Talk

30 September 2020 12:31

Joint work with Angus Macintyre

Adeles of a ~~global~~ field K
number field

$\mathbb{A}_K$

char $p>0$ version

Hrushovski
GVF's
func field

{ # field
func field of curve/$\mathbb{F}_q$
$\mathbb{F}_q(t)$

Weil (3 Nov 1937) letter to Hasse – char $p>0$

another proof of RH curves/$\mathbb{F}_q$

(adelic) Riemann-Roch

Chevalley (20 June 1935) letter to Hasse.

Ideles – invertible adeles – Class field theory

idele class gp algebraically

ext class gp

Artin Whaples – axiomatic treatment of global fields 1945

defined adele as an additive version Idele

gen Weil's notion to number field

Tate Thesis new pf Hecke's zeta functions of K

$\zeta_{\mathbb{Q}}(s) \longrightarrow \zeta_K(s)$

using Fourier analysis on $\mathbb{A}_K \leftarrow GL_1$-automorphic forms

$\Downarrow$

automorphic forms GL_2 – Jacquet-Langlands

G reductive Langlands Program

Galois representations automorphic representations

$\boxed{SL_n(\mathbb{A}_K)/SL_n(K)}$ Weil conjecture on measure of this space

Tamagawa measure

$SL_n(K)$ Tamagawa measure

Siegel Maas formula for quadratic forms

Weil – new adelic pf using volume of above spaces

Birch-Swinnerton Dyer Conjecture arises from an analogue for elliptic curves $\underline{E(\mathbb{A}_{\mathbb{Q}})}$?

Riemann Hypothesis (RH)

Paul Cohen: Def $\boxed{\mathbb{A}_K / K^*}$ quotient by action of K^* by multiplication

Alain Connes: key to RH and Spectral realization of zeros of zeta function $\zeta(s)$

usual absolute value $|\cdot| \cdot \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$

absolute value on K:

$$\begin{cases} |x| = 0 \iff x = 0 \\ |xy| = |x|\,|y| \\ |x+y| \leq |x| + |y| \end{cases}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & x < 0 \end{cases}$$

$\boxed{\text{Arch}}$

(p-adic) $|\cdot|_p : K^* \to \mathbb{R}_{\geq 0}$

$$\left\{ |x+y| \leq \max\{|x|, |y|\} \right.$$

non-Arch

$n \in \mathbb{Z}$ $\quad n = p^k n'$ $\quad p \nmid n'$ $\quad \boxed{\text{Non-Arch}}$ $\quad |x|_p = p^{-v_p(x)}$

$$\left\{ v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b) \right.$$

$v_p(n) = k$ $\quad |n|_p = p^{-k}$ $\quad v_p(0) = \infty$

$v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{0\}$

$|\cdot|_p \cdot \mathbb{Q}_p \longrightarrow$ gives a topology where open basis given by balls:

$$B(a, r) = \left\{ x \in K \mid |x - a| < r \right\}$$

$|\cdot|_1$, $|\;|_2$ equivalent if they define same topology.

locally compact fields $\mathbb{R}$, $\mathbb{Q}_p$

$\boxed{K}$ valued field $\quad \mathcal{O}_K = \{ a \in K \mid v(a) \geq 0 \}$

$|a|_p \leq 1$ in p-adic case

unique max ideal $\quad M_K = \{ a \in K \mid v(a) > 0 \}$

$|a|_p < 1$ in p-adic case.

Residue field $\quad K_R = \mathcal{O}_K / M_K$

$\mathbb{Q}_\infty = \mathbb{R}$

Residue field $K_k = \mathcal{O}_K / M_K$

$Q_\infty = R$

Value group Γ

$$\left\{ \begin{array}{l} \mathcal{O}_{Q_p} = \mathbb{Z}_p \\ M_{Q_p} = p\mathbb{Z} \\ K_{Q_p} = \mathbb{F}_p \\ \Gamma = \mathbb{Z} \end{array} \right\}$$

Def $\mathbb{A}_Q$

$$\left\{ (a(p))_p \in \prod_{p \le \infty} Q_p \;\middle|\; \begin{array}{l} a(p) \in \mathbb{Z}_p \text{ for all} \\ \text{but finitely many } p \end{array} \right\} = \mathbb{A}_Q$$

note: $a(\infty) \in \mathbb{R}$

Ring componentwise operations

locally compact.

$$Q \xhookrightarrow{\text{embedding}} \mathbb{A}Q$$

$$a \longmapsto (a, a, \dots)$$

principal adele

$v_p(a) = 0 \quad p \gg 0$

$\mathbb{A}Q/Q$ compact.

$$\mathbb{I}_K = \mathbb{A}_K^{\times} = GL_1(\mathbb{A}_K)$$

$V_K = \{$all $|\cdot|_v$ on $K\}$ up to equivalence

$\exists$ finitely many Archimedean completions K_v

the rest are non-Archim and each K_v is a finite ext of Q_p some p

$$\begin{tikzcd} & K_v \\ K \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \\ & \vdots \end{tikzcd}$$

consider the valuation ring $\mathcal{O}_{K_v}$

$\mathbb{A}K$ is restricted product with respect to $\mathcal{O}_{K_v}$.

$$\mathbb{A}K = \left\{ (a(v))_v \in' \prod_{\in V_K} K_v \;\middle|\; \begin{array}{l} a(v) \in \mathcal{O}_{K_v} \\ \text{all but finitely} \\ \text{many } v \in V_K \end{array} \right\}$$

$$\| \quad \prod_{v \in VK}{}' K_v$$

Topology and measure

$\mathbb{R}$, μ

$\mathbb{Q}_p$, μ_p Haar measure.

$(\mathbb{Q}_p, +)$ locally compact $\longrightarrow$ Haar measure $\mu_p(a+U) = \mu_p(U)$

additive Haar measure

$(\mathbb{Q}_p^\times, \cdot)$ Haar measure. multiplicative

$$\frac{1}{|x|_p}\mu_p(x) = \mu_p^*(x)$$

$$\mu_p^*(aU) = \mu_p^*(U)$$

Fix additive Haar measure on $\mathbb{Q}_p^n$ denoted $dx_1 \dots dx_n$ normalized to give $\mathbb{Z}_p^n$ volume 1.

Similarly for K_v^n (give $\mathcal{O}_{K_v}^n$ volume 1)

$$\Lambda_\infty = \{ \text{Arch abs val} \} \leftarrow \text{only } |\,|_\infty \ K = \mathbb{Q}$$

$$\Lambda^{fin} = \{ \text{valuations corresp to Non-Arch abs. val} \} \leftarrow \text{all } |\cdot|_p \ K = \mathbb{Q}$$

$$\mathbb{A}_K = \{ (a(v))_v \in \prod K_v \mid a(v) \in \mathcal{O}_{K_v} \text{ for all but fin. many } v \}$$

Basis open sets: $\prod_{v \in V_K} \Gamma_v$ s.t $\Gamma_v \subseteq K_v$ is open for all $v \in V_K$ and $\Gamma_v = \mathcal{O}_{K_v}$ for all but fin many non-Arch v

Suppose μ_v is Haar measure on K_v

Then a basis of measurable sets in $\mathbb{A}_K$ given by

$\prod_{v \in V_K} M_v$ where $M_v \subseteq K_v$ is μ_v-measurable and $M_v = \mathcal{O}_{K_v}$ for all but finitely many non-Archimedean v.

Model-theoretically, suppose L is a language $(M_i)_{i \in I}$ an I-indexed family of L-structures.

Let $\Phi(x)$ be a L-formula (in 1-variable). Define restricted product of M_i with respect to $\Phi(x)$ to be

$\prod^{\Phi(x)} M \quad \{ (a(i)) \in \prod M \mid$

$$\prod_{i\in I}^{\Phi(x)} M_i = \left\{ (a(i)) \in \prod_{i \in I} M_i \;\middle|\; M_i \models \Phi(a(i)) \text{ for all but finitely many } i \in I \right\}$$

This is a generalization of the adeles $\mathbb{A}_K$

by taking $I = V_K$, $M_i = K_v$

and $\Phi(x)$ to be the formula from the following theorem defining $\mathcal{O}_{K_v}$ uniformly in K_v independently of v.

<u>Th</u> (Cluckers-D-Leenknegt-Macintyre): $\exists$ $\mathcal{L}_{rings}$-formula $\Phi(x)$ that is $\exists\forall$ and defines $\mathcal{O}_{K_v}$ in K_v for all non-Archimedean K_v

<u>Example.</u>

$\{ \mathbb{Q}_p : p \text{ prime} \}$ $\quad$ $\exists$-formula if const for p.

$\mathcal{L}_{ring}$-formula $\quad$ $\mathcal{L}_{ring} \cup \{t\}$ $\quad$ This is easy

$K \xrightarrow{\text{completion}} K_v$

$\cup$

$\mathbb{Q}_p \supseteq \mathbb{Z}_p$ $\quad$ Need p as a parameter for defining $\mathbb{Z}_p$ in $\mathbb{Q}_p$

<u>In fact</u>: $x \in \mathbb{Z}_p \iff \mathbb{Q}_p \models \exists y\,(1 + xp = y^2)$ if $p \neq 2$ (similar if $p = 2$)

Our Th (CDLM) is more general:

Consider $\{$ Henselian valued fields with finite or pseudo-finite residue field $\}$ $\quad$ p-adic valuation and topology is algebraically definable

Then the Theorem says that the val ring of all members of this class have a uniform definition by an $\mathcal{L}_{rings}$-formula that is $\exists\forall$.

<u>Example</u>: Thm applies to all $\mathbb{Q}_p$ and all $K_v \supseteq \mathbb{Q}_p$ and all $\mathbb{F}((t))$ uniformly for all p, and all $q = p^k$ all $k \geq 1$.

Our model theory of $\mathbb{A}_K$ starts with a definable Boolean algebra:

$$\mathbb{B}_K = \{ a \in \mathbb{A}_K \mid a^2 = a \}$$

ean algebra: all $k \geq 1$.

$$\mathbb{B}_K = \{ a \in \mathbb{A}_K \mid a^2 = a \}$$

$e \leq f \iff ef = e$

Boolean algebra: $e \wedge f = ef$

$e \vee f = e + f - ef$

$\bar{e} = 1 - e$

1-1 Correspondence between subsets of V_K and idempotents e_X (supported on X)

$$V_K \supseteq X \longrightarrow e_X \in \mathbb{B}_K$$

$$e_X(v) = \begin{cases} 1 & \text{if } v \in X \\ 0 & \text{if } v \notin X \end{cases}$$

Conversely, if $e \in \mathbb{A}_K$, let $X = \{ v \mid e(v) = 1 \}$
Then $e = e_X$

e min idempotent, then it corresponds to $\overset{\text{a valuation}}{\{v\}}$, $v \in V_K$

e.g. for $\mathbb{A}_{\mathbb{Q}}$:

$(1, 0, 0, \ldots) \longleftrightarrow$ real $|.|$ 2-adic val

$(1, 1, 0, 0, \ldots, 0) \longleftrightarrow \{ \text{real } |.|, v_2 \text{ (or } |.|_2) \}$

$(1, 1, 1, 0, \ldots, 0, \ldots) \longleftrightarrow \{ \text{real } |.|, v_2, v_3 \}$

Consider

$\{ e \in \mathbb{B}_K \mid e$ is a union of finitely many atoms $\}$ ← call it set of all finite support idempotents

e.g. $\{ (1, 0, 1, 0, 1, \underbrace{\ldots 0, \ldots, 0}_{\text{all } 0}) \}$

$$\mathrm{Fin}_K = \{ \text{fin. supp idempot} \}$$

<u>Th:</u> Fin_K is an $\mathcal{L}_{\text{ring}}$-definable subset of $\mathbb{A}_K$

Now Define definable functions $\llbracket \psi(x_1, \ldots, x_n) \rrbracket : \mathbb{A}_K^n \to \mathbb{A}_K$ by

$\llbracket \psi(a_1, \ldots, a_n) \rrbracket = \sup\{ e \text{ atom} \mid e\mathbb{A}_K^n \models \psi(e\ldots$

$\cdots \cup \mathbb{A}_K \to \mathbb{A}_K$ by

$$[\![\varphi(a_1, \ldots, a_n)]\!] = \sup\{ e \text{ atom} \mid e\mathbb{A}_K^n \models \varphi(ea_1, \ldots, ea_n) \}$$

This is a ring-theoretic version of

$$[\![\varphi(a_1, \ldots, a_n)]\!] = \{ i \in I \mid \mathcal{M}_i \models \varphi(a_1(i), \ldots, a_n(i)) \}$$

$\varphi(x_1, \ldots, x_n)$ is fixed $\mathcal{L}_{rings}$-formula

Example. $a \in \mathbb{A}_\mathbb{Q}$, let $\varphi(x)$ be the formula "x is a square".
Then $[\![\varphi(a)]\!]$ is the idempotent that is the union of all the minimal idempotents e s.t. $a(e)$ is a square in $e\mathbb{A}_\mathbb{Q}$. Note that if e corresponds to $\{p\}$ ($p \leq \infty$) then $e\mathbb{A}_\mathbb{Q} \simeq \mathbb{Q}_p$
and $[\![\varphi(a)]\!]$ is an idempotent supported on the set of all p such that $a(p)$ is a square.

$C_j([\![\varphi(a_1, \ldots, a_n)]\!])$ means

$[\![\varphi(a_1, \ldots a_n)]\!]$ is a union of at least

j many atoms

It says $\varphi(a_1(p), \ldots, a_n(p))$ holds in at least j many $\mathbb{Q}_p$

It is an idempotent that has at least j many 1's.

Th. $\varphi(x_1, \ldots x_n)$ $\mathcal{L}_{rings}$-formula. Then $\exists$ $\mathcal{L}_{rings}$-formula $\theta(x_1, \ldots, x_n)$ that is a Boolean combination of formulas of the form $Fin([\![\psi(x_1, \ldots, x_n)]\!])$ and
$C_j([\![\phi(x_1, \ldots, x_n)]\!])$ such that

$$\mathbb{A}_K \models \forall x_1 \ldots \forall x_n \left(\varphi(x_1, \ldots, x_n) \iff \theta(x_1, \ldots, x_n) \right)$$

where ψ and ϕ are $\mathcal{L}_{rings}$-formulas.

← This is quantifier-elimination for $\mathbb{A}_\mathbb{Q}$

Cor: $X \subseteq \mathbb{A}_K^n$ def $\Rightarrow$ X measurable ←

(ctble unions
& intersec

usr : $x \subseteq \mathbb{A}_K$ [illegible] ⟵

(ctble unions & intersections of open sets)

Q : what are measures of definable sets ?

Ax, in his 1968 Ann Math paper proves that

$\{\mathbb{F}_p : p \text{ prime}\}$
$\{\mathbb{F}_q : q \text{ prime power}\}$ are decidable

(This means given $\mathcal{L}_{rings}$-sentence φ we can decide if it holds in all $\mathbb{F}_p$ or all $\mathbb{F}_q$)

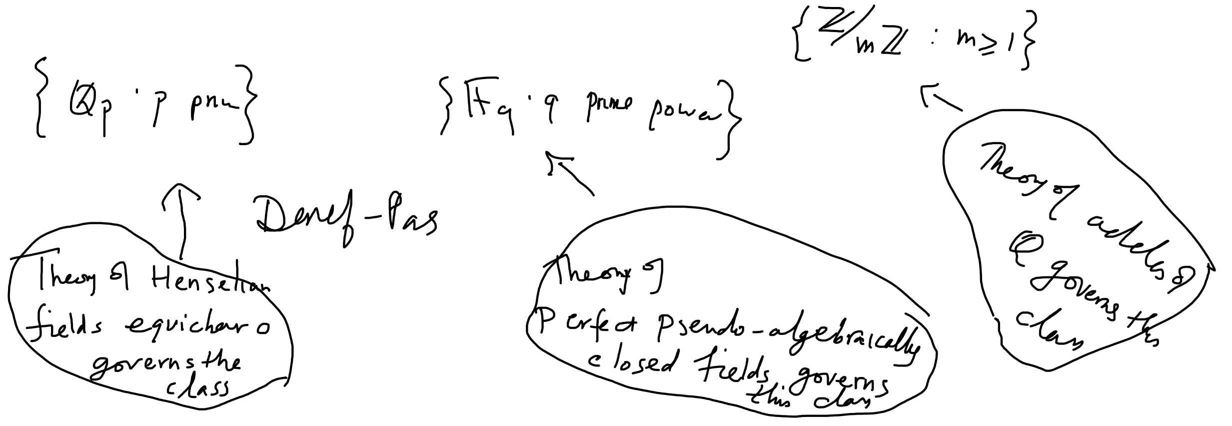
The asymptotic theories are also decidable

He asked the following question

Problem (Ax) . Is the $\mathcal{L}_{rings}$-theory of all the rings $\mathbb{Z}/m\mathbb{Z}$, $m \geq 1$ decidable ?

In other terms, given φ we want to decide if it holds in all $\mathbb{Z}/m\mathbb{Z}$, all $m \geq 1$

We solved Ax's Problem positively



$\{\mathbb{Z}/m\mathbb{Z} : m \geq 1\}$

$\{\mathbb{Q}_p : p \text{ prime}\}$

$\{\mathbb{F}_q : q \text{ prime power}\}$

Denef-Pas

Theory of Henselian fields equichar 0 governs the class

Theory of Perfect pseudo-algebraically closed fields governs this class

Theory of adeles of $\mathbb{Q}$ governs the class

Ax Problem can be reduced to the decidability of $\mathbb{A}_\mathbb{Q}$

Weispfenning (1978) gave first proof of decidability of $\mathbb{A}_\mathbb{Q}$

We give new Pf of decidability only Ax's 1968 paper and our quantifier-elimination above

IV. Nonstandard models of PA   P D'Aquino - Macintyre
using $\mathbb{A}_\mathbb{Q}$

proved that quotients of nonstandard models of PA by principal ideals are "tame".

This uses work of D'Aquino – D – Macintyre

$\mathbb{Z}_p \xrightarrow{v_p} \mathbb{Z}$ on truncations of ordered abelian groups TOAG

$\Rightarrow \mathbb{Z}_p/p^k\mathbb{Z} \xrightarrow{v_{p^k}} \mathbb{Z}/\Delta$ truncation of p-adic valuation and truncations of ordered abelian groups

We study these truncations, give axioms for a class of semi-groups that arise from truncating OAG.

useful in model theory of local rings and their truncations (D'Aquino-Macintyre)