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COMMUNICATIONS IN ALGEBRA
Vol. 30, No. 8, pp. 3687–3702, 2002

THE LICHTENBAUM-HARTSHORNE THEOREM FOR GENERALIZED LOCAL COHOMOLOGY AND CONNECTEDNESS

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ABSTRACT

Let Z be a subset of the spectrum of a local ring R stable under specialization and let N be a d-dimensional finitely generated R-module. It is shown that $H^d_Z(N)$, the dth local cohomology module of the sheaf associated to N with support in Z, vanishes if and only if for every d-dimensional $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}}\hat{N}$, there is a $\mathfrak{q} \in Z$ such that $\dim \hat{R}/(\mathfrak{q}\hat{R} + \mathfrak{p}) > 0$. Applying this criterion for vanishing of $H^d_Z(N)$, several connectedness results for certain algebraic varieties are proved.

DOI: 10.1081/AGB-120005813 0092-7872 (Print); 1532-4125 (Online)
 www.dekker.com

1. INTRODUCTION

It was discovered by J. Rung[15] that local cohomology yields connectedness results. Since then, several authors have used local cohomology as a powerful tool in their investigation of connectedness of algebraic varieties (see e.g., [3,6–8]). What is used from local cohomology are the Lichtenbaum-Hartshorne vanishing theorem and the Mayer-Vietoris sequence. Several proofs for the Lichtenbaum-Hartshorne have been given (see e.g.,[4,5]). On the other hand, G. Lyubeznik[13] extends this result to e′tale cohomology. He also has shown the following generalization of the Lichtenbaum-Hartshorne vanishing theorem for a locally closed subscheme Y of a separated scheme of finite type over a field X. Let $d = \dim X$. Then for all quasi-coherent sheaves $\mathcal{F}$ on X, $H^d_Y(X, \mathcal{F}) = 0$, if and only if every connected component of the preimage of Y in every top-dimensional irreducible component of the normalization of X_{red} is non-proper (see [12]).

Recall that a subset Z of $\operatorname{Spec} R$ is *stable under specialization* (s.u.s. for short) if $V(\mathfrak{p}) \subset Z$, whenever $\mathfrak{p} \in Z$. One can see easily that there is a one to one correspondence between the s.u.s. subsets of $\operatorname{Spec} R$ and the families of supports of $\operatorname{Spec} R$ (see [9, page 218]) for the definition of family of supports). In the case $X = \operatorname{Spec} R$, we focus our attention on s.u.s. subsets of X and prove the Lichtenbaum-Hartshorne vanishing theorem for this class of subsets of X.

Theorem 1.1. *Let N be a d-dimensional finitely generated module over a local ring $(R, \mathfrak{m})$ and let Z be a subset of* $\operatorname{Spec} R$ *stable under specialization. Then the following statements are equivalent:*

(i) $H^d_Z(N) = 0$.
(ii) *For any $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N}$, with* $\dim \hat{R}/\mathfrak{p} = d$, *there is $\mathfrak{q} \in Z$ such that* $\dim \hat{R}/(\mathfrak{q}\hat{R} + \mathfrak{p}) > 0$.

This result will be proved in 2.8. In the proof of 1.1, we use the fact that there is a one to one correspondence between the s.u.s. subsets of $\operatorname{Spec} R$ and the full systems of ideals of R (see 3.1). In this article we use this fact several times. A non-empty subset Φ of ideals of R is called a *system of ideals* if, whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$ there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. A system of ideals Φ is called *full* if, whenever $\mathfrak{a} \in \Phi$ and $\mathfrak{b}$ is an ideal of R with $\mathfrak{a} \subseteq \mathfrak{b}$ then $\mathfrak{b} \in \Phi$.

Our technical tool for proving 1.1 is the following lemma.

Lemma 1.2. *Let Φ be a system of ideals of a local ring $(R, \mathfrak{m})$. For any two finitely generated R-modules N and M, there is a functorial isomorphism*

$$\operatorname{Hom}_R\left(N, \bigcap_{\mathfrak{a}\in\Phi} (\mathfrak{a}M :_M \langle \mathfrak{m} \rangle)\right) \cong \bigcap_{a\in\Phi} (\mathfrak{a}\operatorname{Hom}_R(N, M) :_{\operatorname{Hom_R(N,M)}} \langle \mathfrak{m} \rangle).$$

Here for two R-modules $C \subseteq M$, the union $(\cup_{i\geq 0} C :_M \mathfrak{m}^i)$ is denoted by $C :_M \langle \mathfrak{m} \rangle$. We prove 1.2 in 2.2. For an R-module M, let $\langle \mathfrak{m} \rangle M := \cap_{i\geq 0} \mathfrak{m}^i M$. By applying 1.2, we find the following description of $H_Z^d(N)$:

$$H_Z^d(N) \cong H_{\mathfrak{m}}^d(N) / \sum_{\mathfrak{a}\in\Phi} \langle \mathfrak{m} \rangle (0 :_{H_{\mathfrak{m}}^d(N)} \mathfrak{a}),$$

where Φ denotes the system of ideals of R corresponding to Z (see 2.6 for the proof of this result). This result is not only used in the proof of 1.1, but also it immediately implies that $H_Z^d(N)$ is an Artinian R-module. The Artinianess of $H_Z^d(N)$ is the main result of [2].

In section three, by applying the generalization of the Lichtenbaum-Hartshorne theorem, we deduce several connectedness results. Mainly, we are able to generalize the known results for the closed subsets of the spectrum of a local ring to its stable under specialization subsets. In particular, we deduce the following far reaching generalization of Faltings' connectedness result (see [7,8]), it also extends.[11, Theorem 3.3]

Theorem 1.3. *Let the situation be as in 1.1. Suppose that any minimal prime ideal of* $\operatorname{Ass}_{\hat{R}} \hat{N}$ *is of dimension* d *and that* $H_{\mathfrak{m}}^d(N)$ *is an indecomposable* R*-module. Then* $Z \cap \operatorname{Supp}(N) \backslash \{\mathfrak{m}\}$ *is connected provided* $H_Z^i(N) = 0$ *for* $i = d-1, d$.

All rings considered in this paper are assumed to be commutative and Noetherian (with identity).

2. THE LICHTENBAUM-HARTSHORNE VANISHING THEOREM

The purpose of this section is to give an explicit computation of $H_Z^d(N)$ in terms of a quotient of $H_{\mathfrak{m}}^d(N)$ and, in the same way, to clarify equivalence of the topologies involved. The main results are 2.6 and 2.8.

We shall use the following result in the proof of proposition 2.2, which is in turn our fundamental tool in this section.

Lemma 2.1. *Let* $(R, \mathfrak{m})$ *be a local ring and* Φ *a system of ideals of* R. *Let* $\cap_{i=1}^n Q_i = 0$ *be a minimal primary decomposition of the zero submodule of the finitely generated* R*-module* N. *Then*

$$\bigcap_{\mathfrak{a}\in\Phi} (\mathfrak{a}N :_N \langle \mathfrak{m} \rangle) = \bigcap_{\mathfrak{p}_i \in T} Q_i,$$

where $T = \{\mathfrak{p}_i \in \operatorname{Ass}_R N :$ *there exists* $\mathfrak{a} \in \Phi$ *such that* $\dim R/(\mathfrak{a} + \mathfrak{p}_i) > 0\}$.

Proof. Set $\Omega = \mathrm{Ass}_R N \backslash T$ and $\mathfrak{c} = \cap_{\mathfrak{p}_i \in \Omega} \mathfrak{p}_i$. There is an integer l such that $\mathfrak{c}^l N \subseteq \cap_{\mathfrak{p}_i \in \Omega} Q_i$. For each $\mathfrak{a} \in \Phi$, it follows that the ideal $\mathfrak{c} + \mathfrak{a}$ is $\mathfrak{m}$-primary and so

$$\mathfrak{a}N :_N \langle \mathfrak{m} \rangle = \mathfrak{a}N :_N \langle \mathfrak{c} + \mathfrak{a} \rangle = \mathfrak{a}N :_N \langle \mathfrak{c} \rangle \supseteq 0 :_N \langle \mathfrak{c} \rangle \supseteq \bigcap_{\mathfrak{p}_i \in T} Q_i.$$

Therefore $\cap_{\mathfrak{p}_i \in T} Q_i \subseteq \cap_{\mathfrak{a} \in \Phi} (\mathfrak{a}N :_N \langle \mathfrak{m} \rangle)$.

Conversely, let $x \in \cap_{\mathfrak{a} \in \Phi} (\mathfrak{a}N :_N \langle \mathfrak{m} \rangle)$. Taking $\mathfrak{p}_i \in T$, there is $\mathfrak{a} \in \Phi$ such that $\mathrm{Rad}(\mathfrak{a} + \mathfrak{p}_i) \subsetneqq \mathfrak{m}$. Hence we may choose a prime ideal $\mathfrak{q}$ such that $\mathfrak{a} + \mathfrak{p}_i \subseteq \mathfrak{q} \subsetneqq \mathfrak{m}$. For a given $n \in \mathbb{N}$, we have $\mathfrak{m}^l x \subseteq \mathfrak{a}^n N$ for sufficiently large integer l. It turns out that $x/1 \in \mathfrak{a}^n N_\mathfrak{q}$, for all $n \in \mathbb{N}$. Therefore $x/1 = 0$ in $N_\mathfrak{q}$ by Krull's intersection theorem. Hence $sx = 0$ for some $s \in R \backslash \mathfrak{p}_i$. This implies $x \in Q_i$. Therefore $x \in \cap_{\mathfrak{p}_i \in T} Q_i$. □

The first author would like to thank Professor Peter Schnezel who pointed out that 2.2 holds for the special case $\Phi = \{\mathfrak{a}^n\}_{n \geq 0}$.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a local ring and Φ a system of ideals of R. Let N and M be two finitely generated R-modules. Then there is a functorial isomorphism*

$$\mathrm{Hom}_R\left(N, \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M \langle \mathfrak{m} \rangle)\right) \cong \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}\mathrm{Hom}_R(N, M) :_{\mathrm{Hom_R(N,M)}} \langle \mathfrak{m} \rangle).$$

Proof. It is well known (and can be checked easily) that if $\cap_{i=1}^n Q_i$ is a minimal primary decomposition of the zero submodule of an R-module L, with Q_i a $\mathfrak{p}_i$-primary submodule, and S is a multiplicatively closed subset of R, then

$$\bigcap_{\mathfrak{p}_i \cap S = \emptyset} Q_i = \bigcup_{s \in S} (0 :_L s).$$

Set

$$T = \{\mathfrak{p} \in \mathrm{Ass}_R M : \text{there is } \mathfrak{a} \in \Phi \text{ such that} \dim R/(\mathfrak{a} + \mathfrak{p}) > 0\} \text{ and } S = R \backslash \bigcup_{\mathfrak{p} \in T} \mathfrak{p}.$$

Therefore 2.1 implies that $\cap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M \langle \mathfrak{m} \rangle) = \cup_{s \in S} (0 :_M s)$, and

$$\bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}\mathrm{Hom}_R(N, M) :_{\mathrm{Hom_R(N,M)}} \langle \mathfrak{m} \rangle) = \bigcup_{s \in S} (0 :_{\mathrm{Hom_R(N,M)}} s).$$

Note that $\mathrm{Ass}_R(\mathrm{Hom}_R(N,M)) \subseteq \mathrm{Ass}_R M$ as one can see easily. Since M and $\mathrm{Hom}_R(N,M)$ are Noetherian, there is $t \in S$ such that $(0 :_M t)$ (resp. $(0 :_{\mathrm{Hom}_R(N,M)} t)$) is the largest element of the family $\{(0 :_M s)\}_{s\in S}$ (resp. $\{(0 :_{\mathrm{Hom}_R(N,M)} s)\}_{s\in S}$). Now, the claim follows by the functorial isomorphisms

$$\mathrm{Hom}_R(N, (0 :_M t)) \cong \mathrm{Hom}_R(N \otimes_R R/Rt, M) \cong (0 :_{\mathrm{Hom_R(N,M)}} t). \qquad \square$$

Next let us fix some notation.

Remark and notation 2.3. (i) Let $\mathfrak{a}$ be an ideal of R and N (resp. A) a Noetherian (resp. Artinian) R-module. For a submodule M of N we denote the ultimate constant value of the increasing sequence

$$M \subseteq M :_N \mathfrak{a} \subseteq M :_N \mathfrak{a}^2 \subseteq \cdots \subseteq M :_N \mathfrak{a}^i \subseteq \cdots$$

by $M :_N \langle\mathfrak{a}\rangle$. Also, we denote the least element of the sequence $\{\mathfrak{a}^i A\}_{i\in\mathbb{N}}$ by $\langle\mathfrak{a}\rangle A$.

(ii) Let $(R, \mathfrak{m})$ be a local ring. Denote the faithfully exact functor $\mathrm{Hom}_R(\cdot, E(R/\mathfrak{m}))$ by $(\cdot)^*$. Let M be a submodule of an R-module N. Following the notation of [17, §5.4], the submodule $\{f \in N^* : f(m) = 0, \text{ for all } m \in M\}$ of N^* is denoted by M^λ. Also, for a submodule K of N^*, we denote the submodule

$$\{m \in N : f(m) = 0, \text{ for all } f \in K\}$$

of N by K^μ.

(iii) Let $(R, \mathfrak{m})$ be a complete local ring and M a submodule of a Noetherian R-module N and let K be a submodule of N^*. Then it follows from [17, §5.4] that $M^{\lambda\mu} = M$ and $K^{\mu\lambda} = K$. Moreover one can check easily that, if $\{K_i\}_{i\in\Lambda}$ is a family of submodules of N^* then,

$$\left(\sum_{i\in\Lambda}\right) K_i)^\mu = \bigcap_{i\in\Lambda} K_i^\mu.$$

We shall use the following lemma in the proof of 2.6.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a complete local ring and N a finitely generated R-module. Let Φ be a system of ideals of R. Then*

(i) $(\cap_{\mathfrak{a}\in\Phi}\mathfrak{a}N :_N \langle\mathfrak{m}\rangle)^* \cong N^*/\sum_{\mathfrak{a}\in\Phi}\langle\mathfrak{m}\rangle(0 :_{N^*} \mathfrak{a})$.

(ii) *If* $\Phi \neq \{R\}$, *then* N *is* Φ*-adically complete* (*i.e., the natural map* $N \cong \lim_{\mathfrak{a}\in\Phi} N/\mathfrak{a}N$ *is an isomorphism*).

(iii) *If* $\Phi \overleftarrow{\neq} \{R\}$, *then the inverse system* $\{N/\mathfrak{a}N\}_{\mathfrak{a}\in\Phi}$ *with the natural induced maps denes an inverse system* $\{H^0_{\mathfrak{m}}(N/\mathfrak{a}N)\}_{\mathfrak{a}\in\Phi}$ *such that*

$$\varprojlim_{\mathfrak{a}\in\Phi} H^0_{\mathfrak{m}}(N/\mathfrak{a}N) \cong \bigcap_{\mathfrak{a}\in\Phi} (\mathfrak{a}N :_N \langle\mathfrak{m}\rangle).$$

Proof. (i) Let $M = \cap_{\mathfrak{a}\in\Phi}(\mathfrak{a}N :_N \langle\mathfrak{m}\rangle)$. The injection $M \longrightarrow N$ induces the natural epimorphism $N^* \longrightarrow M^*$. But the kernel of this map is M^λ as can be seen easily (μ, λ are as in 2.3). On the other hand, we have

$$\begin{aligned} M^\lambda &= \left(\bigcap_{\mathfrak{a}\in\Phi}(\mathfrak{a}N :_N \langle\mathfrak{m}\rangle)^{\lambda\mu}\right)^\lambda = \left[\left(\sum_{\mathfrak{a}\in\Phi}(\mathfrak{a}N :_N \langle\mathfrak{m}\rangle)^\lambda\right)^\mu\right]^\lambda \\ &= \sum_{\mathfrak{a}\in\Phi}(\mathfrak{a}N :_N \langle\mathfrak{m}\rangle)^\lambda. \end{aligned}$$

It turns out by [17, Theorem 5.21], that

$$(\mathfrak{a}N :_N \langle\mathfrak{m}\rangle)^\lambda = \langle\mathfrak{m}\rangle(\mathfrak{a}N)^\lambda = \langle\mathfrak{m}\rangle(N^\lambda :_{N^*} \mathfrak{a})$$

for all $\mathfrak{a} \in \Phi$, and hence $M^\lambda = \sum_{\mathfrak{a}\in\Phi}\langle\mathfrak{m}\rangle(0 :_{N^*} \mathfrak{a})$, because $N^\lambda = 0$. This finishes the proof of (i).

(ii) Since N^* is Artinian, it follows that $N^* = \cup_{i\in\mathbb{N}}(0 :_{N^*} \mathfrak{m}^i)$. For each $i \in \mathbb{N}$, there exists $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \mathfrak{m}^i$. Hence

$$N^* = \bigcup_{\mathfrak{a}\in\Phi} (0 :_{N^*} \mathfrak{a}) \cong \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Hom}_R(R/\mathfrak{a}, N^*).(\dagger)$$

Because

$$\operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{a}, N^*), E(R/\mathfrak{m})) \cong R/\mathfrak{a} \otimes_R N^{**}$$

and $N^{**} \cong N$, by applying $(.)^*$ to $(\dagger)$, we deduce that $N \cong \varprojlim_{\mathfrak{a}\in\Phi} N/\mathfrak{a}N$ as required.

(iii) Let $\mathfrak{b}$ be a proper ideal of R such that $\mathfrak{b} \in \Phi$. It is easy to see that $\cap_{\mathfrak{a}\in\Phi} \mathfrak{a}N \subseteq \cap_{n\geq 0} \mathfrak{b}^n N$. Hence, by Krull's intersection theorem, $\cap_{\mathfrak{a}\in\Phi} \mathfrak{a}N = 0$. Hence, in view of (ii) the proof is a straightforward modification of the proof of [16, Lemma 2.3]. □

Let $\mathcal{C}_R$ denote the category of all R-modules and R-homomorphisms. Let Φ be a system of ideals of R. Such a system of ideals Φ determines the Φ-torsion functor $\Gamma_\Phi(\cdot) : \mathcal{C}_R \longrightarrow \mathcal{C}_R$. This is the subfunctor of the identity functor on $\mathcal{C}_R$ for which $\Gamma_\Phi(M) := \{x \in M : \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}$, for each R-module M. For each $i \in \mathbb{N}_0$, let $H^i_\Phi(\cdot) := \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Ext}^i_R(R/\mathfrak{b}, \cdot)$, a functor which (see [4, Remarks 1.3.7]) is naturally equivalent to the i-th right derived functor of $\Gamma_\Phi(\cdot)$. We summarize some useful properties of the functors $H^i_\Phi(\cdot)$ in the following remark.

Remark 2.5. (i) For each $i \in \mathbb{N}_0$, the functors $H^i_\Phi(\cdot)$ and $\varinjlim_{\mathfrak{a}\in\Phi} H^i_\mathfrak{a}(\cdot)$ (from $\mathcal{C}_R$ to itself) are naturally equivalent (see [1, 2.1]). (Here $H^i_\mathfrak{a}(\cdot)$ is the i-th local cohomology functor with respect to $\mathfrak{a}$.)

(ii) Let $f : R \longrightarrow R'$ be a homomorphism of Noetherian commutative rings. Set $\Phi R' := \{\mathfrak{a}R' : \mathfrak{a} \in \Phi\}$. Then $\Phi R'$ is a system of ideals of R'. For any $i \in \mathbb{N}_0$, it follows from the independence theorem for local cohomology[4, Theorem 4.2.1] and (i) that $H^i_\Phi(M) \cong H^i_{\Phi R'}(M)$, for any R'-module M.

Now, we are ready to state and prove the main result of this section. This result extends the main result of the third section of [5] (see [5, Theorem 3.2]). Also it generalizes Bijan-Zadeh's result concerning Artinianess of generalized local cohomology modules (see [2, Theorem 3.1]).

Theorem 2.6. *Let Φ be a system of ideals of $(R, \mathfrak{m})$ such that $\Phi \neq \{R\}$. For a finitely generated R-module N, there is a functorial isomorphism*

$$H^d_\Phi(N) \cong H^d_\mathfrak{m}(N) \Big/ \sum_{\mathfrak{a}\in\Phi} \langle \mathfrak{m} \rangle (0 :_{H^d_\mathfrak{m}(N)} \mathfrak{a}),$$

where $d = \dim N$. In particular $H^d_\Phi(N)$ is an Artinian R-module.

Proof. Assume that $(R, \mathfrak{m})$ is a d-dimensional complete Gorenstein local ring. It follows from [2, Proposition 2.1] that

$$(H^d_\Phi(N))^* \cong (H^d_\Phi(R) \otimes_R N)^* \cong \operatorname{Hom}_R(N, H^d_\Phi(R)^*).$$

Hence it turns out, by the Local Duality Theorem (see e.g. [4, 11.2.5]), that

$$\begin{aligned}(H^d_\Phi(N))^* &\cong \operatorname{Hom}_R\left(N, \varprojlim_{\mathfrak{a}\in\Phi} (\operatorname{Ext}^d_R(R/\mathfrak{a}, R)^*\right)\\ &\cong \operatorname{Hom}_R\left(N, \varprojlim_{\mathfrak{a}\in\Phi} H^0_\mathfrak{m}(R/\mathfrak{a})\right).\end{aligned}$$

Therefore 2.4(iii) and 2.2 imply that

$$
\begin{aligned}
(H^d_\Phi(N))^* &\cong \mathrm{Hom}_R\left(N, \bigcap_{\mathfrak{a}\in\Phi}(\mathfrak{a} : \langle\mathfrak{m}\rangle)\right)\\
&\cong \bigcap_{\mathfrak{a}\in\Phi}(\mathfrak{a}\mathrm{Hom}_R(N,R) :_{\mathrm{Hom_R(N,R)}} \langle\mathfrak{m}\rangle).
\end{aligned}
$$

This yields that $H^d_\Phi(N)$ is Artinian, and so by applying 2.4(i), we deduce that $H^d_\Phi(N) \cong H^d_\mathfrak{m}(N)/\sum_{\mathfrak{a}\in\Phi}\langle\mathfrak{m}\rangle(0 :_{H^d_\mathfrak{m}(N)} \mathfrak{a})$, because $\mathrm{Hom}_R(N,R)^* = H^d_\mathfrak{m}(N)$, as follows from the Local Duality Theorem.

Finally, we treat the case of an arbitrary local ring $(R,\mathfrak{m})$. To do so, put $R_1 := R/\mathrm{Ann}_R N$. By the Cohen structure theorem there exist a d-dimensional complete Gorenstein local ring S such that $\hat{R}_1 \cong S/\mathfrak{b}$ for a certain ideal $\mathfrak{b}$ of S. For any ideal $\mathfrak{a}$ of R, let $\mathfrak{a}'$ denote the preimage of $\mathfrak{a}\hat{R}_1$ in S. Let Ψ be the set of all finite products of elements of $\{\mathfrak{a}' : \mathfrak{a}\in\Phi\}$. Then it is easy to see that Ψ is a system of ideals of S and that the system of ideals $\Phi\hat{R}_1 := \{\mathfrak{a}\hat{R}_1 : \mathfrak{a}\in\Phi\}$ is cofinal in $\Psi\hat{R}_1$. It follows from 2.5 (ii) and [2, Lemma 2.3 (i)] that

$$
H^d_\Phi(N)\otimes_{R_1}\hat{R}_1 \cong H^d_{\Phi\hat{R}_1}(\hat{N}) \cong H^d_{\Psi\hat{R}_1}(\hat{N}) \cong H^d_\Psi(\hat{N}).
$$

Since $H^d_\Psi(\hat{N})$ is an Artinian S-module, by the first part of the proof, it follows that $H^d_{\Phi\hat{R}_1}(\hat{N})$ is Artinian as an $\hat{R}_1$-module and therefore $H^d_\Phi(N)$ is an Artinian R_1-module. Consequently, $H^d_\Phi(N)\otimes_{R_1}\hat{R}_1 \cong H^d_\Phi(N)$. Therefore the situation reduces to the case where the underlying ring is local complete Gorenstein of dimension d, and so the proof is complete by the first part.

Now let N_1 and N_2 be two d-dimensional R-modules. Set $R_1 = R/(\mathrm{Ann}_R N_1 \cap \mathrm{Ann}_R N_2)$. Then R_1 is a d-dimensional Noetherian ring. Let S be a d-dimensional complete Gorenstein ring such that $\hat{R}_1 \cong S/\mathfrak{b}$ for a certain ideal $\mathfrak{b}$ of S. Thus, we can use R_1 and S for both N_1 and N_2 in order to proceed as in the previous paragraph. Therefore the above isomorphism is functorial. □

Corollary 2.7. *Let the situation be as in* 2.6. *Then*

$$
\begin{aligned}
\mathrm{Att}_{\hat{R}}(H^d_\Phi(N)) = \{\mathfrak{p}\in\mathrm{Ass}_{\hat{R}}\hat{N} : \dim\hat{R}/\mathfrak{p} = d \text{ and } \mathrm{Rad}(\mathfrak{a}\hat{R}+\mathfrak{p})\\
= \mathfrak{m}\hat{R}, \text{for all } \mathfrak{a}\in\Phi\}.
\end{aligned}
$$

Proof. Let $(\cdot)^*$ denote the functor $\mathrm{Hom}_{\hat{R}}(\cdot, E(R/\mathfrak{m}))$. Then $H^d_{\mathfrak{m}}(N)^{**} \cong H^d_{\mathfrak{m}}(N)$ by Matlis duality, so 2.4(i) implies that

$$H^d_{\Phi}(N) \cong H^d_{\mathfrak{m}}(N)^{**} \Big/ \sum_{\mathfrak{a}\in\Phi} \langle \mathfrak{m}\rangle (0 :_{H^d_{\mathfrak{m}}(N)^{**}} \mathfrak{a})$$
$$\cong \left(\bigcap_{\mathfrak{a}\in\Phi} \mathfrak{a} H^d_{\mathfrak{m}}(N)^* :_{H^d_{\mathfrak{m}}(N)^*} \langle \mathfrak{m}\rangle \right)^*.$$

So, we have

$$\mathrm{Att}_{\hat{R}}(H^d_{\Phi}(N)) = \mathrm{Ass}_{\hat{R}}\left(\bigcap_{\mathfrak{a}\in\Phi} \mathfrak{a} H^d_{\mathfrak{m}}(N)^* :_{H^d_{\mathfrak{m}}(N)^*} \langle \mathfrak{m}\rangle \right).$$

Also, by[4, Theorem 7.3.2],

$$\mathrm{Ass}_{\hat{R}} H^d_{\mathfrak{m}}(N)^* = \mathrm{Att}_{\hat{R}} H^d_{\mathfrak{m}}(N) = \{\mathfrak{p} \in \mathrm{Ass}_{\hat{R}} \hat{N} : \dim \hat{R}/\mathfrak{p} = d\}.$$

Consequently, the claim results from 2.1 and the following easy observation. Let M be a finitely generated R-module and let $0 = \cap_{i=1}^n Q_i$ be a minimal primary decomposition of the zero submodule of M. Set $L = \cap_{i=1}^m Q_i$, for $0 \le m < n$. Then $0 = \cap_{i=m+1}^n (L \cap Q_i)$ is a minimal primary decomposition of the zero submodule of L such that $\mathrm{Rad}(L \cap Q_i :_R L) = \mathrm{Rad}(Q_i :_R M)$. □

The following result extends the Lichtenbaum-Hartshorne vanishing theorem to generalized local cohomology. In view of 3.2, this result implies 1.1.

Theorem 2.8. *Let Φ denote a system of ideals of a local ring $(R, \mathfrak{m})$ such that $\Phi \neq \{R\}$. For a d-dimensional finitely generated R-module N, the following conditions are equivalent:*

(i) $H^d_{\Phi}(N) = 0$.
(ii) $H^d_{\mathfrak{m}}(N) = \sum_{\mathfrak{a}\in\Phi} \langle \mathfrak{m}\rangle (0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a})$.
(iii) *For any $\mathfrak{p} \in \mathrm{Ass}_{\hat{R}} \hat{N}$ with $\dim \hat{R}/\mathfrak{p} = \dim N$, there is $\mathfrak{a} \in \Phi$ such that $\dim \hat{R}/(\mathfrak{a}\hat{R} + \mathfrak{p}) > 0$.*
(iv) *For any $\mathfrak{a} \in \Phi$ there is $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} K_{\hat{N}} :_{K_{\hat{N}}} \langle \mathfrak{m}\rangle \subseteq \mathfrak{a} K_{\hat{N}}$, (here $K_{\hat{N}} = (H^d_{\mathfrak{m}}(N))^*$ is the canonical module of $\hat{N}$).*

Proof. First we observe that the equivalence of (i), (ii) and (iii) follows from 2.6 and 2.7. Note that for an Artinian R-module A, we have $A = 0$ if and only if $\mathrm{Att}_R(A) = \emptyset$.

To complete the proof, it is enough to show the equivalence of (iii) and (iv). Set $M := H^d_{\mathfrak{m}}(N)^*$. As we have mentioned in the proof of 2.7,

$$\mathrm{Ass}_{\hat{R}}M = \{\mathfrak{p} \in \mathrm{Ass}_{\hat{R}}\hat{N} : \dim \hat{R}/\mathfrak{p} = d\}.$$

Suppose (iv) holds. Then $\cap_{\mathfrak{a}\in\Phi}(\mathfrak{a}M :_M \langle\mathfrak{m}\rangle) \subseteq \cap_{\mathfrak{a}\in\Phi}\mathfrak{a}M$. The second term is zero by Krull's intersection theorem and, so (iii) holds by 2.1. Let (iii) hold. For a given $\mathfrak{a} \in \Phi$, the subset

$$\Psi = \{\mathfrak{b} \in \Phi : \mathfrak{b} \subseteq \mathfrak{a}\}$$

of Φ forms a system of ideals of R. Since for any $\mathfrak{b} \in \Phi$, there is a $\mathfrak{c} \in \Psi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$, we can and do replace Φ by Ψ. Hence, by 2.1,

$$\bigcap_{\mathfrak{b}\in\Psi} (\mathfrak{b}M :_M \langle\mathfrak{m}\rangle) = 0.$$

Now, because the module $\mathfrak{a}M :_M \langle\mathfrak{m}\rangle/\mathfrak{a}M$ has finite length, (iv) follows from the following proposition. □

Note that 2.8 generalizes [5, Corollary 3.4].

The following proposition extends a version of Chevalley's theorem which is proved in [14, Lemma 3.3].

Proposition 2.9. *Let $(R, \mathfrak{m})$ be a complete local ring and M a submodule of a finitely generated R-module N. Let $\{N_j\}_{j\in J}$ be a collection of submodules of N such that for each $j, k \in J$, there is $l \in J$ with $N_l \subseteq N_j \cap N_k$. Assume that the family $\{M + N_j\}_{j\in J}$ has a minimal element. Then there is $j_0 \in J$ such that $N_{j_0} \subseteq M + \cap_{j\in J} N_j$.*

Proof. Replacing N by $N/\cap_{j\in J} N_j$, we can assume that $\cap_{j\in J} N_j = 0$. By assumption, there exists $k \in J$ such that $M + N_k$ is a minimal element of the family $\{M + N_j\}_{j\in J}$. In fact the hypothesis on $\{N_j\}_{j\in J}$ implies that $M + N_k$ is the least element of this family. Hence $M + N_k = \cap_{j\in J}(M + N_j)$. To complete the proof, it is enough to show that this intersection is equal to M. To this end, note that if M_1 and M_2 are submodules of N, then $(M_1 + M_2)^\lambda = M_1^\lambda \cap M_2^\lambda$. Now, by 2.3(iii),

$$\bigcap_{j\in J}(M + N_j) = \bigcap_{j\in J}(M + N_j)^{\lambda\mu} = \left(\sum_{j\in J}(M + N_j)^\lambda\right)^\mu$$
$$= \left(\sum_{j\in J}(M^\lambda \cap N_j^\lambda)\right)^\mu.$$

The hypothesis on $\{N_j\}_{j\in J}$ implies that

$$\sum_{j\in J}(M^\lambda \cap N_j^\lambda) = M^\lambda \cap \left(\bigcup_{j\in J} N_j^\lambda\right).$$

On the other hand, since by 2.3(iii), $(\cup_{j\in J} N_j^\lambda)^\mu = \cap_{j\in J} N_j = 0 = (N^*)^\mu$, it follows that $\cup_{j\in J} N_j^\lambda = N^*$. Therefore

$$\bigcap_{j\in J}(M + N_j) = (M^\lambda \cap N^*)^\mu = M. \qquad \square$$

3. CONNECTEDNESS THEOREMS

In this section we examine connectedness of certain subsets of Spec R. In fact via the generalized Lichtenbaum-Hartshorne vanishing theorem and the generalized Mayer-Vietoris sequence (see 3.4), we are able to extend some previously known connectedness results. To this end we recall some notation and definitions for use in the sequel. Recall that a subset Z of Spec$R$ is *stable under specialization* (s.u.s. for short), if whenever $\mathfrak{p} \in Z$ and $\mathfrak{q}$ is a prime ideal of R with $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{q} \in Z$. A system of ideals Φ is called *full* if, whenever $\mathfrak{a} \in \Phi$ and $\mathfrak{b}$ is an ideal of R with $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathfrak{b} \in \Phi$. The following result illustrates a close relationship between full systems of ideals and s.u.s. subsets of R (see e.g., [18, Lemma 2.3]).

Lemma 3.1. *The maps $\Phi \longrightarrow V(\Phi) := \cup_{\mathfrak{a}\in\Phi} V(\mathfrak{a})$ and $Z \longrightarrow F(Z) := \{\mathfrak{b} : \mathfrak{b}$ is an ideal of R with $V(\mathfrak{b}) \subseteq Z\}$ are inverse bijections between the set of full systems of ideals of R and the set of s.u.s. subsets of* Spec R.

For a s.u.s. subset Z of $X =$ Spec R, let $\Gamma_Z(X, \cdot)$ denote the section functor with support in Z, from the category of sheaves on X to the category of abelian groups. We denote the right derived functors of $\Gamma_Z(X, .)$, by $H^i_Z(X, .)$, $i \geq 0$. These are called the cohomology groups of X with support in Z. Specially for an R-module M, the R-module $H^i_Z(X, \tilde{M})$ is denoted by $H^i_Z(M)$, where $\tilde{M}$ denotes the sheaf associated to M on X (for more details about the cohomology of sheaves see [10, Ch.3]). The following, which can be deduced by 2.5(i), [9, page 219] and [10, Ch.3, Ex.3.3(b)], is another connection between the notion of "s.u.s. subsets" and that of "full systems of ideals".

Lemma 3.2. *Let Z be a s.u.s. subset of* Spec R *and Φ be its corresponding full system of ideals* (*see* 3.1). *Then for each R-module M, the R-modules $H^i_Z(M)$ and $H^i_\Phi(M)$ are isomorphic.*

In the sequel we shall use the following version of Mayer-Vietoris sequence, which can be proved by a slight modification of the proof of [4, 3.2.3]. For presenting this result, we need the following remark.

Remark 3.3. (i) There is another definition of system of ideals,[4, Definition 2.1.10] which obviously coincides with our definition. Let $(I, \leq)$ be a (non-empty) directed partially ordered set. A system of ideals of R over I is a family $\Phi = \{\mathfrak{a}_i\}_{i\in\Lambda}$ of ideals of R satisfying the following conditions:

1) if $i, j \in I$ with $j \leq i$, then $\mathfrak{a}_i \subseteq \mathfrak{a}_j$ and,
2) for all $i, j \in I$, there exists $k \in I$ such that $k \geq i, k \geq j$ and $\mathfrak{a}_k \subseteq \mathfrak{a}_i\mathfrak{a}_j$.

(ii) Let $\Phi_1 = \{\mathfrak{a}_i\}_{i\in I}$ and $\Phi_2 = \{\mathfrak{b}_i\}_{i\in I}$ be two systems of ideals of R. It easily can be checked that $\Phi_1 + \Phi_2 = \{\mathfrak{a}_i + \mathfrak{b}_i\}_{i\in I}$ is a system of ideals of R. Next, we show that $\Phi_1 \cap \Phi_2 := \{\mathfrak{a}_i \cap \mathfrak{b}_i\}_{i\in I}$ is also a system of ideals. In view of [4, Proposition 3.1.1(iii)] and the fact that $\Phi_1\Phi_2 := \{\mathfrak{a}_i\mathfrak{b}_i\}_{i\in I}$ is a system of ideals, it is enough to show that for each $i \in I$, there is $j \in I$ such that $\mathfrak{a}_j \cap \mathfrak{b}_j \subseteq \mathfrak{a}_i\mathfrak{b}_i$. To this end, note that for a given $i \in I$, by the Artin-Rees lemma, there exists $c \in \mathbb{N}$ such that $\mathfrak{a}_i^m \cap \mathfrak{b}_i = \mathfrak{a}_i^{m-c}(\mathfrak{a}_i^c \cap \mathfrak{b}_i)$ for all $m > c$. Hence

$$\mathfrak{a}_i^{1+c} \cap \mathfrak{b}_i^{1+c} \subseteq \mathfrak{a}_i^{1+c} \cap \mathfrak{b}_i = \mathfrak{a}_i(\mathfrak{a}_i^c \cap \mathfrak{b}_i) \subseteq \mathfrak{a}_i\mathfrak{b}_i.$$

Now, there is $j \in I$ such that $\mathfrak{a}_j \subseteq \mathfrak{a}_i^{1+c}$ and $\mathfrak{b}_j \subseteq \mathfrak{b}_i^{1+c}$. Consequently, $\mathfrak{a}_j \cap \mathfrak{b}_j \subseteq \mathfrak{a}_i\mathfrak{b}_i$, as required.

(iii) For a system of ideals $\Phi = \{\mathfrak{a}_i\}_{i\in I}$ and an ideal $\mathfrak{b}$ of R, put

$$\Phi_{\mathfrak{b}} := \{\mathfrak{a}_i + \mathfrak{b}^n : (i, n) \in I \times \mathbb{N}\} \text{ and } \Phi^{\mathfrak{b}} := \{\mathfrak{a}_i\mathfrak{b}^n : (i, n) \in I \times \mathbb{N}\}.$$

With pointwise ordering, the set $I \times \mathbb{N}$ becomes a directed partially ordered set. It is easy to see that $\Phi_{\mathfrak{b}}$ and $\Phi^{\mathfrak{b}}$ are systems of ideals and that $V(\Phi_{\mathfrak{b}}) = V(\Phi) \cap V(\mathfrak{b})$ and $V(\Phi^{\mathfrak{b}}) = V(\Phi) \cup V(\mathfrak{b})$.

Lemma 3.4. *Let $\Phi_1 = \{\mathfrak{a}_i\}_{i\in I}$ and $\Phi_2 = \{\mathfrak{b}_i\}_{i\in I}$ be systems of ideals of R. For each R-module M, there is a functorial long exact sequence*

$$\begin{aligned} \ldots \longrightarrow H^i_{\Phi_1+\Phi_2}(M) &\longrightarrow H^i_{\Phi_1}(M) \oplus H^i_{\Phi_2}(M) \\ \longrightarrow H^i_{\Phi_1\cap\Phi_2}(M) &\longrightarrow H^{i+1}_{\Phi_1+\Phi_2}(M) \longrightarrow \ \ldots . \end{aligned}$$

Now, we are ready to establish our first connectedness result. Let Φ be a system of ideals of R and N a finitely generated R-module. We shall denote $\min\{\text{grade}(\mathfrak{a}, N) : \mathfrak{a} \in \Phi\}$ by grade (Φ, N). Since $H^i_\Phi(N) = \lim_{\longrightarrow\, \mathfrak{a}\in\Phi} H^i_{\mathfrak{a}}(N)$, it follows that $H^i_\Phi(N) = 0$, for all $i < \text{grade}(\Phi, N)$.

Proposition 3.5. *Let N be an indecomposable finitely generated module over a local ring $(R, \mathfrak{m})$. Let Z be a s.u.s. subset of* $\operatorname{Spec} R$ *such that grade $(F(Z), N) > 1$. Then the space* $\operatorname{Supp} N \backslash Z$ *is connected.*

Proof. Taking $Z' := Z \cap \operatorname{Supp} N$, we have $\operatorname{Supp} N \backslash Z = \operatorname{Supp} N \backslash Z'$. Thus we may and do assume that $Z \subseteq \operatorname{Supp} N$. Note that, in view of 3.2 and 2.5(ii), $H^i_Z(N) \cong H^i_{Z'}(N)$ for all $i \geq 0$ and since grade $(F(Z), N) > 1$, we have that $\operatorname{Supp} N \backslash Z \neq \emptyset$. Suppose $\operatorname{Supp} N \backslash Z$ is disconnected. Then there are ideals $\mathfrak{a}, \mathfrak{b} \supseteq \operatorname{Ann}_R N$ with the following properties:

i) $Z \cup V(\mathfrak{a}) \subsetneqq \operatorname{Supp} N$ and $Z \cup V(\mathfrak{b}) \subsetneqq \operatorname{Supp} N$,
ii) $V(\mathfrak{a} + \mathfrak{b}) \subseteq Z$; and
iii) $Z \cup V(\mathfrak{a} \cap \mathfrak{b}) = \operatorname{Supp} N$.

Let $F(Z) = \Phi$. Then the first part of the Mayer-Vietoris sequence yields an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0_{\Phi^{\mathfrak{a}} + \Phi^{\mathfrak{b}}}(N) &\longrightarrow H^0_{\Phi^{\mathfrak{a}}}(N) \oplus H^0_{\Phi^{\mathfrak{b}}}(N) \\ &\longrightarrow H^0_{\Phi^{\mathfrak{a}} \cap \Phi^{\mathfrak{b}}}(N) \longrightarrow H^1_{\Phi^{\mathfrak{a}} + \Phi^{\mathfrak{b}}}(N). \end{aligned}$$

The condition (ii) together with 3.3(iii) and 3.1, imply that $\Phi^{\mathfrak{a}} + \Phi^{\mathfrak{b}} = F(Z)$. Thus, it follows that $H^i_{\Phi^{\mathfrak{a}} + \Phi^{\mathfrak{b}}}(N) = 0$ for $i = 0, 1$. Hence $H^0_{\Phi^{\mathfrak{a}}}(N) \oplus H^0_{\Phi^{\mathfrak{b}}}(N) \cong H^0_{\Phi^{\mathfrak{a}} \cap \Phi^{\mathfrak{b}}}(N)$. From the condition (iii) and 3.3(iii), we deduce that $V(\Phi^{\mathfrak{a}} \cap \Phi^{\mathfrak{b}}) = \operatorname{Supp} N$. Hence $H^0_{\Phi^{\mathfrak{a}} \cap \Phi^{\mathfrak{b}}}(N) \cong N$ and so $H^0_{\Phi^{\mathfrak{a}}}(N) \oplus H^0_{\Phi^{\mathfrak{b}}}(N) \cong N$. Because of the indecomposability assumption on N, it follows that $H^0_{\Phi^{\mathfrak{a}}}(N) \cong N$ or $H^0_{\Phi^{\mathfrak{b}}}(N) \cong N$. This implies that either $V(\Phi^{\mathfrak{a}}) = \operatorname{Supp} N$ or $V(\Phi^{\mathfrak{b}}) = \operatorname{Supp} N$. Therefore, by 3.3(iii), $Z \cup V(\mathfrak{a}) = \operatorname{Supp} N$ or $Z \cup V(\mathfrak{b}) = \operatorname{Supp} N$. Hence we arrived at a contradiction, by condition (i), so $\operatorname{Supp} N \backslash Z$ is connected.

Note that 3.5 extends [5, Lemma 4.5]. In the rest of this section, we use $\operatorname{Min}_R M$ (resp. $\operatorname{Assh}_R N$) to denote the subset $\{\mathfrak{p} \in \operatorname{Ass}_R N : \mathfrak{p}$ is minimal in $\operatorname{Ass}_R N\}$ (resp. $\{\mathfrak{p} \in \operatorname{Ass}_R N : \dim R/\mathfrak{p} = \dim N\}$) of $\operatorname{Ass}_R N$.

Theorem 3.6. *Let Z be a s.u.s. subset of* $\operatorname{Spec} R$ *and N a finitely generated R-module with $d = \dim N$. Suppose that $\operatorname{Min}_{\hat{R}} \hat{N} = \operatorname{Assh}_{\hat{R}} \hat{N}$ and that $H^d_{\mathfrak{m}}(N)$ is an indecomposable R-module. Then $Z \cap \operatorname{Supp}(N) \backslash \{\mathfrak{m}\}$ is connected provided $H^i_Z(N) = 0$ for $i = d-1, d$.*

Proof. First of all note that, since $H^d_Z(N) = 0$, it follows that $Z \cap \operatorname{Supp}(N) \backslash \{\mathfrak{m}\} \neq \emptyset$. Suppose that $Z \cap \operatorname{Supp}(N) \backslash \{\mathfrak{m}\}$ is disconnected. Then there are ideals $\mathfrak{b}, \mathfrak{c} \supseteq \operatorname{Ann}_R N$ such that for $X = Z \cap \operatorname{Supp} N$, the following conditions are satisfied:

1) $\{\mathfrak{m}\} = X \cap V(\mathfrak{b} + \mathfrak{c})$,

2) $X \backslash V(\mathfrak{b}) \subsetneqq X \backslash \{\mathfrak{m}\}, X \backslash V(\mathfrak{c}) \subsetneqq X \backslash \{\mathfrak{m}\}$; and
3) $X \subseteq V(\mathfrak{b} \cap \mathfrak{c})$.

Set $F(Z) = \Phi$. We shall denote $\operatorname{Ann}_R N$ by $\mathfrak{a}$. The above conditions imply that

i) $V(\Phi_{\mathfrak{b}} + \Phi_{\mathfrak{c}}) = \{\mathfrak{m}\}$,
ii) $\{\mathfrak{m}\} \subsetneqq V(\Phi_{\mathfrak{b}}), \{\mathfrak{m}\} \subsetneqq V(\Phi_{\mathfrak{c}})$, and
iii) $V(\Phi_{\mathfrak{b}} \cap \Phi_{\mathfrak{c}}) = V(\Phi_{\mathfrak{a}})$, where $\Phi_{\mathfrak{b}}$ and $\Phi_{\mathfrak{c}}$ are the same as in 3.3 (iii).

The Mayer-Vietoris sequence

$$H^{d-1}_{\Phi_{\mathfrak{a}}}(N) \longrightarrow H^d_{\mathfrak{m}}(N) \longrightarrow H^d_{\Phi_{\mathfrak{b}}}(N) \oplus H^d_{\Phi_{\mathfrak{c}}}(N) \longrightarrow H^d_{\Phi_{\mathfrak{a}}}(N),$$

provides an isomorphism $H^d_{\mathfrak{m}}(N) \cong H^d_{\Phi_{\mathfrak{b}}}(N) \oplus H^d_{\Phi_{\mathfrak{c}}}(N)$. Recall that by 2.5(ii), $H^i_{\Phi_{\mathfrak{a}}}(N) = H^i_{\Phi}(N)$ for all $i \geq 0$. By the indecomposablity assumption on $H^d_{\mathfrak{m}}(N)$ one of the direct summands, say $H^d_{\Phi_{\mathfrak{b}}}(N)$, has to be zero. Hence $H^d_{\mathfrak{m}}(N) \cong H^d_{\Phi_{\mathfrak{c}}}(N)$. By applying 2.7 to the system of ideals $\Phi_{\mathfrak{c}}$ and $\{\mathfrak{m}^k\}_{k \in \mathbb{N}}$ we deduce

$$\begin{aligned} \operatorname{Min}_{\hat{R}} \hat{N} = \{\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N} : \dim \hat{R}/\mathfrak{p} = d \text{ and} \dim \hat{R}/(\mathfrak{a}\hat{R} + \mathfrak{p}) \\ = 0 \text{ for all } \mathfrak{a} \in \Phi_{\mathfrak{c}}\}. \end{aligned}$$

This implies that $V(\Phi_{\mathfrak{c}}) = \{\mathfrak{m}\}$, which contradicts the fact that $\{\mathfrak{m}\} \subsetneqq V(\Phi_{\mathfrak{c}})$. Therefore $Z \cap \operatorname{Supp}(N) \backslash V(\mathfrak{m})$ is connected. □

Now, we state our last connectedness result.

Theorem 3.7. *Let $(R, \mathfrak{m})$ denote a local ring and let N be a finitely generated R-module with $d = \dim N$ such that $\operatorname{Min}_{\hat{R}} \hat{N}$ consists of a single prime $\mathfrak{p}$. Let Z be a s.u.s. subset of $\operatorname{Spec} R$ such that $H^i_Z(N) = 0$ for $i = d-1, d$. Then $Z \cap \operatorname{Supp} N \backslash \{\mathfrak{m}\}$ is a connected subset of $Z \cap \operatorname{Supp} N$.*

Proof. Let $X = Z \cap \operatorname{Supp} N$. If $X \backslash \{\mathfrak{m}\}$ is disconnected, then there are ideals $\mathfrak{b}, \mathfrak{c} \supseteq \operatorname{Ann}_R N$ satisfying the following conditions:

1) $\{\mathfrak{m}\} = X \cap V(\mathfrak{b} + \mathfrak{c})$.
2) $X \backslash V(\mathfrak{b}) \subsetneqq X \backslash \{\mathfrak{m}\}, X \backslash V(\mathfrak{c}) \subsetneqq X \backslash \{\mathfrak{m}\}$, and
3) $X \subseteq V(\mathfrak{b} \cap \mathfrak{c})$.

Let $\Phi = F(Z)$. Suppose $\Phi_{\mathfrak{a}}$, $\Phi_{\mathfrak{b}}$ and $\Phi_{\mathfrak{c}}$ are as in the proof of 3.6. As in the proof of 3.6, we have the following Mayer-Vietoris sequence

$$H^{d-1}_Z(N) \longrightarrow H^d_{\mathfrak{m}}(N) \longrightarrow H^d_{\Phi_{\mathfrak{b}}}(N) \oplus H^d_{\Phi_{\mathfrak{c}}}(N) \longrightarrow H^d_Z(N).$$

By our vanishing assumption on $H^i_Z(N)$, it follows that $H^d_{\mathfrak{m}}(N)$ is embedded in $H^d_{\Phi_{\mathfrak{b}}}(N) \oplus H^d_{\Phi_{\mathfrak{c}}}(N)$. Because $X\backslash V(\mathfrak{c}) \subsetneqq X\backslash\{\mathfrak{m}\}$, it follows that there is $(i, n) \in I \times \mathbb{N}$ such that $\dim R/(\mathfrak{a}_i + \mathfrak{c}^n) > 0$. This implies that $\dim \hat{R}/(\mathfrak{a}_i + \mathfrak{c}^n)\, \hat{R} + \mathfrak{p} > 0$. Recall that $\mathrm{Min}_{\hat{R}}\hat{N} = \{\mathfrak{p}\}$. Consequently, $H^d_{\Phi_{\mathfrak{c}}}(N) = 0$, by the generalized Lichtenbaum-Hartshorne vanishing theorem (see 2.8.).

Similarly $H^d_{\Phi_{\mathfrak{b}}}(N) = 0$. Thus $H^d_{\mathfrak{m}}(N) = 0$, which contradicts Grothendieck's non-vanishing theorem (see [4, Theorem 6.1.4]). □

ACKNOWLEDGMENTS

The research of the first and third authors was partially supported by a grant from Shahid Behashti University. That of the second author was partially supported by a grant from Tabriz University and IPM.

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Received February 2001
Revised September 2001