ON VANISHING OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let $\mathfrak{a}$ denote an ideal of a d-dimensional Gorenstein local ring R and M and N two finitely generated R-modules with $\operatorname{pd} M < \infty$. It is shown that $H^d_{\mathfrak{a}}(M,N) = 0$ if and only if $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} > 0$ for all $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{M} \cap \operatorname{Supp}_{\hat{R}} \hat{N}$.

1. Introduction

A generalization of local cohomology functors has been given by J. Herzog in [**6**]. Let $\mathfrak{a}$ denote an ideal of a commutative Noetherian ring R. For each $i \geq 0$, the functor $H^i_{\mathfrak{a}}(.,.)$ defined by $H^i_{\mathfrak{a}}(M,N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$, for all R-modules M and N. Clearly, this notion is a generalization of the usual local cohomology functor. The study of this concept was continued in the articles [**8**], [**2**],[**9**], [**1**] and [**10**].

Two important type of theorems concerning local cohomology are finiteness and vanishing results. We collect the known vanishing results for generalized local cohomology in the following theorem.

Theorem 1.1. *Let M and N be two non-zero finitely generated R-modules such that* $\operatorname{pd} M < \infty$.

(i) *([**9**, Theorem 3.7]) Suppose* $\dim N < \infty$. *Then $H^i_{\mathfrak{a}}(M,N) = 0$ for all $i >$* $\operatorname{pd} M + \dim(M \otimes_R N)$.

(ii) *([**2**, Proposition 5.5]) Let* $t = \operatorname{grade}_N(M/\mathfrak{a}M) = \inf\{i : \operatorname{Ext}^i_R(M/\mathfrak{a}M, N) \neq 0\}$. *If $t < \infty$, then $H^i_{\mathfrak{a}}(M,N) = 0$ for all $i < t$ and $H^t_{\mathfrak{a}}(M,N) \neq 0$.*

(iii) *([**9**, Theorem 2.5]) $H^i_{\mathfrak{a}}(M,N) = 0$, for all $i > ara(\mathfrak{a}) + \operatorname{pd} M$, where $ara(\mathfrak{a})$ the arithmetic rank of the ideal $\mathfrak{a}$ is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$.*

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(iv) *([**8**, Theorem 2.3]) Let $(R,\mathfrak{m})$ be a local ring. Then depth$N$ is the least integer i such that $H^i_{\mathfrak{m}}(M,N)\neq 0$.*

As the main result of this paper, we generalize the Lichtenbum-Hartshorne vanishing theorem to generalized local cohomology in the certain case, where R is Gorenstein. Namely, we prove:

Theorem 1.2. *Let $\mathfrak{a}$ denote an ideal of a d-dimensional Gorenstein local ring $(R,\mathfrak{m})$. Let M and N be two finitely generated R-modules with $\operatorname{pd} M<\infty$. Then $H^d_{\mathfrak{a}}(M,N)$ is an Artinian R-module. Moreover the following are equivalent:*

(i) $H^d_{\mathfrak{a}}(M,N)=0$.

(ii) $\dim \hat{R}/\mathfrak{a}\hat{R}+\mathfrak{p}>0$ *for all* $\mathfrak{p}\in \operatorname{Ass}_{\hat{R}}\hat{M}\cap \operatorname{Supp}_{\hat{R}}\hat{N}$.

Having 1.1(i) in mind, one may think that $\operatorname{pd} M+\dim(M\otimes_R N)$ or $\operatorname{Max}\{\operatorname{pd} M,\dim N\}$ is the last integer i such that $H^i_{\mathfrak{a}}(M,N)\neq 0$. We provide examples which shows that this is not true even in the case R is local and $\mathfrak{a}$ is the maxima ideal of R.

All rings considered in this paper are assumed to be commutative Noetherian with identity. In our terminology we follow that of the text book [**4**].

2. Main result

Let $\mathfrak{a}$ denote an ideal of a ring R. The generalized local cohomology defined by

$$H^i_{\mathfrak{a}}(M,N)=\varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^nM,N)$$

for all R-modules M and N. Note that this is in fact a generalization of the usual local cohomology, because if $M=R$, then $H^i_{\mathfrak{a}}(R,N)=H^i_{\mathfrak{a}}(N)$. We use the following lemma several times in this paper. Its proof is easy and we lift it to the reader.

Lemma 2.1. *Let M and N be two R-modules. The following are hold.*

(i) *Let $0\longrightarrow N\longrightarrow E^{\cdot}$ be an injective resolution of N. Then*

$$H^i_{\mathfrak{a}}(M,N)\cong H^i((\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M,E^{\cdot})))\cong H^i(H^0_{\mathfrak{a}}(M,E^{\cdot})).$$

Moreover, if M is finitely generated, then $H^i_{\mathfrak{a}}(M,N)\cong H^i(\operatorname{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^{\cdot})))$.

(ii) *If $f:R\longrightarrow S$ is a flat ring homomorphism, then*

$$H^i_{\mathfrak{a}}(M,N)\otimes_R S\cong H^i_{\mathfrak{a}S}(M\otimes_R S,N\otimes_R S).$$

Theorem 2.2. *Let $\mathfrak{m}$ be a maximal ideal of R and M,N be two finitely generated R-modules. Then $H^i_{\mathfrak{m}}(M,N)$ is Artinian for all $i\geq 0$.*

Proof. Let $0 \longrightarrow N \longrightarrow E^{\cdot}$ be a minimal injective resolution of N. By 2.1 (i), it follows that $H^i_{\mathfrak{m}}(M,N) = H^i(\operatorname{Hom}_R(M,\Gamma_{\mathfrak{m}}(E^{\cdot})))$. Denote the k-th term of $E^{\cdot}$ by E^k. Because any subquotient of an Artinian R-module is also Artinian, it is enough to show that for each k, the module $\operatorname{Hom}_R(M,\Gamma_{\mathfrak{m}}(E^k))$ is Artinian. One can see easily that for any prime ideal $\mathfrak{p}$ of R,

$$\operatorname{Hom}_R(M,\Gamma_{\mathfrak{m}}(E(R/\mathfrak{p}))) = \begin{cases} 0 & ,\mathfrak{p} \neq \mathfrak{m} \\ \operatorname{Hom}_R(M,E(R/\mathfrak{p})) & ,\mathfrak{p} = \mathfrak{m}. \end{cases}$$

Thus $\operatorname{Hom}_R(M,\Gamma_{\mathfrak{m}}(E^k))$ is equal to the direct sum of $\mu^k(\mathfrak{m},N)$ copies of $\operatorname{Hom}_R(M,E(R/\mathfrak{m}))$. Here $\mu^k(\mathfrak{m},N) = \dim_{R/\mathfrak{m}}(\operatorname{Ext}^k_R(R/\mathfrak{m},N))_{\mathfrak{m}}$ is the k-th Bass number of N with respect to $\mathfrak{m}$, which is clearly finite. Next, it is easy to see that $\operatorname{Hom}_R(M,E(R/\mathfrak{m}))$ is Artinian and so $\operatorname{Hom}_R(M,\Gamma_{\mathfrak{m}}(E^k))$ is Artinian as required. □

The following is our technical tool throughout this paper.

Proposition 2.3. *Let $\mathfrak{a}$ be an ideal of a d-dimensional Gorenstein local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Then the following statements hold.*

(i) $H^i_{\mathfrak{m}}(M,R) = \begin{cases} \operatorname{Hom}_R(M,E(R/\mathfrak{m})), & i = d \\ 0 & ,i \neq d \end{cases}$. *In particular,* $H^d_m(R) = E(R/\mathfrak{m})$.

(ii) *Assume that* $\operatorname{pd} M < \infty$. *For any R-module N and any $i > d$,* $H^i_{\mathfrak{a}}(M,N) = 0$ *and so $H^d_{\mathfrak{a}}(M,.)$ is a right exact functor.*

Proof. (i) Let $0 \longrightarrow R \longrightarrow E^{\cdot}$ be a minimal injective resolution of R. Then $E^i \cong \oplus_{htp=i} E(R/\mathfrak{p})$, for each $0 \leq i \leq d$ and $E^i = 0$ for all $i > d$, and so

$$\operatorname{Hom}_R(M,\Gamma_{\mathfrak{m}}(E^i)) = \begin{cases} \operatorname{Hom}_R(M,E(R/\mathfrak{m})), & i = d \\ 0 & ,i \neq d \end{cases}.$$

Hence $H^i_{\mathfrak{m}}(M,R) = H^i(\operatorname{Hom}_R(M,\Gamma_{\mathfrak{m}}(E^{\cdot}))) = \begin{cases} \operatorname{Hom}_R(M,E(R/\mathfrak{m})), & i = d \\ 0 & ,i \neq d \end{cases}$. For $M = R$, we have $H^d_{\mathfrak{m}}(R) = H^d_{\mathfrak{m}}(R,R) = \operatorname{Hom}_R(R,E(R/\mathfrak{m})) = E(R/\mathfrak{m})$.

(ii) Because $\operatorname{injdim} R = d$, it follows that $H^i_{\mathfrak{a}}(M,R) = 0$ for all $i > d$. Now, by decreasing induction on $i > d$, we show that $H^i_{\mathfrak{a}}(M,N) = 0$ for all finitely generated R-modules N. This will complete the proof, because any R-module is the direct limit of a direct system consisting of finitely generated R-modules and the functor $H^i_{\mathfrak{a}}(M,\cdot)$, commutes with direct limits. The claim clearly holds for $i = \operatorname{pd} M + \dim R + 1$ by 1.1 (i). Assume that $i > d$ and that the claim holds for $i + 1$. Now we prove it for i. Let N be a finitely generated R-module. There is a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,$$

where F is a finitely generated free R-module. We deduce the long exact sequence

$$\ldots \longrightarrow H^i_{\mathfrak{a}}(M,K) \longrightarrow H^i_{\mathfrak{a}}(M,F) \longrightarrow H^i_{\mathfrak{a}}(M,N) \longrightarrow H^{i+1}_{\mathfrak{a}}(M,K) \longrightarrow \ldots.$$

By assumption $H^{i+1}_{\mathfrak{a}}(M,K) = 0$. On the other hand, we have $H^i_{\mathfrak{a}}(M,F) = 0$, because the functor $H^i_{\mathfrak{a}}(M,.)$ is additive and $H^i_{\mathfrak{a}}(M,R) = 0$. Thus $H^i_{\mathfrak{a}}(M,N) = 0$. □

The theory of attached prime ideals for Artinian modules is dual of the theory of primary decomposition for Noetherian modules. For an account of this theory, we refer the reader to [**4**, Chapter 7]. The following may be regarded as a generalization of Grothendieck non-vanishing theorem, in the case R is Gorenstein.

Lemma 2.4. *Let $(R,\mathfrak{m})$ be* $\mathfrak{a}$ *d-dimensional Gorenstein local ring and let M and N be two finitely generated R-modules with* $\operatorname{pd} M < \infty$. *Then* $\operatorname{Att}_R(H^d_{\mathfrak{m}}(M,N)) = \operatorname{Ass}_R M \cap \operatorname{Supp}_R N$. *In particular* $H^d_{\mathfrak{m}}(M,N) \neq 0$ *if and only if* $\operatorname{Ass}_R M \cap \operatorname{Supp}_R N \neq \emptyset$.

Proof. Since the functor $H^d_{\mathfrak{m}}(M,.)$ is right exact, it follows from 2.3 that $H^d_{\mathfrak{m}}(M,N) \cong N \otimes_R H^d_{\mathfrak{m}}(M,R) \cong N \otimes_R \operatorname{Hom}_R(M, E(R/\mathfrak{m}))$. Thus $H^d_{\mathfrak{m}}(M,N) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(N,M), E(R/\mathfrak{m}))$, by [**7**, Lemma 3.60]. For a finitely generated R-module C, it is known and one can check easily that $\operatorname{Att}_R(\operatorname{Hom}_R(C, E(R/\mathfrak{m}))) = \operatorname{Ass}_R C$. Therefore, by [**3**, p.267, Proposition 10],

$$\operatorname{Att}_R(H^d_m(M,N)) = \operatorname{Ass}_R(\operatorname{Hom}_R(N,M)) = \operatorname{Supp}_R N \cap \operatorname{Ass}_R M.$$

The last assertion follows immediately, because for an Artinian R-module A, $\operatorname{Att}_R A$ is empty if and only if A is zero. □

Let $\mathfrak{a}$ be an ideal of R. For an Artinian R-module A, we put $< \mathfrak{a} > A = \cap_{n\in\mathbb{N}} \mathfrak{a}^n A$.

Theorem 2.5. *Let the situation be as in 2.4. Let* $\mathfrak{a}$ *be an ideal of* R*. Then there is a natural isomorphism*

$$H^d_{\mathfrak{a}}(M,N) \cong H^d_{\mathfrak{m}}(M,N)/\sum_{n\in\mathbb{N}} <\mathfrak{m}> (0:_{H^d_{\mathfrak{m}}(M,N)} \mathfrak{a}^n).$$

In particular $H^d_{\mathfrak{a}}(M,N)$ *is Artinian.*

Proof. Denote $E(R/\mathfrak{m})$, by E. It follows from the local duality theorem [**4**, 11.2.5] that

$$\operatorname{Ext}^d_R(M/\mathfrak{a}^n M, R) \cong \operatorname{Hom}_R(H^0_{\mathfrak{m}}(M/\mathfrak{a}^n M), E).$$

Let $A = \operatorname{Hom}_R(M,E)$. For a fixed integer n, let $t(n) \in \mathbb{N}$ be such that $H^0_{\mathfrak{m}}(M/\mathfrak{a}^n M) = \operatorname{Hom}_R(R/\mathfrak{m}^{t(n)}, M/\mathfrak{a}^n M)$ and $<\mathfrak{m}> (0:_A \mathfrak{a}^n) = \mathfrak{m}^{t(n)}(0:_A \mathfrak{a}^n)$. Then by [**7**, Lemma 3.60], it follows that

$$\operatorname{Hom}_R(H^0_{\mathfrak{m}}(M/\mathfrak{a}^n M), E)) \cong R/\mathfrak{m}^{t(n)} \otimes_R \operatorname{Hom}_R(M/\mathfrak{a}^n M, E)$$

$$\cong R/\mathfrak{m}^{t(n)} \otimes_R \operatorname{Hom}_R(R/\mathfrak{a}^n, A) \cong (0:_A \mathfrak{a}^n)/ <\mathfrak{m}> (0:_A \mathfrak{a}^n).$$

One can check easily that $\varinjlim_n (0:_A \mathfrak{a}^n)/<\mathfrak{m}>(0:_A \mathfrak{a}^n) \cong A/\sum_{n\in\mathbb{N}} <\mathfrak{m}> (0:_A \mathfrak{a}^n)$. Hence

$$H^d_{\mathfrak{a}}(M,R) \cong A/\sum_{n\in\mathbb{N}} <\mathfrak{m}> (0:_A \mathfrak{a}^n).$$

It follows by [**5**, Lemma 3.1] that $H^d_{\mathfrak{a}}(M,R)\otimes_R N \cong (A\otimes_R N)/\sum_{n\in\mathbb{N}} <\mathfrak{m}> (0:_{A\otimes_R N} \mathfrak{a}^n)$. But 2.3 (ii) implies that $H^d_{\mathfrak{a}}(M,R)\otimes_R N \cong H^d_{\mathfrak{a}}(M,N)$ and $A\otimes_R N \cong H^d_{\mathfrak{m}}(M,N)$. Note that $A \cong H^d_{\mathfrak{m}}(M,R)$, by 2.3 (i). This finishes the proof. □

The following is an extension of the Lichetenbum-Hartshorne vanishing theorem for generalized local cohomology.

Corollary 2.6. *Let* $\mathfrak{a}$ *denote an ideal of a* d*-dimensional Gorenstein local ring* $(R,\mathfrak{m})$*. Let* M *and* N *be two finitely generated* R*-modules with* $\operatorname{pd} M < \infty$*. Then* $\operatorname{Att}_{\hat{R}}(H^d_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{M} \cap \operatorname{Supp}_{\hat{R}} \hat{N} : \dim \hat{R}/\mathfrak{a}\hat{R}+\mathfrak{p} = 0\}$*. Thus the following statements are equivalent.*

(i) $H^d_{\mathfrak{a}}(M,N) = 0$.
(ii) $\dim \hat{R}/\mathfrak{a}\hat{R}+\mathfrak{p} > 0$, *for all* $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{M} \cap \operatorname{Supp}_{\hat{R}} \hat{N}$.

Proof. Let $A = H^d_{\mathfrak{m}}(M,N)$ and $B = \sum_{n\in\mathbb{N}} < \mathfrak{m} > (0 :_{H^d_{\mathfrak{m}}(M,N)} \mathfrak{a}^n)$. Let $A = \sum_{n\in\mathbb{N}} A_i$ be a minimal secondary representation of the Artinian $\hat{R}$-module A, where A_i is $\mathfrak{p}_i$-secondary. We may assume that $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}_i > 0$ for $i = 1, \ldots, k$ and that $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}_i = 0$ for $i = k+1, \ldots, t$. Then by [**5**, Theorem 2.8], $\sum_{i=1}^k A_i$ is a minimal secondary representation of B. It is easy to see that $A/B = \sum_{i=k+1}^t (A_i + B)/B$ is a minimal secondary representation of A/B. Therefore, it follows from 2.4 and 2.5 that

$$\operatorname{Att}_{\hat{R}}(H^d_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{M} \cap \operatorname{Supp}_{\hat{R}} \hat{N} : \dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} = 0\}.$$

Note that because $H^d_{\mathfrak{m}}(M,N)$ is an Artinian R-module, we have

$$H^d_{\mathfrak{m}}(M,N) \cong H^d_{\mathfrak{m}}(M,N) \otimes_R \hat{R} \cong H^d_{\mathfrak{m}\hat{R}}(\hat{M},\hat{N}).\square$$

Question 2.7. Let $(R,\mathfrak{m})$ be a local ring and M and N two finitely generated R-modules with $\operatorname{pd} M < \infty$. Describe the last integer i such that $H^i_{\mathfrak{m}}(M,N) \neq 0$.

The following examples shows that the above mentioned integer is neither $\operatorname{pd} M + \dim(M \otimes_R N)$ nor $\operatorname{Max}\{\operatorname{pd} M, \dim N\}$.

Example 2.8. Suppose that$(R,\mathfrak{m})$ is a regular local ring of dimension d.

(i) Suppose that $d > 1$. Let $\mathfrak{p} \neq \mathfrak{m}$ be a non-zero prime ideal and x a non-zero element in $\mathfrak{p}$. Set $N = R/xR$ and $M = R/\mathfrak{p}$. Then by 2.4,

$$\operatorname{Att}_R(H^d_{\mathfrak{m}}(M,N)) = \operatorname{Ass}_R M \cap \operatorname{Supp}_R N = \{\mathfrak{p}\}.$$

Hence $H^d_{\mathfrak{m}}(M,N) \neq 0$. On the other hand, since by Auslander-Buchsbaum formula $\operatorname{pd} M = \operatorname{depth} R - \operatorname{depth} M$, we have $\operatorname{Max}\{\operatorname{pd} M, \dim N\} = d - 1$.

(ii) Suppose $M \neq 0$ is a non-Cohen-Macaulay finitely generated R-module. Then

$$\operatorname{depth} R = \operatorname{pd} M + \operatorname{depth} M < \operatorname{pd} M + \dim(M \otimes_R R) = l.$$

Thus $H^l_{\mathfrak{m}}(M,R) = 0$, by 2.3(i).

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