A note on the dual of Burch’s inequality

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Abstract

The aim of this paper is to improve the main result of [5] (Theorem 3.5). We show that if \( A \) is a non-zero Artinian module over a commutative ring \( R \) and \( a \subseteq b \) are ideals of \( R \) such that \((0 :_b b) \neq 0\), then the dual of Burch’s inequality

\[ S_b(a,A) \leq \text{Kdim}_R(A) - \text{width}_b(0 :_a a') \ (i \geq 0) \]

holds (the dual notions \( S_b(a,A), \text{Kdim}_R(A), \text{width}_b(A) \) are explained in [5]). © 2000 Elsevier Science B.V. All rights reserved.

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Let \( N \) be a finitely generated module over a commutative Noetherian ring \( R \) and \( I \subseteq J \) be ideals in \( R \) satisfying \( N \neq JN \). We denote the graded \( R \)-algebra \( \bigoplus_{i \geq 0} I^i \) (the Rees ring of \( I \)) by \( \hat{R} \). In [1], the analytic spread of \( I \) at \( J \) with respect to \( N \), which is denoted by \( l_I(I,N) \), was defined as the Krull dimension of the \( \hat{R} \)-annihilator of the graded \( \hat{R} \)-module \( \bigoplus_{0 \leq i \leq l_I(I,N)/I^iN} \) and the generalization of Burch’s inequality [2, Corollary (i)]

\[ l_I(I,N) \leq \dim_R(N) - \text{depth}_J(N/I^iN) \ (i \geq 0) \]

was proved. Here, \( \dim_R(N) \) means the Krull dimension of \( N \) and \( \text{depth}_J(N/I^iN) \) denotes the length of a maximal \( N/I^iN \)-sequences in \( J \).

Let \( A \) be a non-zero Artinian module over a commutative ring \( R \) and let \( a \subseteq b \) be ideals of \( R \). The dual notions \( S_b(a,A), \text{Kdim}_R(A), \text{width}_b(A), \) to those of

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\[ l_b(a, N) \dim_R(N) \text{ and } \dim_b(N) \] were studies in [5, 9], [4, 7] and [6, 10], respectively. The aim of [5] is to prove the dual version of Burch's inequality

\[ S_b(a, A) \leq K \dim_R(A) - \text{width}_b(0 :_A a^i) \quad (i \geq 0), \]

in the following conditions: \( R \) is a Noetherian ring, \( R/b \) is Artinian and \( b \) is contained in any associated prime of \( A \).

In this paper we shall improve this result by showing that if \( (0 :_A a^i) \neq 0 \), then the above inequality holds without any restriction on \( R \) (see Theorem 6(ii)). Also, we prove that \( \text{width}_b(0 :_A a^i) \) takes a constant value for large \( i \) (see Theorem 6(i)).

Throughout \( R \) will denote a (non-trivial) commutative ring with identity and \( A \) will be a non-zero Artinian \( R \)-module.

**Remark 1** (See [8, Theorem 3.2 and the proof of Lemma 2.2]). There exist only finitely many maximal ideals \( m \) of \( R \) for which \( \text{Soc}(A) \) has a submodule isomorphic to \( R = m \). Let the distinct such maximal ideals be \( m_1, \ldots, m_s \). It is easy to see that \( \text{Ass}_R(A) = \{m_1, \ldots, m_s\} \). Set \( J = \bigcap_{i=1}^s m_i \), and let \( \hat{R} \) be the \( J \)-adic completion of \( R \).

(i) The ring \( R' := \hat{R}/(0 :_{\hat{R}} A) \) is a semi-local commutative Noetherian ring which is complete in the topology defined by its Jacobson radical.

(ii) For \( \hat{f} = (r_j + J')_{j\geq 1} \in \hat{R} \) and \( x \in A \), the sequence \( (r_j x)_{j\geq 1} \) is ultimately constant and \( \hat{f} x \) is defined as the ultimate constant value of the above sequence. Hence the module \( A \) is, in a natural way, a faithful Artinian module over \( R' \), and a subset of \( A \) is an \( R \)-module if and only if it is an \( R' \)-module. Moreover, if \( \psi : R \to R' \) is the natural map, then, for any \( r \in R \), the multiplication by \( r \) on \( A \) has the same effect as multiplication by \( \psi(r) \) on \( A \).

**Notation 2.** Throughout the remainder of the paper, \( R' \) and \( \hat{R} \) are as in Remark 1 and the following notations will be used.

Set \( E := \bigoplus_{m' \in \text{Max}(R')} E(R'/m'), \) where \( \text{Max}(R') \) is the set of all maximal ideals of \( R' \) and \( E(R'/m') \) is the injective envelope of \( R'/m' \). We shall use \( D(\cdot) \) to denote the additive, exact, \( R \)-linear functor \( \text{Hom}_{R'}(\cdot, E) \) from the category of all \( R' \)-modules and \( R' \)-homomorphisms to itself. Also, for any ideal \( I \) of \( R \), we use \( IR' \) to denote the extension of \( I \) to \( R' \) under \( \psi \).

We recall some facts in the following.

**Remark 3.** (i) (See the proof of [3, Theorem 2.1].) Let \( I \) be an ideal of \( R \) such that \( (0 :_A I) \neq 0 \). Then \( \text{width}_I(A) = \text{depth}_{IR'}(D(A)) \).

(ii) (See [11, Lemma 2.13(i)].) Let \( L \) be an \( R \)-submodule of \( A \). Then \( K \dim_R(L) = \dim_{R'}(D(L)) \).

For the proof of Theorem 5, we need the following lemma.
Lemma 4. Let $R$ be a Noetherian ring. Then $K \dim_R(A) = \dim(R/(0 :_R A))$.

Proof. In view of the proof of [5, Corollary 1.11], we may assume that $A$ is a faithful $R$-module and that

$$K \dim_R(A) = \dim \left( \frac{\hat{R}}{(0 :_\hat{R} A)} \right) \leq \dim(\hat{R}) = \dim(R).$$

Now, we prove that $\dim(\hat{R}) \leq \dim(\hat{R}/(0 :_\hat{R} A))$. It is enough to show that if $q \in \text{Spec}(\hat{R})$, then $(0 :_\hat{R} A) \subseteq q$. Let $\mathfrak{p} \in \text{Spec}(\hat{R})$, and set $\varphi^{-1}(\mathfrak{p}) = p$, where $\varphi : R \rightarrow \hat{R}$ is the natural map. Then, by [12, Corollary 2.4 and Lemma 2.5], there exists a maximal ideal $m$ of $R$ such that $p \in \text{Supp}_R(D_m(A))(D_m(\cdot))$ is the functor $\text{Hom}_R(\cdot,E(R/m))$. Since $\hat{R}_q$ is a faithfully flat $\hat{R}_q$-module, we have $D_m(A)_p \otimes_{R_q} \hat{R}_q \neq 0$. On the other hand, we have

$$D_m(A)_p \otimes_{R_q} \hat{R}_q \cong (D_m(A) \otimes_{\hat{R}} S^{-1}\hat{R}) \otimes_{R_q} \hat{R}_q \cong D_m(A)_q \otimes_{R_q} S^{-1}\hat{R},$$

where $S = \varphi(R - p)$. Therefore $q \in \text{Supp}_R(D_m(A))$ and consequently $(0 :_\hat{R} A) \subseteq (0 :_R D_m(A)) \subseteq q$. \qed

Theorem 5. Let $a \subseteq b$ be ideals of $R$. Then $S_b(a,A) = l_{bR'}(aR',D(A))$. (Note that, by [6, 1.6(3)], $D(A)$ is a finitely generated $R'$-module.)

Proof. Let $S = \bigoplus_{0 \leq i} a^i \rightarrow S' = \bigoplus_{0 \leq i} a^i R'$ be the natural map from the Rees ring of $a$ to that of $aR'$. Set $G = \bigoplus_{n \leq 0} G_n$, where $G_n = (0 :_A a^ib)/(0 :_A a^i) = (0 :_A a^i R')/ (0 :_A a^i R')$ and $n = -i \leq 0$. By Remark 1(ii), any $S$-submodule of $G$ is an $S'$-submodule. So, by [5, Definition 2.5 and Lemma 1.2], we have

$$S_b(a,A) = K \dim_S(G) = K \dim_{S'}(G).$$

Put $G'_n = D(G_n)_{\leq m}$ for $n \leq 0$ and $G'_n = \bigoplus_{0 \leq m} G'_n$. Then, by [5, Lemma 1.9], $G'_m = a^m R'D(A)/a^m b R'D(A)$ for $m \geq 0$. Since $(0 :_{R'} M) = (0 :_{R'} D(M))$ for any $R'$-module $M$ (see [6, Theorem 1.6(8)]), we have $(0 :_{S'} G) = (0 :_{S'} G')$. It therefore follows from [5, Lemma 2.4] and Lemma 4 that

$$K \dim_{S'}(G) = \dim(S'/0 :_{S'} G)) = \dim(S'/0 :_{S'} G))) = l_{bR'}(aR',D(A)). \qed$$

Theorem 6. Let $a$ and $b$ be ideals of $R$ such that $(0 :_A a + b) \neq 0$. The following statements hold:

(i) $\text{width}_b(0 :_A a^i)$ becomes for large $i$ eventually constant.

(ii) If $a \subseteq b$, then

$$S_b(a,A) \leq K \dim_R(A) - \text{width}_b(0 :_A a^i) \quad (i \gg 0).$$
Proof. (i) Follows from Remark 3(i), [5, Lemma 1.9] and [1, Theorem (2)(i)].
(ii) Follows from Remark 3 and Theorem 5.

Acknowledgements

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