Cardinalities of definable sets in finite structures

Dugald Macpherson

Joint work with Anscombe, Steinhorn, Wolf; IPM Tehran

July 29, 2020
Structure of Talk

1. Chatzidakis - van den Dries - Macintyre Theorem on definability in finite fields, asymptotic classes, measurable structures.
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3. Multidimensional EXACT classes and approximations of homogeneous structures.
CDM Theorem

Theorem [Chatzidakis, van den Dries and Macintyre 1992] Let
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\( \varphi(x_1, \ldots, x_n; y_1, \ldots, y_m) \) be a formula in the language \( L_{\text{rings}} = (+, -, \times, 0, 1) \) of rings. Then

(i) there is a positive constant \( C \) and finitely many pairs \( (d_i, \mu_i) \), with \( d_i \in \{0, 1, \ldots, n\} \) and \( \mu_i \in \mathbb{Q}^>0 \) such that for each finite field \( \mathbb{F}_q \), and each \( \bar{a} \in \mathbb{F}_q^m \), if the set

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\varphi(\mathbb{F}_q^n, \bar{a}) := \{ \bar{b} \in \mathbb{F}_q^n : \mathbb{F}_q \models \varphi(\bar{b}, \bar{a}) \}
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is nonempty, then

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||\varphi(\mathbb{F}_q^n, \bar{a})| - \mu_i q^{d_i}| < Cq^{d_i-(1/2)}
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\[ ||\varphi(\mathbb{F}_q^n, \bar{a}) | - \mu_i q^{d_i} || < C q^{d_i - (1/2)} \]
for some \( i \);

(ii) for each pair \( (d_i, \mu_i) \), there is a formula \( \psi_i(y_1, \ldots, y_m) \) in the language of rings such that \( \psi_i(\mathbb{F}_q^m) \) consists of those \( \bar{a} \in \mathbb{F}_q^m \) for which the corresponding inequality holds.
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Corollary: There is no \( L_{\text{rings}} \)-formula (even with parameters) which uniformly in all finite fields \( \mathbb{F}_p^2 \) defines the prime subfield \( \mathbb{F}_p \).
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Ryten (PhD thesis 2007): For any fixed Lie type \(\tau\), possibly twisted, the class of all finite simple groups of type \(\tau\) is an asymptotic class.

**Corollary.** Fix a Lie type \(\tau\), giving an \(N\)-dimensional asymptotic class of finite simple groups. There is a finite set \(E\) of pairs \((d_i,\mu_i)\in\{0,...,N\}\times\mathbb{Q}_{\geq 0}\) such that for any finite simple group \(G\) of Lie type \(\tau\), each conjugacy class of \(G\) has cardinality roughly \(\mu|G|^{d/N}\) for some \((d,\mu)\in E\).

**Proof:** Apply Ryten result to formula \(\phi(x,y)\) of form \(\exists z (z−1xz=y)\).
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**Proof:** Apply Ryten result to formula $\phi(x, y)$ of form $\exists z (z^{-1} x z = y)$. 
Measurable structures

Corresponding notion of **measurable structure** \((M + \text{Steinhorn})\), for *infinite* structures. This means we can assign pairs \((d, \mu)\) to definable sets, with similar counting properties as for asymptotic classes.

Fact:

1. Any ultraproduct of an asymptotic class is measurable. So every pseudofinite field is measurable. (The idea: an ultraproduct of sets of size roughly \(\mu^\text{d}\) is assigned the pair \((d, \mu)\)).

2. Every measurable structure is supersimple of finite SU-rank. (Recall that the class of simple theories contains the stable theories, that forking gives a nice notion of independence in simple theories, and that supersimple + stable = superstable.)

3. \((\mathbb{C}, +, \times)\) is not measurable, due to the 2-1 surjection \(x \mapsto x^2\) \(\mathbb{C}\{0\} \to \mathbb{C}\{0\}\).
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**Note:** Our framework will NOT include the class of total orders, due to the formula $x < y$. 
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3. Disjoint unions of complete graphs all of same size (\(n\) copies of \(K_m\), so 2 parameters varying freely).

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4. Finite abelian groups.
5. Finite graphs of bounded degree.
For a class $C$ of finite $\mathcal{L}$-structures and a tuple $\bar{y}$ of variables, we denote by $(C, \bar{y})$ the set $\{ (M, \bar{a}) \mid M \in C, \bar{a} \in M^{\bar{y}} \}$ of pairs (‘pointed structures’) consisting of a structure in $C$ and a $\bar{y}$-tuple from that structure.

The idea: we partition $\bar{y}$-space uniformly (across $C$) into a fixed finite number of parts, each part (uniformly) $\emptyset$-definable in each structure in $C$. 

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A finite partition $\Phi$ of $(C, \bar{y})$ (i.e. finitely many parts) is Ø-definable if for each $P \in \Phi$ there exists an $\mathcal{L}$-formula $\phi_P(\bar{y})$ without parameters such that

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for each $M \in C$. 

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Definition of $R$-m.a.c.

Let $R$ be ANY set of functions $\mathcal{C} \to \mathbb{R}^{\geq 0}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an $R$-multidimensional asymptotic class (or an $R$-m.a.c. for short) if for every formula $\phi(\bar{x}, \bar{y})$ there is a finite $\emptyset$-definable partition $\Phi$ of $(\mathcal{C}, \bar{y})$ and a set $H_\Phi := \{ h_P \in R \mid P \in \Phi \}$ of functions such that:

$$\left| \left| \phi(M|\bar{x}; \bar{b}) - h_P(M) \right| \right| = o(h_P(M)) \quad (1)$$

for $(M, \bar{b}) \in P$ as $|M| \to \infty$. 
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**The idea:** The size of $\phi(M^{\bar{x}}, \bar{b})$ is a function of $M$, the function depending just on the part of $\bar{b}$ (i.e. the part of $(M, \bar{b})$). The notion of $R$-m.e.c. is a strengthening of $R$-m.a.c.
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weak $R$-m.a.c. (or $R$-m.e.c.) – drop the definability clause on the partition $\Phi$. 
Observations

Basic facts about $R$-m.a.c.s and $R$-m.e.c.s.

1. To prove a class $\mathcal{C}$ is an $R$-m.a.c. or $R$-m.e.c it suffices to prove the condition for formulas $\phi(x, \bar{y})$ (with $x$ a single variable), replacing $R$ by the ring generated by $R$. (Fibering argument, using definability. Compare how o-minimality is a one-variable condition but implies cell decomposition.)
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2. (Wolf) If $C$ is a m.a.c. or m.e.c. then so is any class of finite structures uniformly bi-interpretable with $C$. (Note: These conditions are not closed under interpretability or taking reducts, as the definability clause may be lost.)
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3. Any class uniformly interpretable in a m.a.c. is a weak m.a.c..
Examples

1. (Garcia, M, Steinhorn) Class of 2-sorted structures \((V, \mathbb{F}_q)\), with \(V\) a finite-dimensional vector space over \(\mathbb{F}_q\). Here, given a formula \(\phi(\bar{x}, \bar{y})\) there is a finite set \(E_\phi\) of polynomials \(g(V, F)\) over \(\mathbb{Q}\) such that if \(M = (V, F)\) then each \(h_P(M)\) has form \(g(|V|, |F|)\) for some \(g \in E_\phi\).
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2. More generally, fix a quiver \(Q\) (digraph) of finite representation type \((A_n, D_n, E_6, E_7, E_8)\). Over the field \(F\), this has a finite-dimensional path algebra \(FQ\), which has finitely many isomorphism types of indecomposable representations. Let

\[ C_Q := \{(V, FQ, F) : F \text{ finite field}, V \text{ finite module for } FQ\} \]

(3-sorted, with the natural language). Then \(C_Q\) is an \(R\)-mac with the functions \(h\) given by polynomials \(g(F, W_1, \ldots, W_t)\), where the \(W_i\) variables correspond to the indecomposables for the quiver \(Q\).
3. (Bello Aguirre) In the language of rings, for fixed $d \in \mathbb{N}$, let $C_d$ be the collection of all finite residue rings $\mathbb{Z}/n\mathbb{Z}$, where $n$ is a product of powers of at most $d$ primes, each with exponent at most $d$. Then $C_d$ is a weak m.a.c., and is a m.a.c. after appropriate expansion by unary predicates. If just one prime is involved, this is an asymptotic class (e.g. $\{\mathbb{Z}/p^2\mathbb{Z} : p \text{ prime}\}$ is a 2-dim asymptotic class). Ultraproducts are supersimple of finite SU-rank. (Idea: $\mathbb{Z}/p^d\mathbb{Z}$ is coordinatised uniformly by $\mathbb{Z}/p\mathbb{Z}$.)
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Generalised measurable structures
Let \((S, +, \cdot, 0, 1, <)\) be a (commutative) ordered semiring (so \((S, +, 0)\), \((S, \cdot, 1)\) are commutative monoids, least element 0, etc.).
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\[\forall x, y, z \in S((x < y \land d(y) = d(z)) \to x + z < y + z).\]
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Let \(S\) be a measuring semiring and let \(M\) be an \(L\)-structure. We say that \(M\) is **\(S\)-measurable** if there is a function \(h : \text{Def}(M) \to S\) such that

1. **finite sets**: \(h(X) = |X|\) for finite \(X\);
2. **finite additivity**: if \(X, Y \in \text{Def}(M)\) are disjoint, \(h(X \cup Y) = h(X) + h(Y)\);
3. **mac condition**: for each definable family \(\mathcal{X}\) (given by a formula \(\phi(\bar{x}, \bar{y})\)) there exists a finite set \(F \subseteq S\) such that \(h(\mathcal{X}) = F\) and for each \(f \in F\), \(h^{-1}(f)\) is a \(\emptyset\)-definable family; and
4. **Fubini**: suppose \(p : X \to Y\) is a definable function and there exists \(f \in S\) such that for all \(\bar{a} \in Y\), \(h(p^{-1}(\bar{a})) = f\); then we have \(h(X) = f \cdot h(Y)\).
In finite fields, by CDM, definable sets had size roughly $\mu q^d$. Likewise, if $M$ is $S$-measurable, and $D$ is formed as above via $d_S : S \to D$, then $(D, \text{max}, \oplus, -\infty, 0, <)$ has a semiring structure (with max and $\oplus$ induced from $+$ and $\times$ via $d_S$), and we may form

$$E = \mathbb{R}^{\geq 0} X^D = \{ \mu X^d : \mu \in \mathbb{R}^{\geq 0}, d \in D \}.$$
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**Proposition.** If $M$ is $S$-measurable, and the associated set of dimensions $(d_S \circ h)(\Def(M))$ is well-ordered, then $M$ is supersimple. (Idea: Forking ensures drop in dimension.)
**Proposition.** Let $M$ be (weakly) generalised measurable. Then

(i) $M$ does not have the strict order property (i.e. there is no definable partial order on any $M^n$ with an infinite totally ordered subset);
Generalised measurable structures

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**Example** (Anscombe). If $M$ is a Fraïssé limit of a free amalgamation class then $M$ is generalised measurable (note for example the generic triangle-free graph is such a Fraïssé limit and has TP1 and TP2 theory).
Generalised measurable structures

**Proposition.**
1. If $\mathcal{C}$ is a m.a.c. then any ultraproduct is generalised measurable (so NSOP, functionally unimodular, etc.)

Note.
The above supersimplicity result applies to ultraproducts of examples like $\{ (V, F_q) : q$ prime power, $V$ finite dim. over $F_q \}$ and the quiver example, where the defining functions are given by polynomials in several variables, so the corresponding set of dimensions is well-ordered (they are given by the polynomial degrees, which are ordered like $\mathbb{N}^d$).
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1. If $C$ is a m.a.c. then any ultraproduct is generalised measurable (so NSOP, functionally unimodular, etc.)

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Examples of m.e.c.s

1. (Essentially by Pillay) Let $M$ be any pseudofinite strongly minimal set. Then there is a m.e.c. whose infinite ultraproducts are all elementarily equivalent to $M$, with the functions determining cardinalities given as polynomials (over $\mathbb{Z}$) in the cardinalities of the finite structures.
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(If $(m, q) = 1$ and $|m| \leq 2\sqrt{q}$ there is an elliptic curve $E$ over $\mathbb{F}_q$ with $q + 1 - m \mathbb{F}_q$-rational points.)
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Problem. Find a m.e.c. with ultraproduct having non-simple theory.
Homogeneous structures as limits of m.e.c.s

**Conjecture.** If $M$ is a homogeneous structure over a finite relational language, then the following are equivalent.

1. There is a m.e.c with ultraproduct elementarily equivalent to $M$.
2. $M$ is stable.

**Remarks.**
1. The direction (2) $\Rightarrow$ (1) follows from Lachlan + Wolf.
2. The Paley graphs form a m.a.c (but not m.e.c) with limit the random graph, which is unstable. (Paley graph $P_{q}$ has vertex set $F_{q}$ where $q \equiv 1 \pmod{4}$, with $x, y$ adjacent iff $x - y$ is a square.)
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Theorem. Let $M$ be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to $M$.


(ii) Any homogeneous tournament (digraph such that for $x \neq y$, exactly one of $x \rightarrow y$ or $y \rightarrow x$ holds).

(iii) The digraph $D_n$ for each $n \geq 3$ (universal subject to omitting an independent set $I_n$).

(iv) The generic bipartite graph.

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For a contradiction, consider a m.e.c. $C$ of finite tournaments with all non-principal ultraproducts ≡ the random tournament.

1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices (count in 2 ways the pairs $(x, y)$ with $x \rightarrow y$).

2. For any formula $\phi(\bar{x}, \bar{y})$, in a large enough finite tournament $M \in C$ the cardinality $|\phi(M, \bar{a})|$ depends just on the isomorphism type of $\bar{a}$ (uses QE, + definability clause of m.e.c.).

3. If $M \in C$ is large enough then $M$ is regular, so $|M|$ is odd, by (1).

4. If $M$ is large enough finite and $a, b$ are distinct vertices, then the tournaments $M$ and on the sets $\{x: a, b \rightarrow x\}$, $\{x: x \rightarrow a, b\}$, $\{x: a \rightarrow x \rightarrow b\}$ and $\{x: b \rightarrow x \rightarrow a\}$ are all regular, so all of odd size.

5. $|M| = \text{the sum of four odd numbers} + 2$, which is even – contradicting (3)!
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1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices (count in 2 ways the pairs $(x, y)$ with $x \rightarrow y$).

2. For any formula $\phi(x, y)$, in a large enough finite tournament $M \in C$ the cardinality $|\phi(M, \bar{a})|$ depends just on the isomorphism type of $\bar{a}$ (uses QE, + definability clause of m.e.c.).
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3. If $M \in C$ is large enough then $M$ is regular, so $|M|$ is odd, by (1).
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4. If $M$ is large enough finite and $a, b$ are distinct vertices, then the tournaments $M$ and on the sets $\{x : a, b \to x\}$, $\{x : x \to a, b\}$, $\{x : a \to x \to b\}$ and $\{x : b \to x \to a\}$ are all regular, so all of odd size.
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5. $|M| = \text{the sum of four odd numbers } + 2$, which is even – contradicting (3)!