Minicourse

Numerical Semigroups and Applications

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Abstract. Let $\mathbb{N}$ be the set of nonnegative integers and let S be a subset of $\mathbb{N}$. We say that S is a monoid if $0 \in S$ and for all $a, b \in S, a + b \in S$. We say that S is a numericalsemigroup if the gcd of S is 1. If S is a numericl semigroup then $\mathbb{N} \setminus S$ is a finite set. Furthermore, S is finitely generated, i.e. there exist $a_1, \ldots, a_r \in S$ such that for all $s \in S, s = \sum_{i=1}^{r} \lambda_i a_i$ with $\lambda_1, \ldots, \lambda_r \in \mathbb{N}$. Here are some invariants associated with S:

1. The cardinality of $\mathbb{N} \setminus S$, denoted $g(S)$, and called the genus of S.
2. $F(S) = \max(\mathbb{N} \setminus S)$, called the Frobenius number of S.
3. The cardinality of a minimal system of generators of S, denoted $e(S)$, called the embedding dimension of S.
4. The cardinality of $\{s \in S : s < F(S)\}$, denoted $n(S)$.
5. The minimal element of $S^* = S \setminus \{0\}$, denoted $m(S)$, called the multiplicity of S.

These invariants are connected by equalities and inequalities. For example, $F(S)+1 = g(S)+n(S)$, and $e(S) \leq m(S)$. Also, their combinatorics leads to some conjectures. Here are two examples:

1. Conjecture 1 (Wilf conjecture): $F(S) + 1 \leq n(S)e(S)$.
2. Let $n(g)$ be the cardinality of the set of numerical semigroups with genus g.
   Conjecture 2: $n(g) \leq n(g + 1)$.

Numerical semigroups are also related to some problems in commutative algebra and algebraic geometry. Let $\mathbb{K}$ be a field and let $A = \mathbb{K}[t^{a_1}, \ldots, t^{a_r}]$ be the $\mathbb{K}$-algebra of polynomials in $t^{a_1}, \ldots, t^{a_r}$. The ring A is the coordinate ring of the curve parametrized by $t^{a_1}, \ldots, t^{a_r}$ and information about A can be derived from the properties of the numerical semigroup generated by $a_1, \ldots, a_n$. Thus in many cases the names of invariants in numerical semigroups are inherited from commutative algebra and algebraic geometry.

Numerical semigroups are also useful in the study of singularities of plane algebraic curves. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $f(x, y)$ be an element of $\mathbb{K}[\![x, y]\!]$. Given another element $g \in \mathbb{K}[\![x, y]\!]$, we define the local intersection multiplicity of f with g to be the rank of the $\mathbb{K}$-vector space $\mathbb{K}[\![x, y]\!]/(f, g)$. When g runs over the set of elements of $\mathbb{K}[\![x, y]\!] \setminus (f)$, these numbers define a semigroup. If furthermore f is irreducible, then this semigroup is a numerical semigroup. This leads to a classification of irreducible formal power series in terms of their associated numerical semigroups. This classification can be generalized to polynomials with one place at infinity. The arithmetic properties of numerical semigroups have been in this case the main tool in the proof of Abhyankar-Moh lemma which says that a coordinate has a unique embedding in the plane.

Reference. Assi, Abdallah; García-Sánchez, Pedro A. Numerical semigroups and applications. RSME Springer Series, 1. Springer, 2016.