Tensor products of some special rings

Masoud Tousi a,b and Siamak Yassemi a,c

a Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran
b Department of Mathematics, Shahid Beheshti University, Tehran, Iran
c Department of Mathematics, University of Tehran, Tehran, Iran

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Abstract

In this paper we solve a problem, originally raised by Grothendieck, on the properties, i.e., complete intersection, Gorenstein, Cohen–Macaulay, that are conserved under tensor product of algebras over a field k.

Keywords: Regular; Complete intersection; Gorenstein; Cohen–Macaulay; Flat homomorphism of rings

Introduction

Throughout this note all rings and algebras considered in this paper are commutative with identity elements, and all ring homomorphisms are unital. Throughout, k stands for a field.

Among local rings there is a well-known chain

Regular ⇒ Complete intersection ⇒ Gorenstein ⇒ Cohen–Macaulay.

These concepts are extended to non-local rings: for example, a ring is regular if for all prime ideal p of R, R_p is a regular local ring.

In this paper, we shall investigate if these properties are conserved under tensor product operations. It is well-known that the tensor product R ⊗_A S of regular rings is not regular in general, even if we assume R and S are A-algebra and A is a field, see Remark 7. In [5],

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E-mail addresses: tousi@ipm.ir (M. Tousi), yassemi@ut.ac.ir (S. Yassemi).

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Watanabe, Ishikawa, Tachibana, and Otsuka, showed that under a suitable condition tensor products of regular rings are complete intersections. It is proved in [3], that the tensor product
\[ R \otimes_A S \]
of Cohen–Macaulay rings are again Cohen–Macaulay if we assume \( R \) is flat \( A \)-module and \( S \) is a finitely generated \( A \)-module, and in [5], it is shown that the same is true for Gorenstein rings. Recently, in [1], Bouchiba and Kabbaj showed that if \( R \) and \( S \) are \( k \)-algebras such that \( R \otimes_k S \) is Noetherian then \( R \otimes_k S \) is a Cohen–Macaulay ring if and only if \( R \) and \( S \) are Cohen–Macaulay rings.

In this paper we shall show that the same is true for complete intersection and Gorenstein rings. Also it is shown that \( R \otimes_k S \) satisfies Serre’s condition \((S_n)\) if and only if \( R \) and \( S \) satisfy \((S_n)\).

Main results

A Noetherian local ring \( R \) is a complete intersection (ring) if its completion \( \hat{R} \) is a residue class ring of a regular local ring \( S \) with respect to an ideal generated by an \( S \)-sequence. We say that a Noetherian ring is locally a complete intersection if all its localizations are complete intersections.

A Noetherian ring \( R \) satisfies Serre’s condition \((S_n)\) if depth \( R_p \geq \min\{n, \dim R_p\} \) for all prime ideal \( p \) of \( R \). Also, a Noetherian ring \( R \) satisfies Serre’s normality condition \((R_n)\) if \( R_p \) is a regular local ring for all prime ideal \( p \) with \( \dim R_p \leq n \).

The following theorem is collected from [2, Remark 2.3.5, Corollary 3.3.15, Theorem 2.1.7, and Theorem 2.2.12].

**Theorem 1.** Let \( \varphi: (R, m) \rightarrow (S, n) \) be a flat local homomorphism of Noetherian local rings. Then the following hold:

(a) \( S \) is a complete intersection (resp. Gorenstein, Cohen–Macaulay) \( \iff \) \( R \) and \( S/mS \) are complete intersections (resp. Gorenstein, Cohen–Macaulay).

(b) If \( S \) is regular then \( R \) is regular.

(b2) If \( R \) and \( S/mS \) are regular then \( S \) is regular.

**Corollary 2.** Let \( \varphi: R \rightarrow S \) be a flat homomorphism of Noetherian rings. Then the following hold:

(a) If \( R \) and the fibers \( R_p/pR_p \otimes_R S, p \in \text{Spec}(R) \), are regular (resp. locally complete intersections, Gorenstein, Cohen–Macaulay) then \( S \) is regular (resp. locally complete intersection, Gorenstein, Cohen–Macaulay).

(b) If \( S \) is locally complete intersection (resp. Gorenstein, Cohen–Macaulay) then the fibers \( R_p/p\hat{R}_p \otimes_R S, p \in \text{Spec}(R) \), are locally complete intersections (resp. Gorenstein, Cohen–Macaulay).

**Proof.** (a) Let \( q \in \text{Spec}(S) \). Set \( p = q \cap R \in \text{Spec}(R) \). The induced homomorphism \( \hat{\varphi}: R_p \rightarrow S_q \) is flat and local. It is clear that \( S_q/p\hat{R}_q S_q \) is a localization of \( R_p/pR_q \otimes_R S \). Now the assertion follows from Theorem 1.
Proposition 5. Now the assertion follows from Theorems 1 and 3.

\[ \text{(2, Theorem 2.3.3(c)).} \]

Let

Proof.

\[ \text{Corollary 4.} \]

\[ (a) \text{ Let } q \in \text{Spec}(S) \text{ and } p = q \cap R. \text{ If } S_q \text{ satisfies } (S_n) \text{ (resp. } (R_n)) \text{ then } R_p \text{ satisfies } (S_n) \text{ (resp. } (R_n)). \]

\[ (b) \text{ If } R \text{ and the fibers } R_p/pR_p \boxtimes_R S, \text{ p } \in \text{Spec}(R), \text{ satisfy } (S_n) \text{ (resp. } (R_n)) \text{ then } S \text{ satisfies } (S_n) \text{ (resp. } (R_n)). \]

Theorem 3 (see [2], Propositions 2.1.16 and 2.2.21). Let \( \varphi : R \to S \) be a flat homomorphism of Noetherian rings. Then the following hold:

\[ (a) \text{ If } S \text{ is regular (resp. locally complete intersection, Gorenstein, Cohen–Macaulay), then so is } R. \]

\[ (b) \text{ If } S \text{ satisfies } (S_n) \text{ (resp. } (R_n)) \text{, then so does } R. \]

Proof. Let \( p \in \text{Spec}(R). \) Since \( \varphi \) is faithfully flat there exists \( q \in \text{Spec}(S) \) such that 

\[ p = q \cap R. \] 

Consider the flat local homomorphism \( \hat{\varphi} : R_p \to S_q \) where \( \hat{\varphi}(r/s) = \varphi(r)/\varphi(s). \)

Now the assertion follows from Theorems 1 and 3.

Proposition 5. Let \( k \) be a field, \( L \) and \( K \) be two extension fields of \( k. \) Suppose that \( L \otimes_k K \) is Noetherian. Then the following hold:

\[ (a) L \otimes_k K \text{ is locally complete intersection.} \]

\[ (b) \text{ If } k \text{ is perfect then } L \otimes_k K \text{ is regular.} \]

Proof. (a) With the same method in the proof of [4, Theorem 2.2], we can assume that \( K \) is a finitely generated extension field of \( k \) (note that, in view of Theorem 1, [4, Lemma 2.1] is true with “Gorenstein ring” replaced by “complete intersection”). Now using [2, Proposition 2.1.11] we have that \( L \otimes_k K \) is isomorphic to

\[ A = T^{-1}(L[x_1, x_2, \ldots, x_n])/(f_1, f_2, \ldots, f_m)T^{-1}(L[x_1, x_2, \ldots, x_n]), \]

where \( T \) is a multiplicatively closed subset of \( L[x_1, x_2, \ldots, x_n] \) and \( f_1, f_2, \ldots, f_m \) is a \( T^{-1}(L[x_1, x_2, \ldots, x_n]) \)-sequence. Therefore \( A \) is locally complete intersection, cf. [2, Theorem 2.3.3(c)].

(b) The assertion follows from the note on page 49 of [4].

Theorem 6. Let $R$ and $S$ be non-zero $k$-algebras such that $R \otimes_k S$ is Noetherian. Then the following hold:

(a) $R \otimes_k S$ is locally complete intersection (resp. Gorenstein, Cohen–Macaulay) if and only if $R$ and $S$ are locally complete intersections (resp. Gorenstein, Cohen–Macaulay).

(b) $R \otimes_k S$ satisfies $(S_n)$ if and only if $R$ and $S$ satisfy $(S_n)$.

(c) If $R \otimes_k S$ is regular then $R$ and $S$ are regular.

(d) If $R \otimes_k S$ satisfies $(R_n)$ then $R$ and $S$ satisfy $(R_n)$.

(e) The converse of parts (c) and (d) hold if $\text{char}(k) = 0$ or $\text{char}(k) = p$ such that $k = \{a^p \mid a \in k\}$. 

Proof. Consider two faithfully flat homomorphisms

$$\varphi : R \rightarrow R \otimes_k S \quad \text{and} \quad \psi : S \rightarrow R \otimes_k S$$

of Noetherian rings.

If $R \otimes_k S$ is regular (resp. locally complete intersection, Gorenstein, Cohen–Macaulay) then by Corollary 4 we have $R$ and $S$ are regular (resp. locally complete intersections, Gorenstein, Cohen–Macaulay). Also if $R \otimes_k S$ satisfies $(S_n)$ (resp. $(R_n)$) then by Corollary 4, $R$ and $S$ satisfy $(S_n)$ (resp. $(R_n)$).

Now let $R$ and $S$ be locally complete intersection (resp. Gorenstein, Cohen–Macaulay). By Corollary 2 it is enough to show that the fibres $(R \otimes_k S) \otimes_R R_p / pR_p \cong R_p / pR_p \otimes_k S$ over every prime ideal $p$ of $R$ is locally complete intersection (resp. Gorenstein, Cohen–Macaulay). Consider the flat homomorphism $\gamma : S \rightarrow R_p / pR_p \otimes_k S$. Using Corollary 2, it is enough to show that the fibres $(R_p / pR_p \otimes_k S) \otimes_S S_q / qS_q \cong R_p / pR_p \otimes_k S_q / qS_q$ over every prime $q$ of $S$ is locally complete intersection (resp. Gorenstein, Cohen–Macaulay). But it is clear to see that $R_p / pR_p \otimes_k S_q / qS_q$ is Noetherian, since it is a localization of $R / p \otimes_k S / q \cong R \otimes_k S / (p \otimes_k S + R \otimes_k q)$, which is Noetherian. Now the assertion follows from Proposition 5.

If $R$ and $S$ satisfy $(S_n)$, with the same proof $R \otimes_k S$ satisfies $(S_n)$. By using the Proposition 5 the proof of part (e) is the same. \(\Box\)

Remark 7. The converse of part (c) in Theorem 6 is not true. For example, let $k$ be an imperfect field of characteristic 3, let $a \in k$ be an element with no cube root in $k$. Then $K = k[x]/(x^3 - a)k[x]$ is a splitting field of $x^3 - a$ over $k$. Thus $K \otimes_k K \cong K[x]/(x^3 - a)K[x]$, which is not regular.

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