ELSEVIER

Journal of Pure and Applied Algebra 155 (2001) 17–28

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

A generalization of the dualizing complex structure and its applications

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Received 1 July 1998; received in revised form 20 May 1999

Comunicated by C.A. Weibel

Abstract

The structural property of dualizing complex is generalized and its applications are given to show that (1) over a ring possessing a dualizing complex, the conditions of Serre on the canonical module are connected with the local cohomologies of the ring itself; (2) the Cousin complexes of certain modules over a ring possessing a dualizing complex have finitely generated cohomologies; (3) a quotient of a Cohen–Macaulay local ring which satisfies (S_2) and admits canonical module, also possesses a dualizing complex. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 13D25; 13H10; 13D45

0. Introduction

In [4], the authors have described that, for an (S_2) local ring A, the fundamental dualizing complex for A, if it exists, is isomorphic to the Cousin complex of the canonical module K of A with respect to the height filtration (= the dimension filtration) of A. This structural property of dualizing complex has been proved to be useful in applications. In this paper we generalize this structural property in the following direction: if $(A, \mathfrak{m})$ possesses a fundamental dualizing complex $I^{\cdot}$, M is a finitely generated A-module satisfying (S_2) and $\mathrm{Min}_A(M) = \mathrm{Assh}_A(M)$, then

$$0 \to H^{d-s}(\mathrm{Hom}_A(M, I^{\cdot})) \to \mathrm{Hom}_A(M, I^{d-s}) \to \mathrm{Hom}_A(M, I^{d-s+1}) \to \cdots$$

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PII: S0022-4049(99)00160-7

is the Cousin complex of $H^{d-s}(\mathrm{Hom}_A(M, I^{\cdot}))$ with respect to some appropriate filtration, where $d = \dim A$ and $s = \dim_A(M)$; this is the main objective of Section 1. We will find some applications of this structural property in Sections 2–4.

In Section 2, for an (S_2) local ring $(A, \mathfrak{m})$ which possesses a dualizing complex, we examine the effect of Serre condition (S_n) on its canonical module K. We show among other things that K is (S_n) if and only if $H_{\mathfrak{m}}^{\dim A - i}(A) = 0$, for $i = 1, \ldots, n-2$, which is equivalent to saying that $\mu^i(\mathfrak{p}, K) = \delta_{i\, \mathrm{ht}\, \mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$ and $0 \le i < n$, where $\mu^i(\mathfrak{p}, K)$ is the Bass number of K and δ_{ij} is the Kronecker delta. It is also shown that if $(A, \mathfrak{m})$ is a complete and (S_2) local domain such that $H_{\mathfrak{m}}^{-1+\dim A}(A) = 0$ then A satisfies the Canonical Element Conjecture.

In Section 3, the main result is Theorem 3.2: Assume that $(A, \mathfrak{m})$ is a local ring possessing a dualizing complex. Then for each finitely generated A-module M where $\mathrm{Min}_A(M) = \mathrm{Assh}_A(M)$ and M satisfies (S_2), all cohomology modules of the Cousin complex of M are finitely generated.

In Section 4, we prove that if A is an (S_2) local ring admitting canonical module such that all of its formal fibres are Cohen–Macaulay, then A also possesses a dualizing complex.

Throughout A is a commutative Noetherian ring with non-zero identity, and M is an A-module. For each $n \ge 0$, M is said to satisfy (S_n) whenever $\mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \ge \min\{\mathrm{ht}_M \mathfrak{p}, n\}$ for all $\mathfrak{p} \in \mathrm{Supp}_A(M)$. Denote, by $\mathrm{Min}_A(M)$, the set of all minimal elements of $\mathrm{Supp}_A(M)$. For a finite-dimensional module M, $\mathrm{Assh}_A(M) = \{\mathfrak{p} \in \mathrm{Supp}_A(M): \dim(A/\mathfrak{p}) = \dim M\}$.

1. A generalization of dualizing complex structure

The main purpose of this section is to find a generalization of [4, Theorem 2.4 and Corollary 2.5]. Let us fix our notations.

Dualizing complexes 1.1. (See [13, Further definitions 2.4; 6, Theorem 3.6; 15, Definition 1.1 and Theorem 1.2]). A *dualizing complex* $I^{\cdot}$ for A is a complex of A-modules and A-homomorphisms

$$I^{\cdot} : 0 \longrightarrow I^0 \xrightarrow{\delta^0} I^1 \longrightarrow \cdots \xrightarrow{\delta^{l-1}} I^l \longrightarrow 0$$

such that

(i) $I^{\cdot}$ is a bounded complex of injective A-modules and $I^0 \neq 0$, $I^l \neq 0$;
(ii) for each i, $H^i(I^{\cdot})$, the ith homology module of $I^{\cdot}$, is a finitely generated A-module;
(iii) $\bigoplus_{0 \le i \le l} I^i \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(A)} E(A/\mathfrak{p})$, where $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$ as A-module, that is, each prime ideal of A occurs in exactly one term of $I^{\cdot}$, and occurs exactly once.

For each $\mathfrak{p} \in \mathrm{Spec}(A)$, denote $t(\mathfrak{p}; I^{\cdot})$ by the unique integer t for which $E(A/\mathfrak{p})$ is a summand of I^t (see [14, p. 208]).

Lemma 1.2 (Sharp [14, Lemma 3.3]). *With the above notation, suppose that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals of* A *such that* $\mathfrak{p} \subset \mathfrak{q}$ *and there is no prime ideal strictly between* $\mathfrak{p}$ *and* $\mathfrak{q}$. *Then*

$$t(\mathfrak{q}; I^{\cdot}) = t(\mathfrak{p}; I^{\cdot}) + 1.$$

Cousin complexes 1.3 (see [16]). A filtration of $\mathrm{Spec}(A)$ is a descending sequence $\mathscr{F} = (F_i)_{i \geq 0}$ of subsets of $\mathrm{Spec}(A)$, so that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots$$

with the property that, for each $i \geq 0$, each member of $\partial F_i = F_i - F_{i+1}$ is a minimal member of F_i, with respect to inclusion. We say that $\mathscr{F}$ admits M if $\mathrm{Supp}_A(M) \subseteq F_0$. Suppose that $\mathscr{F}$ is a filtration of $\mathrm{Spec}(A)$ that admits M. The Cousin complex $C(\mathscr{F}, M)$ for M with respect to $\mathscr{F}$ has the form

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \to \cdots \to M^n \xrightarrow{d^n} M^{n+1} \to \cdots$$

where $M^n = \bigoplus_{\mathfrak{p} \in \partial F_n} (\mathrm{Coker}\, d^{n-2})_{\mathfrak{p}}$ for all $n \geq 0$. The homomorphisms in this complex have the following properties: for $m \in M$ and $\mathfrak{p} \in \partial F_0$, the component of $d^{-1}(m)$ in $M_{\mathfrak{p}}$ is $m/1$; for $n > 0$, $x \in M^{n-1}$ and $\mathfrak{q} \in \partial F_n$, the component of $d^{n-1}(x)$ in $(\mathrm{Coker}\, d^{n-2})_{\mathfrak{q}}$ is $\bar{x}/1$, where $- : M^{n-1} \to \mathrm{Coker}\, d^{n-2}$ is the canonical epimorphism.

Assume that A possesses a dualizing complex $I^{\cdot}$. Let M be a finitely generated A-module, we set $k = \min\{ j\colon \mathrm{Supp}_A(M) \cap \mathrm{Ass}_A(I^j) \neq \phi\}$, and, for each $i \geq 0$,

$$T_i := \{\mathfrak{p} \in \mathrm{Supp}_A(M)\colon t(\mathfrak{p}; I^{\cdot}) \geq i + k\}.$$

By Lemma 1.2, $\mathscr{T}_M^{I^{\cdot}} := (T_i)_{i \geq 0}$ is a filtration of $\mathrm{Spec}(A)$ that admits $K_M^{I^{\cdot}} = H^k(\mathrm{Hom}_A(M, I^{\cdot}))$. Note that we have $\mathrm{Ass}_A(\mathrm{Hom}_A(M,N)) = \mathrm{Supp}_A(M) \cap \mathrm{Ass}_A(N)$ for all A-module N, so $\mathrm{Hom}_A(M, I^j) = 0$ for all $j < k$ and $K_M^{I^{\cdot}} \cong \mathrm{Hom}_A(M, \mathrm{Ker}\, \delta^k)$. We are now in a position to give a generalization of [4, Theorem 2.4].

Theorem 1.4. *Assume that the ring* A *possesses a dualizing complex*

$$I^{\cdot} : 0 \to I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots \to I^l \to 0.$$

For a finitely generated A*-module* M, *let the situations and notations be as above. Set* $\mathrm{Hom}_A(M, I^{\cdot})^*$:

$$0 \to K_M^{I^{\cdot}} \to \mathrm{Hom}_A(M, I^k) \to \mathrm{Hom}_A(M, I^{k+1}) \to \cdots \to \mathrm{Hom}_A(M, I^l) \to 0,$$

such that $(\mathrm{Hom}_A(M, I^{\cdot})^*)^{-1} = K_M^{I^{\cdot}}$, *and* $(\mathrm{Hom}_A(M, I^{\cdot})^*)^i = \mathrm{Hom}(M, I^{k+i})$, $i = 0, 1, \ldots$. *The following statements are true*:

(i) *There exists a* (*unique*) *homomorphism of complexes*

$$\Psi_M^{I^{\cdot}} = (\psi^i)_{i \geq -2} : C(\mathscr{T}_M^{I^{\cdot}}, K_M^{I^{\cdot}}) \to \mathrm{Hom}_A(M, I^{\cdot})^*$$

(*over* $\mathrm{Id}_{K_M^{I^{\cdot}}}$) *from the Cousin complex of* $K_M^{I^{\cdot}}$ *with respect to* $\mathscr{T}_M^{I^{\cdot}}$ *to the induced complex* $\mathrm{Hom}_A(M, I^{\cdot})^*$ *of* $\mathrm{Hom}_A(M, I^{\cdot})$.

(ii) $\mathrm{Min}_A(M)=\mathrm{Ass}_A(K_M^{I^\cdot})$ *if and only if* $\mathscr{T}_M^{I^\cdot}=\mathscr{H}_M$, *where* $\mathscr{H}_M=(H_i)_{i\geq 0}$ *is the height filtration of* M *that is* $H_i=\{\mathfrak{p}\in \mathrm{Supp}_A(M)\colon\ \mathrm{ht}_M(\mathfrak{p})\geq i\}$ *for all* $i\geq 0$. *In this situation* $\mathrm{Supp}_A(M)=\mathrm{Supp}_A(K_M^{I^\cdot})$.
(iii) *If* $\mathrm{Min}_A(M)=\mathrm{Ass}_A(K_M^{I^\cdot})$, *then* M *is* (S_1) *if and only if* $\Psi_M^{I^\cdot}$ *is an epimorphism.*
(iv) *If* $\mathrm{Min}_A(M)=\mathrm{Ass}_A(K_M^{I^\cdot})$, *then* M *is* (S_2) *if and only if* $\Psi_M^{I^\cdot}$ *is an isomorphism.*

Proof. (i) For each $i\geq 0$, set $X^i=\mathrm{Hom}_A(M,I^{k+i})$ and $X^{-1}=K_M^{I^\cdot}$. We show that the complex

$$X^\cdot : 0\xrightarrow{e^{-2}}X^{-1}\xrightarrow{e^{-1}}X^0\xrightarrow{e^0}\cdots,$$

where e^{-2} is the zero map, e^{-1} is inclusion map and $e^i=\mathrm{Hom}_A(\mathrm{Id}_M,\delta^{i+k})$ for $i\geq 0$, satisfies the conditions of [4, Proposition 2.1]. The condition $\mathrm{Supp}_A(X^i)\subseteq T_i$ for all $i\geq 0$, is clear. For the second condition we proceed as follows. Let $i\geq 0$. Recall that for an A-module X with $\mathrm{Supp}_A(X)\subseteq T_i$, we denote by $\xi(X)\colon X\to\bigoplus_{\mathfrak{p}\in\partial T_i}X_{\mathfrak{p}}$ the natural map, for which, if $x\in X$ then the component of $\xi(X)(x)$ in $X_{\mathfrak{p}}$ is $x/1$. Set $\mathscr{F}=(F_j)_{j\geq 0}$ to be such that $F_j=\{\mathfrak{p}\in\mathrm{Spec}(A)\colon\ t(\mathfrak{p},I^\cdot)\geq j\}$ and $I=I^{i+k}$. Thus $\partial T_i=\mathrm{Supp}_A(M)\cap\mathrm{Ass}_A(I)$. Since M is a finitely generated A-module, for each $\mathfrak{p}\in\partial T_i$, the natural map $\theta_{\mathfrak{p}}:\mathrm{Hom}_A(M,I_{\mathfrak{p}})\to\mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}},I_{\mathfrak{p}})$, where, for each $f\in\mathrm{Hom}_A(M,I_{\mathfrak{p}})$, $\theta_{\mathfrak{p}}(f)(m/s)=(1/s)f(m)$ for all $m/s\in M_{\mathfrak{p}}$, is an isomorphism. We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_A(M,I) & & \xrightarrow{\xi(\mathrm{Hom}_A(M,I))} & & \bigoplus_{\mathfrak{p}\in\partial T_i}(\mathrm{Hom}_A(M,I))_{\mathfrak{p}} \\
\downarrow{\scriptstyle \mathrm{Hom}_A(\mathrm{Id}_M,\xi(I))} & & & & \\
\mathrm{Hom}_A\Big(M,\bigoplus_{\mathfrak{p}\in\partial F_{i+k}} I_{\mathfrak{p}}\Big) & & & & \uparrow{\scriptstyle \text{natural map}} \\
\downarrow{\scriptstyle \text{natural map}} & & & & \\
\bigoplus_{\mathfrak{p}\in\partial F_{i+k}}\mathrm{Hom}_A(M,I_{\mathfrak{p}}) & \xrightarrow{\text{natural map}} & \bigoplus_{\mathfrak{p}\in\partial T_i}\mathrm{Hom}_A(M,I_{\mathfrak{p}}) & \xrightarrow{\oplus_{\mathfrak{p}\in\partial T_i}\theta_{\mathfrak{p}}} & \bigoplus_{\mathfrak{p}\in\partial T_i}\mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}},I_{\mathfrak{p}})
\end{array}$$

Since $\xi(I)$ is an isomorphism (see [4, Lemma 2.2]), $\xi(\mathrm{Hom}_A(M,I))$ is an isomorphism.

(ii) For each $\mathfrak{p}\in\mathrm{Supp}_A(M)$ so that $t(\mathfrak{p};I^\cdot)=k$, we have $(K_M^{I^\cdot})_{\mathfrak{p}}\cong\mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}},E(A/\mathfrak{p}))$ so that $\mathrm{Ass}_A(K_M^{I^\cdot})=\mathrm{Supp}_A(M)\cap\mathrm{Ass}_A(I^k)$. It is clear, by Lemma 1.2, that $\mathrm{Ass}_A(K_M^{I^\cdot})\subseteq\mathrm{Min}_A(M)$. Hence $\mathrm{Min}_A(M)=\mathrm{Ass}_A(K_M^{I^\cdot})$ if and only if $t(\mathfrak{p};I^\cdot)-k=\mathrm{ht}_M(\mathfrak{p})=0$ for all $\mathfrak{p}\in\mathrm{Min}_A(M)$. Now the first assertion follows from Lemma 1.2. The second assertion is easy.

The rest may be treated in the same way as in [4, Theorem 2.4(iii) and (iv)], but we bring a proof here for the convenience of the reader.

(iii) Let M be (S_1). By [4, Proposition 2.1(i)], it is enough to show that $\mathrm{Supp}_A(\mathrm{Coker}\, e^{i-1}) \subseteq T_{i+1}$ for each $i \geq 0$. Since $\mathrm{Im}\, e^{-1} = \mathrm{Ker}\, e^0$, the assertion is clear for $i = 0$. Assume that $i > 0$. We have

$$\mathrm{Supp}_A(\mathrm{Coker}\, e^{i-1}) \subseteq \mathrm{Supp}_A(\mathrm{Hom}_A(M, I^{i+k})) \subseteq T_i.$$

Suppose that $\mathfrak{p} \in \partial T_i$, so that $t(\mathfrak{p}; I^{\cdot}) = i + k$. Therefore we have $\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, I^j_\mathfrak{p}) = 0$ for all $j > i + k$. Hence

$$\begin{aligned}(\mathrm{Coker}\, e^{i-1})_\mathfrak{p} \cong \mathrm{Coker}\, e^{i-1}_\mathfrak{p} &\cong H^{i+k}((\mathrm{Hom}_A(M, I^{\cdot}))_\mathfrak{p}) \\ &\cong H^{i+k}(\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, I^{\cdot}_\mathfrak{p})).\end{aligned}$$

Since $I^{\cdot}_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ (see [13, Theorem 4.2]) and $t(\mathfrak{p}A_\mathfrak{p}, I^{\cdot}_\mathfrak{p}) = i + k$, we have, by Sharp [15, Theorem (2.6)],

$$\max\{j\colon H^j(\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, I^{\cdot}_\mathfrak{p})) \neq 0\} = i + k - \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}).$$

As $\mathrm{ht}_M(\mathfrak{p}) = i > 0$, we have $\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) > 0$. Thus $H^{i+k}(\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, I^{\cdot}_\mathfrak{p})) = 0$ and so $\mathfrak{p} \notin \mathrm{Supp}_A(\mathrm{Coker}\, e^{i-1})$. This shows that $\mathrm{Supp}_A(\mathrm{Coker}\, e^{i-1}) \subseteq T_{i+1}$.

Conversely, assume that $\Psi^{I^{\cdot}}_M$ is an epimorphism. Let $\mathfrak{p} \in \mathrm{Supp}_A(M)$. We may assume that $\mathrm{ht}_M(\mathfrak{p}) > 0$. Set $i = \mathrm{ht}_M(\mathfrak{p})$. Then $\mathfrak{p} \in \partial H_i$; so, by [4, Proposition 2.1(i)], $\mathfrak{p} \notin \mathrm{Supp}_A(\mathrm{Coker}\, e^{i-1})$. Thus

$$\mathrm{Hom}_{A_\mathfrak{p}}(\mathrm{Id}_{M_\mathfrak{p}}, \delta^{i-1+k}_\mathfrak{p}) : \mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, I^{i-1+k}_\mathfrak{p}) \to \mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, I^{i+k}_\mathfrak{p})$$

is an epimorphism. It therefore follows from the fact that $t(\mathfrak{p}; I^{\cdot}) = i+k$ that $H^j(\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, I^{\cdot}_\mathfrak{p})) = 0$ for all $j \geq i+k$. Hence, by [15, Theorem (2.6)], $i+k-\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) < i+k$; that is $\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq 1$. Thus M is (S_1).

(iv) Assume that M is (S_2). We have, by (iii) and [4, Proposition 2.1(i)], that

$$\mathrm{Supp}_A(H^{i-1}(X^{\cdot})) \subseteq \mathrm{Supp}_A(\mathrm{Coker}\, e^{i-2}) \subseteq T_i$$

for all $i \geq 0$. Consequently, in view of [4, Proposition 2.1(ii)], it is enough to show that $\mathrm{Supp}_A(H^{i-1}(X^{\cdot})) \subseteq T_{i+1}$ for all $i \geq 0$. We have $H^{-1}(X^{\cdot}) = 0 = H^0(X^{\cdot})$. Let us assume $i \geq 2$ and $\mathfrak{p} \in \partial T_i$ so that $\mathrm{ht}_M(\mathfrak{p}) = i$. Since M is (S_2), we may use [15, Theorem (2.6)] again to deduce that $\mathfrak{p} \notin \mathrm{Supp}_A(H^{i-1}(X^{\cdot}))$. This shows that $\mathrm{Supp}_A(H^{i-1}(X^{\cdot})) \subseteq T_{i+1}$.

Conversely, assume that $\Psi^{I^{\cdot}}_M$ is an isomorphism of complexes. Let $\mathfrak{p} \in \mathrm{Supp}_A(M)$. By (iii), we may assume that $\mathrm{ht}_M(\mathfrak{p}) > 1$. Set $i = \mathrm{ht}_M(\mathfrak{p})$. Then $\mathfrak{p} \in \partial T_i$; so, from [4, Proposition 2.1(ii)] it follows that $\mathfrak{p} \notin \mathrm{Supp}_A(H^j(X^{\cdot})))$ for all $j \geq i - 1$. This, again, shows that $i + k - \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) < i + k - 1$, i.e. $\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq 2$. Therefore M is (S_2). □

In the following corollary, we are interested in the particular case of Theorem 1.4 in which A is local. The dualizing complex of a local ring is of the form

$$I^{\cdot} : 0 \to I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{d-1}} I^d \to 0, \qquad (*)$$

where $d = \dim A$ and $I^i = \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(A), \dim(A/\mathfrak{p}) = d-i} E(A/\mathfrak{p})$ (see [6, Definition 4.3]). Note that the dualizing complex of a local ring is unique up to isomorphism of complexes (see [13, Theorem 4.5; 6, 4.2]).

Corollary 1.5. *Let A be a local ring and* $\dim A = d$. *Suppose that A possesses a dualizing complex* (*). *Let M be a non-zero finitely generated A-module with* $\dim_A(M) = s$. *Set $\mathscr{D}_M = (D_i)_{i\geq 0}$ be the dimension filtration of* $\mathrm{Supp}_A(M)$, *i.e.*

$$D_i = \{\mathfrak{p} \in \mathrm{Supp}_A(M) \colon \dim(A/\mathfrak{p}) \leq s - i\}, \quad i = 0, 1, \dots .$$

We choose the notations $K_M := H^{d-s}(\mathrm{Hom}_A(M, I^{\cdot}))$ *and* $\mathrm{Hom}_A(M, I^{\cdot})^*$ *to be the complex*

$$0 \to K_M \hookrightarrow \mathrm{Hom}_A(M, I^{d-s}) \xrightarrow{(\delta^0)^*} \cdots \to \mathrm{Hom}_A(M, I^d) \to 0,$$

so that $(\mathrm{Hom}_A(M, I^{\cdot})^*)^{-1} = K_M$ *and* $(\mathrm{Hom}_A(M, I^{\cdot})^*)^i = \mathrm{Hom}_A(M, I^{d-s+i})$ $i = 0, 1, \dots$. *Then the following statements hold*:

(i) *There exists a* (*unique*) *homomorphism of complexes*

$$\Psi_M = (\psi^i)_{i\geq -2} : C(\mathscr{D}_M, K_M) \to \mathrm{Hom}_A(M, I^{\cdot})^*$$

(*over* Id_{K_M}), *from the Cousin complex of K_M with respect to $\mathscr{D}_M$ to* $\mathrm{Hom}_A(M, I^{\cdot})^*$.

(ii) $\mathrm{Min}_A(M) = \mathrm{Assh}_A(M)$ *if and only if* $\mathscr{D}_M = \mathscr{H}_M$, *where $\mathscr{H}_M$ is as in* 1.4. *In this situation* $\mathrm{Supp}_A(M) = \mathrm{Supp}_A(K_M)$.

(iii) *If* $\mathrm{Min}_A(M) = \mathrm{Assh}_A(M)$, *then M is* (S_1) *if and only if Ψ_M is an epimorphism.*

(iv) *If* $\mathrm{Min}_A(M) = \mathrm{Assh}_A(M)$, *then M is* (S_2) *if and only if Ψ_M is an isomorphism.*

Proof. In view of Theorem 1.4, it is enough to prove that $\mathscr{D}_M = \mathscr{T}_M$, where $\mathscr{T}_M$ is as in the paragraph preceding Theorem 1.4, and $\mathrm{Ass}_A(K_M) = \mathrm{Assh}_A(M)$.

Note that $\min\{j\colon \dim(A/\mathfrak{p}) = d - j$ for some $\mathfrak{p} \in \mathrm{Supp}_A(M)\} = d - s$. This shows that the integer k, introduced in the paragraph preceding 1.4, is equal to $d - s$. It follows, by elementary argument, that $\mathscr{D}_M = \mathscr{T}_M$.

As in the proof of 1.4(ii), we have $\mathrm{Ass}_A(K_M) = \mathrm{Supp}_A(M) \cap \mathrm{Ass}_A(I^{d-s})$. Therefore $\mathrm{Ass}_A(K_M) = \mathrm{Assh}_A(M)$. □

2. Serre condition and canonical module

Throughout this section $(A, \mathfrak{m})$ is a local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. A finitely generated A-module K is called a *canonical module* of A precisely when

$$K \otimes_A \hat{A} \cong \mathrm{Hom}_A(H^d_{\mathfrak{m}}(A), E(A/\mathfrak{m})),$$

where $\hat{A}$ is the completion of A with respect to $\mathfrak{m}$-adic topology, and $H^d_{\mathfrak{m}}(-)$ is the dth local cohomology functor with respect to $\mathfrak{m}$. Note that canonical module of A, if it exists, is unique up to isomorphism of modules. It is known that if a local ring possesses a dualizing complex $I^{\cdot}$, then $H^0(I^{\cdot})$ is the canonical module of A.

We will use the structural Corollary 1.5 in studying the effect of Serre condition (S_n) on canonical module.

The following remark will be used frequently in the rest of this paper.

Remark 2.1. If A is an (S_2) local ring such that it admits canonical module K, then we have the following facts:

(i) $\mathrm{Assh}(A) = \mathrm{Min}(A) = \mathrm{Ass}(A)$ (by [2, Lemma 1.1]);
(ii) $\mathrm{Supp}_A(K) = \mathrm{Spec}(A)$ (by [1, 1.7]);
(iii) $\mathrm{Assh}_A(K) = \mathrm{Min}_A(K) = \mathrm{Ass}_A(K)$ (by (i), (ii) and [1, 1.7]);
(iv) $\dim A_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = \dim A = \dim_A(K)$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$ (by (ii) and [1, 1.9]).

Lemma 2.2. *Assume that A possesses a dualizing complex $I^{\cdot}$. Let M be a finitely generated A-module which satisfies the condition (S_2) such that*

$$\mathrm{Min}_A(M) = \mathrm{Assh}_A(M).$$

Set $s = \dim_A(M)$ and $K_M = H^{d-s}(\mathrm{Hom}_A(M, I^{\cdot}))$. Then the following statements are true:

(i) *K_M is (S_2).*
(ii) *For $n \geq 3$, K_M is (S_n) if and only if $H^{d-s+t}(\mathrm{Hom}_A(M, I^{\cdot})) = 0$ for all t, $1 \leq t \leq n-2$.*
(iii) *Let K_M be (S_n) with $n \geq 3$ and $x_1, \ldots, x_{n-3}$ be an M-sequence. Then $H^{d-s+t}(\mathrm{Hom}_A(M/(x_1, \ldots, x_i)M, I^{\cdot})) = 0$ for all i and t with $1 \leq i+1 \leq t \leq n-2$.*

Proof. (i) and (ii). By Corollary 1.5(ii) and (iv), $\mathrm{Hom}_A(M, I^{\cdot})^*$ is isomorphic to $C(K_M)$, the Cousin complex of K_M with respect to $\mathscr{H}_{K_M}$, the height filtration of K_M. Since $\mathrm{Hom}_A(M, I^{\cdot})^*$ is exact at terms -1 and 0, the same is true for $C(K_M)$. Now the claims follow from [17, Example 4.4].

(iii) By induction on i. For $i = 0$, it is true by (ii). Assume $0 < i \leq n-3$ and the result is known for all j, $0 \leq j < i$. Let $i+1 \leq t \leq n-2$. By induction hypothesis, we have

$$H^{d-s+t-1}(\mathrm{Hom}_A(M/(x_1, \ldots, x_{i-1})M, I^{\cdot})) = 0$$

and

$$H^{d-s+t}(\mathrm{Hom}_A(M/(x_1, \ldots, x_{i-1})M, I^{\cdot})) = 0.$$

Using the exact sequence

$$0 \to M/(x_1, \ldots, x_{i-1})M \xrightarrow{x_i} M/(x_1, \ldots, x_{i-1})M \to M/(x_1, \ldots, x_i)M \to 0, \qquad (*)$$

we get the exact sequence

$$H^{d-s+t-1}(\mathrm{Hom}_A(M/(x_1, \ldots, x_{i-1})M, I^{\cdot})) \to H^{d-s+t}(\mathrm{Hom}_A(M/(x_1, \ldots, x_i)M, I^{\cdot})) \to H^{d-s+t}(\mathrm{Hom}_A(M/(x_1, \ldots, x_{i-1})M, I^{\cdot})),$$

from which the result follows. □

Proposition 2.3. *Assume that the situation and notation are as in Lemma 2.2. Let $n \geq 2$, K_M be (S_n) and $x_1, \ldots, x_{n-2}$ be an M-sequence. Then $x_1, \ldots, x_{n-2}$ is K_M-sequence and, for each i, $0 \leq i \leq n-2$,*

$$K_{M/(x_1, \ldots, x_i)M} \cong K_M/(x_1, \ldots, x_i)K_M.$$

Proof (*By induction on i*)**.** Let $0 < i \leq n-2$ and the result is true for all j, $0 \leq j < i$. Since $H^{d-s+i-1}(\mathrm{Hom}_A(M/(x_1,\ldots,x_i)M, I^\cdot)) = 0$, from Lemma 2.2(iii) and the exact sequence $(*)$ in it, we have the exact sequence

$$0 \to H^{d-s+i-1}(\mathrm{Hom}_A(M/(x_1,\ldots,x_{i-1})M, I^\cdot)) \xrightarrow{x_i} H^{d-s+i-1}(\mathrm{Hom}_A(M/(x_1,\ldots,x_{i-1})M, I^\cdot)) \to H^{d-s+i}(\mathrm{Hom}_A(M/(x_1,\ldots,x_i)M, I^\cdot)) \to 0.$$

The result follows. □

Corollary 2.4. *Assume that A is (S_2) and it possesses a dualizing complex $I^\cdot$. Set $K = H^0(I^\cdot)$. Let $n \geq 2$ and K satisfies (S_n). If $x_1,\ldots,x_{n-2}$ is an A-sequence then it is a K-sequence and the canonical module of $A/(x_1,\ldots,x_i)A$ is $K/(x_1,\ldots,x_i)K$, for all i, $0 \leq i \leq n-2$.*

Proof. By Remark 2.1(i), $\mathrm{Min}(A) = \mathrm{Assh}(A)$. Let $x_1,\ldots,x_{n-2}$ be an A-sequence and $0 \leq i \leq n-2$. Since $\mathrm{Hom}_A(A/(x_1,\ldots,x_i)A, I^\cdot)$ provides the dualizing complex for $A/(x_1,\ldots,x_i)A$ (see [13, Theorem 3.9]), the result follows by Proposition 2.3. □

The continuation of this study is motivated by the work of Dutta who proves that "A satisfies Canonical Element Conjecture if A is a complete local normal domain whose canonical module is (S_3)" (see [5, Theorem 2.6]). In this connection we will find the relationship between the Serre condition on canonical module and vanishing of local cohomology modules of the ring itself. Thus, in view of Proposition 2.5 and [5, Theorem 1.4], we may extend result [5, Theorem 2.6] in the following form: If A is a complete local domain which satisfies (S_2) and $H_{\mathfrak{m}}^{\dim A-1}(A) = 0$, then A satisfies the Canonical Element Conjecture.

Proposition 2.5. *Let A satisfy (S_2) and admit dualizing complex $I^\cdot$ (see the paragraph preceding Corollary 1.5) with the canonical module K. For $n \geq 2$ the following statements are equivalent*:

(i) *K is (S_n)*;

(ii) $0 \to K \overset{\delta^{-1}}{\hookrightarrow} I^0 \xrightarrow{\delta^0} I^1 \to \cdots \to I^{n-1}$ *is part of a minimal injective resolution of K where δ^{-1} is the inclusion map*;

(iii) $\mu^i(\mathfrak{p}, K) = \delta_{i\,\mathrm{ht}\,\mathfrak{p}}$ (*Kronecker delta*) *for all* $\mathfrak{p} \in \mathrm{Spec}(A)$ *and* $0 \leq i < n$, *where $\mu^i(\mathfrak{p}, K)$ is the ith Bass number of K*;

(iv) $H_{\mathfrak{m}}^{d-i}(M) = D(\mathrm{Ext}_A^i(M,K))$ *for* $i = 0,1,\ldots,n-2$ *and for all finitely generated A-modules M, where* $D(-) = \mathrm{Hom}_A(-, E(A/\mathfrak{m}))$;

(v) $H_{\mathfrak{m}}^{d-i}(A) = 0$ *for* $i = 1,\ldots,n-2$.

Proof. (i)⇒(ii) By Corollary 1.5(ii) and (iv) (or [4, Corollary 2.5(iv)]),

$$I^* : 0 \to K \hookrightarrow I^0 \xrightarrow{\delta^0} I^1 \to \cdots \xrightarrow{\delta^{d-1}} I^d \to 0$$

is the Cousin complex of K with respect to the dimension (=height) filtration of K. By [12, Proposition 5.3] each I^i is an essential extension of $\operatorname{Im}\delta^{i-1}$. Also by Lemma 2.2(ii), I^* is exact at $K, I^0, \ldots, I^{n-2}$. The claim follows.

(ii)⇒(iii) By Remark 2.1(iv), $\dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}) = \dim A$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Now, the claim is clear.

(iii)⇒(i) We have $\operatorname{depth}_{A_\mathfrak{p}}(K_\mathfrak{p}) = \min\{j\colon \mu^j(\mathfrak{p}, K) \neq 0\}$ for all $\mathfrak{p} \in \operatorname{Supp}_A(K)$. By Remark 2.1(ii), $\operatorname{Supp}_A(K) = \operatorname{Spec}(A)$ and that $\operatorname{ht}_K(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Hence from the assumption it follows that $\operatorname{depth}_{A_\mathfrak{p}}(K_\mathfrak{p}) \geq \min\{n, \operatorname{ht}_K(\mathfrak{p})\}$ for all $\mathfrak{p} \in \operatorname{Supp}_A(K)$.

(ii)⇒(iv) follows from the definition of Ext functor and Grothendieck's local duality theorem [3, Corollary 2.5].

(iv)⇒(v) is clear.

(v)⇒(i) follows from [3, Corollary 2.2] and Lemma 2.2(ii). □

Note that if M is an A-module with situation Lemma 2.2, then, for an integer $n \geq 3$, the following conditions are equivalent: (i) K_M satisfies (S_n); (ii) $H^{s-i}_\mathfrak{m}(M) = 0$ for all i with $1 \leq i \leq n-2$. This is clear from [3, Corollary 2.2] and Lemma 2.2(ii). This fact is known when A is an epimorphic image of a local Gorenstein ring (see [11, Corollary 1.15]). [1]

Corollary 2.6. *Assume that A possesses a dualizing complex and satisfies (S_l) for some $l \geq 2$, and that K is the canonical module A. A necessary and sufficient condition for A to be a Cohen–Macaulay ring is that K is (S_k) for some integer k with $k + l \geq d + 2$.*

Proof. We may assume that $l \leq d$, so that $H^i_\mathfrak{m}(A) = 0$ for $i = 0, 1, \ldots, l-1$. On the other hand, by Proposition 2.5, K satisfying (S_k) is equivalent to $H^{d-i}_\mathfrak{m}(A) = 0$ for $i = 1, \ldots, k-2$. Now A is a Cohen–Macaulay ring if and only if K is (S_k) for some integer k with $k + l \geq d + 2$. □

3. A study of Cousin complexes over rings admitting a dualizing complex

In this section $(A, \mathfrak{m})$ is a local ring. If A possesses a dualizing complex, then we will prove that for any finitely generated A-module M satisfying (S_2) and $\operatorname{Min}_A(M) = \operatorname{Assh}_A(M)$, all cohomology modules of the Cousin complex of M, $C_A(M)$, with respect to the M-height filtration of $\operatorname{Spec}(A)$ are finitely generated.

Lemma 3.1. *Assume that A possesses a dualizing complex $I^\cdot$ and M is a finitely generated A-module with $\dim_A(M) = d = \dim A$, $\operatorname{Min}_A(M) = \operatorname{Assh}_A(M)$, and M is (S_2). Then, all cohomology modules of $C_A(M)$, the Cousin complex of M with respect to the M-height filtration of* $\operatorname{Spec}(A)$, *are finitely generated.*

[1] We thank the referee for quoting it to us.

Proof. We have $H^0(\mathrm{Hom}_A(M,I^\cdot)) = \mathrm{Hom}_A(M,K)$, where $K = H^0(I^\cdot)$. Set $K_M = \mathrm{Hom}_A(M,K)$. By Lemma 2.2(i), K_M is (S_2), also, by Corollary 1.5(ii), $\dim_A(K_M) = d$ and $\mathrm{Min}_A(K_M) = \mathrm{Assh}_A(K_M)$. Consequently, by Corollary 1.5(iv), $C_A(\mathrm{Hom}_A(K_M,K))$, the Cousin complex of $\mathrm{Hom}_A(K_M,K)$ with respect to the height filtration of K_M, is isomorphic to $\mathrm{Hom}_A(K_M,I^\cdot)^*$. But $\mathrm{Hom}_A(K_M,K) = \mathrm{Hom}_A(\mathrm{Hom}_A(M,K),K) \cong M$ (see [1, Proposition 4.4]) and the height filtration of M is equal to the height filtration of K_M, by Corollary 1.5(ii). The result follows from [13, Lemma 3.4(ii)]. □

Now we can prove the following theorem.

Theorem 3.2. *If A possesses a dualizing complex, then for each finitely generated A-module M such that $\mathrm{Min}_A(M) = \mathrm{Assh}_A(M)$ and M is (S_2), all cohomologoy modules of $C_A(M)$, the Cousin complex of M with respect to the M-height filtration of* $\mathrm{Spec}(A)$, *are finitely generated.*

Proof. Take an A-module M with the required conditions, and set $\bar{A} = A/0 : M$. We have $\dim \bar{A} = \dim_{\bar{A}}(M)$; $\bar{A}$ also possesses a dualizing complex (see [13, Theorem 3.9]) and $\mathrm{Min}_{\bar{A}}(M) = \mathrm{Assh}_{\bar{A}}(M)$. It is also straightforward to see that M satisfies (S_2) as $\bar{A}$-module. Therefore, by Lemma 3.1, all cohomology modules of $C_{\bar{A}}(M)$, the Cousin complex of M, as $\bar{A}$-module, are finitely generated $\bar{A}$-modules. The result follows from the fact that there exists a natural isomorphism of complexes, of $\bar{A}$-modules, between $C_{\bar{A}}(M)$ and $C_A(M)$. □

It should be noted that if A is (S_2) and admits a canonical module then the converse of Theorem 3.2 also holds (see [4, Corollary 3.4]).

4. Which local ring possesses a dualizing complex?

It was shown in [9, Section 6, Example 2] that there exists a local ring A with canonical module and non-Gorenstein formal fibres, hence not a homomorphic image of a Gorenstein ring, which, by [7, Corollary 6.2], is equivalent to saying that A does not possess a dualizing complex. The aim of this section is to prove the following theorem (cf. with [9, Theorem 5.2]).

Theorem 4.1. *If an (S_2) local ring, with Cohen–Macaulay formal fibres, admits a canonical module, then it possesses a dualizing complex.*

Lemma 4.2. *Let $\varphi : A \to B$ be a flat ring homomorphism. Assume that M is an A-module and that all fibres $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A B$, for all $\mathfrak{p} \in \mathrm{Supp}_A(M)$, are Cohen–Macaulay. Then*

$$H^i(C_A(M)) \otimes_A B \cong H^i(C_B(M \otimes_A B))$$

as B-modules. (*Here if X is a module over a ring R, then $C_R(X)$ denotes the Cousin complex of X with respect to the X-height filtration of* Spec(R).)

Proof. By [10, Theorem 2.15] there exists a morphism of complexes

$$\Phi : C_A(M \otimes_A B) \to C_B(M \otimes_A B)$$

which is injective. By [10, Theorem 3.5] the induced quotient complex $C_B(M \otimes_A B)/\Phi(C_A(M \otimes_A B))$ is exact. Therefore Φ is a quasi-isomorphism of complexes, that is

$$H^i(C_A(M \otimes_A B)) \cong H^i(C_B(M \otimes_A B))$$

for all $i \geq 0$. Now the result follows from [10, Lemma 2.9]. □

Proof of Theorem 4.1. Assume that A is a local ring satisfying (S_2) and that all fibres of $A \to \hat{A}$ are Cohen–Macaulay. Denote by K the canonical module of A. By Lemma 4.2,

$$H^i(C_A(K)) \otimes_A \hat{A} \cong H^i(C_{\hat{A}}(\hat{K}))$$

for all $i \geq 0$. By [8, Theorem 23.9(iii)], $\hat{A}$ is also (S_2). Hence, by Remark 2.1(iii), $\mathrm{Min}_{\hat{A}}(\hat{K}) = \mathrm{Assh}_{\hat{A}}(\hat{K})$ so that, by Theorem 3.2, $H^i(C_{\hat{A}}(\hat{K}))$ is a finitely generated $\hat{A}$-module for all $i \geq 0$. Now, by [8, Exercise 7.3], $H^i(C_A(K))$ is finitely generated A-module. On the other hand, by 2.1(ii) and (iv), $C_A(K) = C(\mathscr{D}, K)$, where $C(\mathscr{D}, K)$ is the Cousin complex of K with respect to $\mathscr{D} = (D_i)_{i \geq 0}$, the dimension filtration of Spec(A). The claim follows from [4, Corollary 3.4]. □

Corollary 4.3. *Let A be an (S_2) local ring. Then the following statements are equivalent*:
(i) *A possesses a dualizing complex*,
(ii) *A admits a canonical module and all fibres of $A \to \hat{A}$ are Gorenstein*;
(iii) *A admits a canonical module and all fibres of $A \to \hat{A}$ are Cohen–Macaulay.*

Proof. (i) ⇒ (ii) follows from [14, Theorem 3.7]. (ii) ⇒ (iii) is clear, and (iii) ⇒ (i) follows from Theorem 4.1. □

The following corollary is a particular case of (iii) ⇒ (i) above.

Corollary 4.4. *Assume that A is a local ring which satisfies (S_2) and it is a quotient of a Cohen–Macaulay ring. If A admits canonical module*, *then A possesses a dualizing complex.*

Proof. If A is a quotient of a Cohen–Macaulay ring, then, by using [8, Remark in p. 184 and Exercise 23.1], we see that all formal fibres of A are Cohen–Macaulay. Hence the result is clear from (iii) ⇒ (i) of Corollary 4.3. □

Note that, from Corollary 4.4 and [7, Corollary 6.2], any (S_2) local ring which is a quotient of a Cohen–Macaulay ring and which admits canonical module is a homomophic image of a Gorenstein ring.

Acknowledgements

The authors express their gratitude to the referee for careful reading of the manuscript and for giving valuable comments. This research is supported by Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-1795, Tehran, Iran.

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