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A note on the dual of Burch's inequality

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Abstract

The aim of this paper is to improve the main result of [5] (Theorem 3.5). We show that if A is a non-zero Artinian module over a commutative ring R and $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals of R such that $(0 :_A \mathfrak{b}) \neq 0$, then the dual of Burch's inequality

$$S_{\mathfrak{b}}(\mathfrak{a}, A) \leq \mathrm{Kdim}_R(A) - \mathrm{width}_{\mathfrak{b}}(0 :_A \mathfrak{a}^i) \ (i \gg 0)$$

holds (the dual notions $S_{\mathfrak{b}}(\mathfrak{a}, A)$, $\mathrm{Kdim}_R(A)$, $\mathrm{width}_{\mathfrak{b}}(A)$ are explained in [5]).

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Let N be a finitely generated module over a commutative Noetherian ring R and $I \subseteq J$ be ideals in R satisfying $N \neq JN$. We denote the graded R-algebra $\bigoplus_{i \geq 0} I^i$ (the Rees ring of I) by $\tilde{R}$. In [1], the analytic spread of I at J with respect to N, which is denoted by $l_J(I, N)$, was defined as the Krull dimension of the $\tilde{R}$-annihilator of the graded $\tilde{R}$-module $\bigoplus_{0 \leq i} I^i N / I^i JN$ and the generalization of Burch's inequality [2, Corollary (i)]

$$l_J(I, N) \leq \dim_R(N) - \mathrm{depth}_J(N/I^i N) \quad (i \gg 0)$$

was proved. Here, $\dim_R(N)$ means the Krull dimension of N and $\mathrm{depth}_J(N/I^i N)$ denotes the length of a maximal $N/I^i N$-sequences in J.

Let A be a non-zero Artinian module over a commutative ring R and let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R. The dual notions $S_{\mathfrak{b}}(\mathfrak{a}, A)$, $\mathrm{Kdim}_R(A)$ and $\mathrm{width}_{\mathfrak{b}}(A)$, to those of

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$l_{\mathfrak{b}}(\mathfrak{a},N)\dim_R(N)$ and $\operatorname{depth}_{\mathfrak{b}}(N)$ were studies in [5, 9], [4, 7] and [6, 10], respectively. The aim of [5] is to prove the dual version of Burch's inequality

$$S_{\mathfrak{b}}(\mathfrak{a},A) \le \operatorname{Kdim}_R(A) - \operatorname{width}_{\mathfrak{b}}(0 :_A \mathfrak{a}^i) \quad (i \gg 0),$$

in the following conditions: R is a Noetherian ring, $R/\mathfrak{b}$ is Artinian and $\mathfrak{b}$ is contained in any associated prime of A.

In this paper we shall improve this result by showing that if $(0 :_A \mathfrak{b}) \ne 0$, then the above inequality holds without any restriction on R (see Theorem 6(ii)). Also, we prove that $\operatorname{width}_{\mathfrak{b}}(0 :_A \mathfrak{a}^i)$ takes a constant value for large i (see Theorem 6(i)).

Throughout R will denote a (non-trivial) commutative ring with identity and A will be a non-zero Artinian R-module.

Remark 1 (See [8, Theorem 3.2 and the proof of Lemma 2.2]). There exist only finitely many maximal ideals $\mathfrak{m}$ of R for which $\operatorname{Soc}(A)$ has a submodule isomorphic to $R/\mathfrak{m}$. Let the distinct such maximal ideals be $\mathfrak{m}_1,\dots,\mathfrak{m}_s$. It is easy to see that $\operatorname{Ass}_R(A) = \{\mathfrak{m}_1,\dots,\mathfrak{m}_s\}$. Set $J = \bigcap_{i=1}^s \mathfrak{m}_i$, and let $\hat{R}$ be the J-adic completion of R.

(i) The ring $R' := \hat{R}/(0 :_{\hat{R}} A)$ is a semi-local commutative Noetherian ring which is complete in the topology defined by its Jacobson radical.

(ii) For $\hat{r} = (r_i + J^i)_{i\ge 1} \in \hat{R}$ and $x \in A$, the sequence $(r_i x)_{i\ge 1}$ is ultimately constant and $\hat{r}x$ is defined as the ultimate constant value of the above sequence. Hence the module A is, in a natural way, a faithful Artinian module over R', and a subset of A is an R-module if and only if it is an R'-module. Moreover, if $\psi : R \to R'$ is the natural map, then, for any $r \in R$, the multiplication by r on A has the same effect as multiplication by $\psi(r)$ on A.

Notation 2. Throughout the remainder of the paper, R' and $\hat{R}$ are as in Remark 1 and the following notations will be used.

Set $E := \bigoplus_{\mathfrak{m}' \in \operatorname{Max}(R')} E(R'/\mathfrak{m}')$, where $\operatorname{Max}(R')$ is the set of all maximal ideals of R' and $E(R'/\mathfrak{m}')$ is the injective envelope of $R'/\mathfrak{m}'$. We shall use $D(\cdot)$ to denote the additive, exact, R-linear functor $\operatorname{Hom}_{R'}(\cdot, E)$ from the category of all R'-modules and R'-homomorphisms to itself. Also, for any ideal I of R, we use IR' to denote the extension of I to R' under ψ.

We recall some facts in the following.

Remark 3. (i) (See the proof of [3, Theorem 2.1].) Let I be an ideal of R such that $(0 :_A I) \ne 0$. Then $\operatorname{width}_I(A) = \operatorname{depth}_{IR'}(D(A))$.

(ii) (See [11, Lemma 2.13(i)].) Let L be an R-submodule of A. Then $\operatorname{Kdim}_R(L) = \dim_{R'}(D(L))$.

For the proof of Theorem 5, we need the following lemma.

Lemma 4. *Let R be a Noetherian ring. Then* $\mathrm{K}\dim_R(A) = \dim(R/(0 :_R A))$.

Proof. In view of the proof of [5, Corollary 1.11], we may assume that A is a faithful R-module and that

$$\mathrm{Kdim}_R(A) = \dim\left(\hat{R}/\left(0 :_{\hat{R}} A\right)\right) \leq \dim(\hat{R}) = \dim(R).$$

Now, we prove that $\dim(\hat{R}) \leq \dim(\hat{R}/(0 :_{\hat{R}} A))$. It is enough to show that if $\mathfrak{q} \in \mathrm{Spec}(\hat{R})$, then $(0 :_{\hat{R}} A)) \subseteq \mathfrak{q}$. Let $\mathfrak{q} \in \mathrm{Spec}(\hat{R})$, and set $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, where $\varphi : R \to \hat{R}$ is the natural map. Then, by [12, Corollary 2.4 and Lemma 2.5], there exists a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \in \mathrm{Supp}_R(D_{\mathfrak{m}}(A))(D_{\mathfrak{m}}(\cdot)$ is the functor $\mathrm{Hom}_R(\cdot, E(R/\mathfrak{m})))$. Since $\hat{R}_{\mathfrak{q}}$ is a faithfully flat $\hat{R}_{\mathfrak{p}}$-module, we have $D_{\mathfrak{m}}(A)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{q}} \neq 0$. On the other hand, we have

$$\begin{aligned} D_{\mathfrak{m}}(A)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{q}} &\cong (D_{\mathfrak{m}}(A) \otimes_{\hat{R}} S^{-1}\hat{R}) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{q}} \\ &\cong D_m(A)_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} S^{-1}\hat{R}, \end{aligned}$$

where $S = \varphi(R - \mathfrak{p})$. Therefore $\mathfrak{q} \in \mathrm{Supp}_{\hat{R}}(D_m(A))$ and consequently $(0 :_{\hat{R}} A) \subseteq (0 :_{\hat{R}} D_m(A)) \subseteq \mathfrak{q}$. □

Theorem 5. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R. Then* $S_{\mathfrak{b}}(\mathfrak{a}, A) = l_{\mathfrak{b}R'}(\mathfrak{a}R', D(A))$. (*Note that, by* [6, 1.6(3)], *$D(A)$ is a finitely generated R'-module.*)

Proof. Let $S = \bigoplus_{0 \leq i} \mathfrak{a}^i \to S' = \bigoplus_{0 \leq i} \mathfrak{a}^i R'$ be the natural map from the Rees ring of $\mathfrak{a}$ to that of $\mathfrak{a}R'$. Set $G = \bigoplus_{n \leq 0} G_n$, where $G_n = (0 :_A \mathfrak{a}^i\mathfrak{b})/(0 :_A \mathfrak{a}^i) = (0 :_A \mathfrak{a}^i\mathfrak{b}R')/(0 :_A \mathfrak{a}^i R')$ and $n = -i \leq 0$. By Remark 1(ii), any S-submodule of G is an S'-submodule. So, by [5, Definition 2.5 and Lemma 1.2], we have

$$S_{\mathfrak{b}}(\mathfrak{a}, A) = \mathrm{Kdim}_S(G) = \mathrm{Kdim}_{S'}(G).$$

Put $G'_{-n} = D(G_n)$ for $n \leq 0$ and $G' = \bigoplus_{0 \leq m} G'_m$. Then, by [5, Lemma 1.9], $G'_m = \mathfrak{a}^m R' D(A)/\mathfrak{a}^m \mathfrak{b} R' D(A)$ for $m \geq 0$. Since $(0 :_{R'} M) = (0 :_{R'} D(M))$ for any R'-module M (see[6, Theorem 1.6(8)], we have $(0 :_{S'} G) = (0 :_{S'} G')$. It therefore follows from [5, Lemma 2.4] and Lemma 4 that

$$\mathrm{Kdim}_{S'}(G) = \dim(S'/(0 :_{S'} G)) = \dim(S'/(0 :_{S'} G')) = l_{\mathfrak{b}R'}(\mathfrak{a}R', D(A)). \qquad \square$$

Theorem 6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $(0 :_A \mathfrak{a} + \mathfrak{b}) \neq 0$. The following statements hold*:

(i) $\mathrm{width}_{\mathfrak{b}}(0 :_A \mathfrak{a}^i)$ *becomes for large i eventually constant.*

(ii) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then*

$$S_{\mathfrak{b}}(\mathfrak{a}, A) \leq \mathrm{Kdim}_R(A) - \mathrm{width}_{\mathfrak{b}}(0 :_A \mathfrak{a}^i) \quad (i \gg 0).$$

Proof. (i) Follows from Remark 3(i), [5, Lemma 1.9] and [1, Theorem (2)(i)].
(ii) Follows from Remark 3 and Theorem 5. □

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References

[1] M. Brodmann, The asymptotic nature of the analytic spread, Math. Proc. Cambridge Philos. Soc. 86 (1979) 35–39.
[2] L. Burch, Codimension and analytic spread, Proc. Cambridge Philos. Soc. 72 (1972) 369–373.
[3] K. Khashyarmanesh, S.H. Salarian, M. Tousi, On the local homology theory for Artinian modules, Acta Math. Hungar. 81 (1998) 97–107.
[4] D. Kirby, Dimension and length for Artinian modules, Quart. J. Math. Oxford (2) 41 (1990) 419–429.
[5] I. Nishitani, On the dual of Burch's inequality, J. Pure Appl. Algebra 96 (1994) 147–156.
[6] A. Ooishi, Matlis duality and the width of a module, Hiroshima Math. J. 6 (1976) 573–587.
[7] R.N. Roberts, Krull dimension for Artinian modules over quasi local commutative rings, Quart. J. Math. Oxford (2) 26 (1975) 269–273.
[8] R.Y. Sharp, Artinian modules over commutative rings, Math. Proc. Cambridge Philos. Soc. 111 (1992) 25–33.
[9] R.Y. Sharp, A.J. Taherizadeh, Reductions and integral closures of ideals relative to an Artinian module, J. London Math. Soc. (2) 37 (1988) 203–218.
[10] Z. Tang, H. Zakeri, Co-Cohen–Macaulay modules and modules of generalized fractions, Comm. Algebra 22 (1994) 2173–2204.
[11] M. Tousi, A. Tehranian, Co-Cohen–Macaulay modules over commutative rings, Preprint.
[12] S. Yassemi, Coassociated primes, Comm. Algebra 23 (1995) 1499–1502.