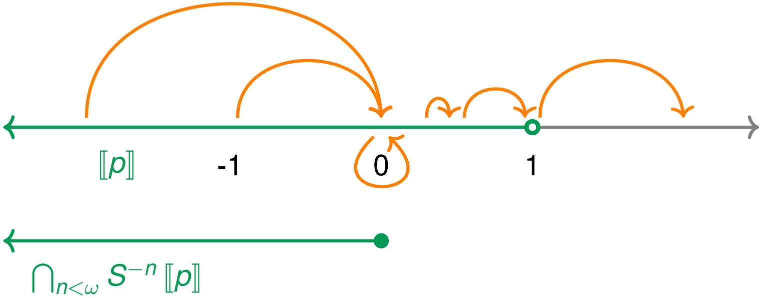

Kremer's counterexample: $\Box p \to \circ\Box p$ fails!

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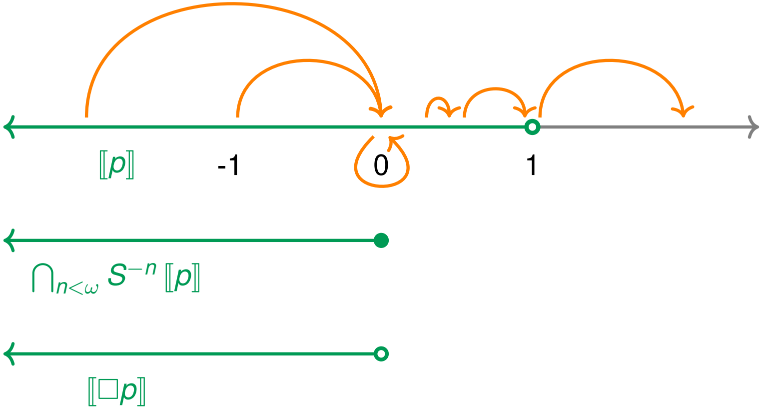

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However, there are Kripke-valid but non-derivable formulas, such as

$$\Box(p \vee q) \to \Diamond p \vee \Box q$$

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Triple $(W, \preccurlyeq, \ell)$ where ℓ assigns a type to each $w \in W$ according to the intuitionistic semantics

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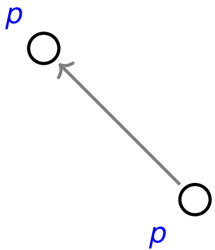

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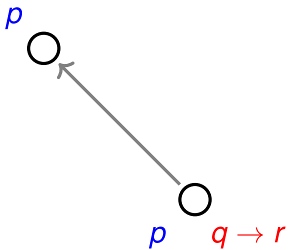

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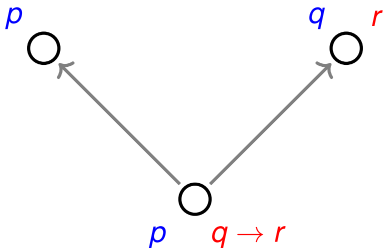

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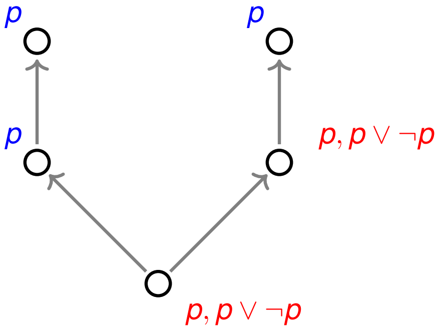

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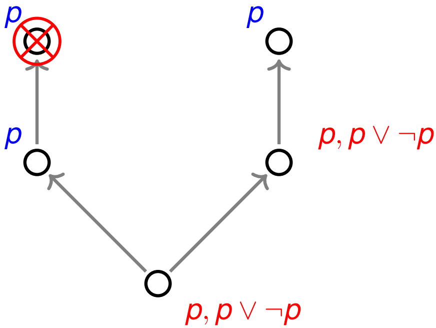

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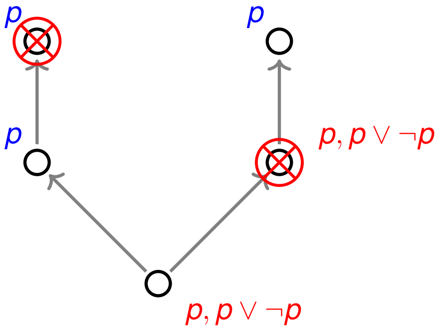

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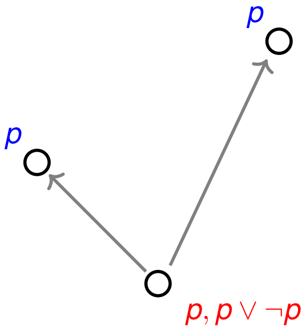

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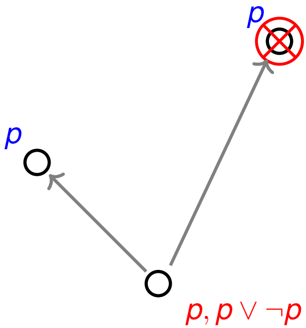

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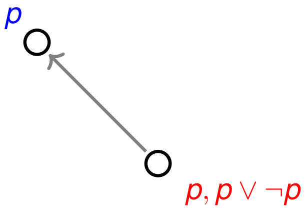

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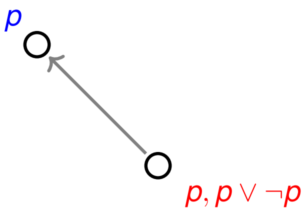

There are finitely many (about 2^n_n) moments of height n up to bisimulation.

The calculus $\mathsf{ITL}^0_\Diamond$

ITaut	Standard intuitionistic propositional axioms

Temporal axioms:

$\mathrm{Next}_\bot$	$\neg \circ \bot$
$\mathrm{Next}_\wedge$	$(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$
$\mathrm{Next}_\vee$	$\circ(\varphi \vee \psi) \to (\circ\varphi \vee \circ\psi)$
$\mathrm{Next}_\to$	$\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$
$\mathrm{Fix}_\Diamond$	$(\varphi \vee \circ\Diamond\varphi) \to \Diamond\varphi$

Rules:

MP $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ Nec $\dfrac{\varphi}{\circ\varphi}$

Mon $\dfrac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$ Ind $\dfrac{\circ\varphi \to \varphi}{\Diamond\varphi \to \varphi}$

The calculus $\mathsf{ITL}^0_{\Diamond\forall}$

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$\mathsf{K}_\forall$	$\forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)$	$\mathsf{EM}_\forall$	$\forall\varphi \vee \neg\forall\varphi$
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Falsifying $\forall(\neg p \vee \Diamond p) \to (\Diamond p \vee \neg\Diamond p)$ topologically

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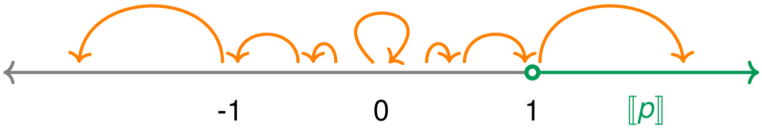

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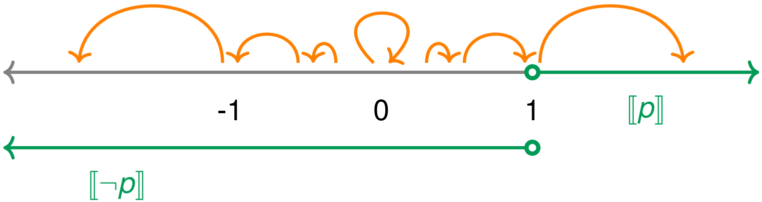

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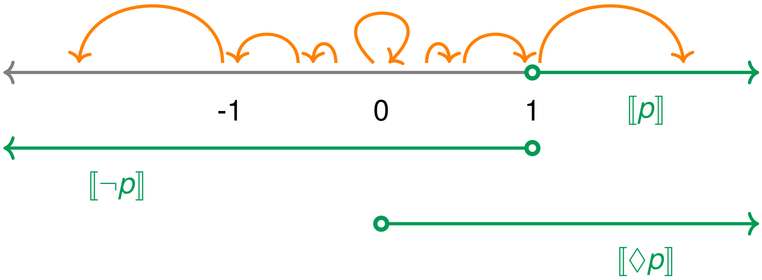

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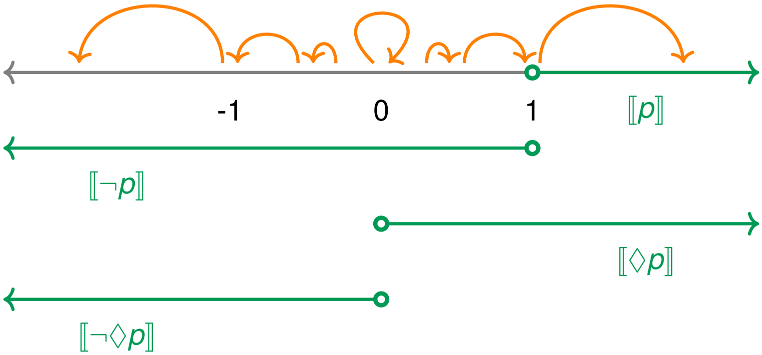

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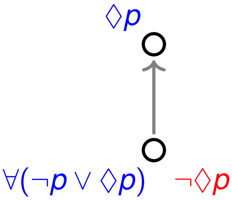

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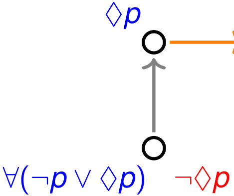

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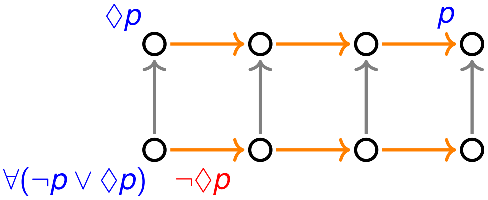

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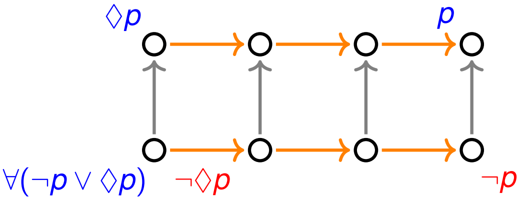

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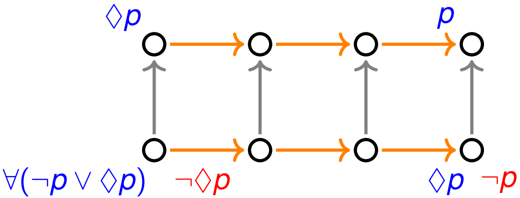

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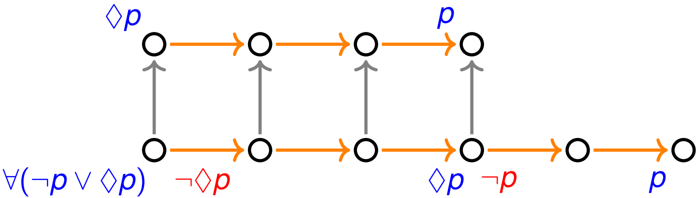

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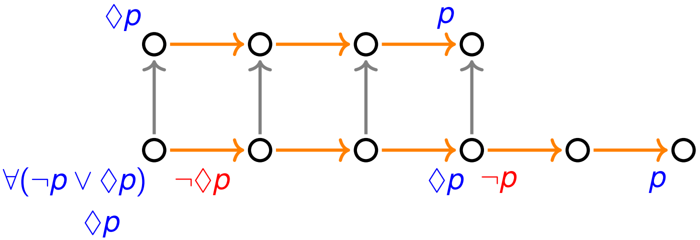

Simulation formulas

Theorem

Given a finite labelled poset $\mathcal{A}$ with domain W, there exist intuitionistic formulas $\left(Sim(w)\right)_{w \in W}$ such that for any model $\mathcal{M} = (X, \preccurlyeq, V)$, tfae:

1. $(\mathcal{M}, x) \not\models Sim(w)$
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(Essentially Jankov-de Jongh formulas)

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Definition

Fix finite Σ closed under subformulas. A world w of $\mathcal{I}_\Sigma$ is possible if $\not\vdash Sim(w)$, and $\mathcal{J}_\Sigma$ is the substructure of possible worlds

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Proof.

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3. We only need to make slight modifications to work with the dual $\Diamond$. $\square$

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Thank you for your attention!

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