Infinite dimensional tilting theory and its applications

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Overview

(Lecture I) Tilting theory and approximations of modules.

(Lecture II) Finite type of tilting classes, and the finitistic dimension conjectures.

(Lecture III) Classification of tilting classes over commutative noetherian rings.

References

- L.Angeleri Hügel, D.Pospíšil, J.Šťovíček, J.Trlifaj, *Tilting, cotilting, and spectra of commutative noetherian rings*, Trans. Amer. Math. Soc. 366(2014), 3487–3517.
- L.Angeleri Hügel, J.Trlifaj, *Tilting theory and the finitistic dimension conjectures*, Trans. Amer. Math. Soc. 354(2002), 4345–4358.
- S.Bazzoni, F.Mantese, A.Tonolo, *Derived equivalences induced by infinitely generated n-tilting modules*, Proc. Amer. Math. Soc. 139(2011), 4225–4234.
- R.Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, 2nd revised and extended ed., Vols. 1 and 2, GEM 41, W. de Gruyter, Berlin 2012.

(Large) Tilting Modules

Let R be a ring and $n < \omega$. A right R-module T is n–tilting provided

(T1) $\mathrm{pd}_R(T) \leq n$,

(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i$ and all κ, (i.e., T is a strong splitter).

(T3) There is a finite exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ with $T_i \in \mathrm{Add}\, T$, (i.e., R has a finite $\mathrm{Add}(T)$-coresolution).

The right tilting class induced by T is
$\mathcal{B}_T = T^\perp = \{M \in \text{Mod-}R \mid \mathrm{Ext}^i_R(T, M) = 0 \text{ for all } i \geq 1\}$.

A tilting module T is good if (T3) holds with $\mathrm{Add}\, T$ replaced by $\mathrm{add}\, T$.

The tilting modules T and T' are equivalent if $\mathcal{B}_T = \mathcal{B}_{T'}$.

Lemma: Each tilting module is equivalent to a good one.

The classic case

T is classical if $T \in \mathrm{mod}\text{–}R$ (i.e., T is strongly finitely presented).

Lemma: Each classical tilting module is good.

Theorem (Miyashita)

Let T be a classical n–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j \leq n, j \neq i} Ker(Ext_R^j(T, -)) \underset{Tor_S^i(-,T)}{\overset{Ext_R^i(T,-)}{\rightleftarrows}} \bigcap_{j \leq n, j \neq i} Ker(Tor_j^S(-, T))$$

where $S = End_R(T)$.

Classical tilting for commutative rings is trivial ...

Lemma

All classical tilting modules over a commutative ring are projective.

The key point:
If R is commutative, and T is strongly finitely presented of projective dimension $n \geq 1$, then $\mathrm{Ext}^n_R(T, T) \neq 0$.

The good case

Theorem (Bazzoni&co.)

Let R be a ring and T be a good n–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j \leq n, j \neq i} Ker(Ext^j_R(T,-)) \underset{Tor^i_S(-,T)}{\overset{Ext^i_R(T,-)}{\rightleftarrows}} \bigcap_{j \leq n, j \neq i} Ker(Tor^S_j(-,T)) \cap \mathcal{E}_\perp$$

where $S = End_R(T)$, $\mathcal{E}_\perp = \{X \in \mathcal{D}(S) \mid Hom_{\mathcal{D}(S)}(\mathcal{E}, X) = 0\}$, and $\mathcal{E}$ is the kernel of the total left derived functor $\mathbb{L}(- \otimes_S T)$.

Some examples in the commutative setting

Dedekind domains

Let R be a Dedekind domain with the quotient field Q, and P be a set of maximal ideals in R. Let R_P be the (unique) submodule of Q containing R such that $R_P/R \cong \bigoplus_{p \in P} E(R/p)$.

Then $T_P = R_P \oplus R_P/R$ is a tilting module, and

$$\mathcal{B}_{T_P} = \{M \in \text{Mod-}R \mid Mp = M \text{ for all } p \in P\}.$$

Iwanaga-Gorenstein rings

Let R be an Iwanaga-Gorenstein ring and

$$0 \to R \to I_o \to \cdots \to I_n \to 0$$

be a minimal injective coresolution of R.

Then $T = \bigoplus_{i \leq n} I_i$ is (obviously) a tilting module.

Tilting cotorsion pairs

Let T be a tilting module. The left tilting class induced by T is $\mathcal{A}_T = {}^{\perp}\mathcal{B}_T = \{A \in \text{Mod-}R \mid \text{Ext}^i_R(A,B) = 0 \text{ for all } B \in \mathcal{B}_T \text{ and } i \geq 1\}$.

Since $\mathcal{A}_T = {}^{\perp}\mathcal{B}_T$ and $\mathcal{B}_T = \mathcal{A}_T^{\perp}$, $(\mathcal{A}_T, \mathcal{B}_T)$ is a (hereditary) cotorsion pair, called the tilting cotorsion pair induced by T.

If T is n-tilting, then $\mathcal{A}_T \cap \mathcal{B}_T = \text{Add}\, T$, $\mathcal{A}_T$ coincides with the class of all modules possessing an $\text{Add}\, T$-coresolution of length $\leq n$, and $\mathcal{B}_T$ with the class of all modules possesing a (possibly infinite) $\text{Add}\, T$-resolution.

In particular, $\mathcal{B}_T$ is closed under arbitrary direct sums, and all modules in $\mathcal{A}_T$ have projective dimension $\leq n$.

Complete cotorsion pairs

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete provided that for each module M, there exists short exact sequences $0 \to M \xrightarrow{f} B \to A \to 0$ and $0 \to B' \to A' \xrightarrow{f'} M \to 0$ such that $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.

f is a special $\mathcal{B}$-preenvelope (= left $\mathcal{B}$-approximation) of M and f' a special $\mathcal{A}$-precover (= right $\mathcal{A}$-approximation) of M.

Theorem (Eklof.-T.)

Let $\mathcal{S}$ be a set of modules. Then the cotorsion pair $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{B} = \mathcal{S}^{\perp}$ is complete.
In particular, each tilting cotorsion pair is complete.

Characterizations of tilting classes and tilting cotorsion pairs

Lemma

Let $\mathcal{T}$ be a torsion class of modules. Then $\mathcal{T}$ is a right 1-tilting class, iff $\mathcal{T}$ is special preenveloping.

Theorem

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then TFAE:

(i) *$\mathfrak{C}$ is tilting.*

(ii) *$\mathcal{A}$ consists of modules of bounded projective dimension, $\mathcal{B}$ is closed under arbitrary direct sums (and $\mathfrak{C}$ is complete).*

Axiomatizability of tilting classes

Definition

Let $\mathcal{S}$ be any set consisting of strongly finitely presented modules of bounded projective dimension. Let $\mathcal{T} = \mathcal{S}^\perp$. Then $\mathcal{T}$ is of finite type.

Lemma

Each class of finite type is axiomatizable by a set of formulas of the first order theory of modules.

Theorem

Let $\mathcal{T}$ be a class of modules. Then TFAE:

(i) *$\mathcal{T}$ is a right tilting class.*

(ii) *$\mathcal{T}$ is of finite type.*

In particular, each right tilting class is axiomatizable.

Tilting classes and resolving subcategories of $\mathrm{mod}\text{–}R$

Definition

A subclass $\mathcal{S}$ of $\mathrm{mod}\text{–}R$ is called resolving provided that it contains all finitely generated projective modules, it is closed under extensions and direct summands, and $A \in \mathcal{S}$ whenever there is an exact sequence $0 \to A \to B \to C \to 0$ with $B, C \in \mathcal{S}$.

Theorem

There is a 1-1 correspondence between

1. *right tilting classes $\mathcal{T}$ in $\mathrm{Mod}\text{–}R$, and*
2. *resolving subclasses $\mathcal{S}$ of $\mathrm{mod}\text{–}R$ consisting of modules of bounded projective dimension.*

It is given by the inverse assignments

$$\mathcal{T} \mapsto \mathcal{A}_{\mathcal{T}} \cap \mathrm{mod}\text{–}R \quad \text{and} \quad \mathcal{S} \mapsto \mathcal{S}^{\perp}.$$

The finitistic dimensions

Notation: $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}_n$ the class of all modules of finite projective dimension, and

$\mathcal{P}^{<\infty}$ the class of all finitely generated modules in $\mathcal{P}$.

$\mathrm{Fdim}(R)$ denotes the big finitistic dimension of R
(= the supremum of projective dimensions of the modules in $\mathcal{P}$

$\mathrm{fdim}(R)$ denotes the little finitistic dimension of R
(= the supremum of projective dimensions of the modules in $\mathcal{P}^{<\infty}$).

Lemma

$fdim(R) \leq Fdim(R) \leq gl.dim(R)$.
If $gl.dim(R) < \infty$, then all these dimensions coincide.

The finitistic dimension conjectures

Let R be a ring.

1. (1st FDC) Does $\mathrm{Fdim}(R) = \mathrm{fdim}(R)$?
2. (2nd FDC) Is $\mathrm{fdim}(R) < \infty$?

The commutative noetherian case

Theorem (Bass, Raynaud-Gruson)

Fdim(R) equals the Krull dimension of R.

Theorem

(i) *(Nagata) There exists rings with* $\mathrm{fdim}(R) = \mathrm{Fdim}(R) = \infty$.
So (2nd FDC) fails in general.

(ii) *(Auslander-Buchsbaum) If R is local, then* $\mathrm{fdim}(R) = \mathrm{depth}(R)$.
In particular, (1st FDC) holds, iff R is Cohen-Macaulay.

The artin algebra case

Theorem (Huisgen-Zimmermann, Smalo)

For each $n > 1$, there exist finite dimensional algebras with fdim$(R) = 1$ and Fdim$(R) = n$. So (1st FDC) fails in general.

However, (2nd FDC) is still an open problem ...

Relations to tilting approximations

Theorem (Angeleri-T.)

Let R be a right noetherian ring.
Then fdim$(R) < \infty$, if and only if $(\mathcal{P}^{<\infty})^\perp$ is a right tilting class. That is, (2nd FDC) holds, iff there is a tilting module T_f with $\mathcal{B}_{T_f} = (\mathcal{P}^{<\infty})^\perp$.
In this case, proj.dim$(T_f) =$ fdim(R), and T_f is unique up to equivalence.

Warning: T_f is not finitely generated in general.

A positive case for artin algebras

Theorem (Angeleri-T.)

Let R be an artin algebra. Then TFAE:

(i) *The tilting module T_f can be chosen finitely generated.*

(ii) *The category $\mathcal{P}^{<\infty}$ is contravariantly finite (= precovering) in $\mathrm{mod}\text{–}R$.*

In this case $\mathcal{P} = \mathcal{A}_{T_f}$, whence $Fdim(R) = fdim(R) = proj.dim(T_f) < \infty$.

The case of Iwanaga-Gorenstein rings

Definition

A ring R is Iwanaga-Gorenstein provided that R is left and right noetherian and of finite left and right injective dimension.

Lemma

If R is Iwanaga-Gorenstein, then the left and right injective dimensions of R coincide (with some $n < \omega$). Moreover, $\mathcal{P} = \mathcal{P}_n = \mathcal{I} = \mathcal{I}_n$.

Theorem (Angeleri-T.)

Let R be an Iwanaga-Gorenstein ring. Then both finitistic dimension conjectures hold for R.
Moreover, the tilting module T_f can be chosen as $T_f = \bigoplus I_i$, where the I_is are the terms of the minimal injective coresolution of R.

The dual setting

Definition

Let R be a ring and $n < \omega$. A left R–module C is n–cotilting provided

(C1) $\mathrm{id}_R(C) \leq n$.

(C2) $\mathrm{Ext}^i_R(C^\kappa, C) = 0$ for all $1 \leq i$ and all cardinals κ.

(C3) There is an injective cogenerator W and a long exact sequence $0 \to C_r \to C_{n-1} \to \cdots \to C_0 \to W \to 0$, with $C_i \in \mathrm{Prod}\, C$.

The class ${}^\perp C = \{M \in R\text{–}\mathrm{Mod} \mid \mathrm{Ext}^i_R(M, C) = 0 \text{ for all } i \geq 1\}$ is the cotilting class induced by C.

The cotilting modules C and C' are equivalent if ${}^\perp C = {}^\perp C'$.

Duality: formal versus explicit

The notions of a cotilting and tilting module are formally dual, but there is also an explicit duality:

Let R be a ring, $n \geq 0$, and T be an n–tilting right R–module. Then the dual module $C = T^* = \operatorname{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is an n–cotilting left R–module.

The tilting right R–modules T and T' are equivalent, iff the dual modules T^* and $(T')^*$ are equivalent cotilting left R–modules.

Moreover, if $\mathcal{S}$ is a set consisting of strongly finitely presented modules of projective dimension $\leq n$ such that $T^\perp = \mathcal{S}^\perp$ is the right tilting class induced by T, then

$$^\perp T^* = \mathcal{S}^\intercal = \{N \in R\text{–Mod} \mid \operatorname{Tor}_i^R(S, N) = 0 \text{ for all } i \geq 1 \text{ and } S \in \mathcal{S}\}$$

is the cotilting class induced by T^*.

Cofinite type

The cotilting modules and classes of the form T^* and ${}^{\perp}T^*$, respectively, are called of cofinite type.
Warning: Not all cotilting modules and classes are of cofinite type in general.

The map $T \mapsto T^*$ induces a bijection between equivalence classes of tilting modules on the one hand, and equivalence classes of cotilting modules of cofinite type on the other hand.

Similarly, the maps

$$\mathcal{T} \mapsto ({}^{\perp}\mathcal{T} \cap \text{mod-}R)^{\intercal}$$

and

$$\mathcal{C} \mapsto ({}^{\intercal}\mathcal{C} \cap \text{mod-}R)^{\perp}$$

provide for a 1–1 correspondence between right tilting classes, and cotilting classes of cofinite type.

The role of associated primes in the noetherian setting

A subset $P \subseteq \mathrm{Spec}(R)$ is closed under generalization provided that $(P, \subseteq)$ is a lower subset in $(\mathrm{Spec}(R), \subseteq)$.

Theorem (The structure of 1–cotilting classes)

Let R be a commutative noetherian ring. Then there is a 1–1 correspondence between

1. *the 1–cotilting classes $\mathcal{C}$ in Mod-R, and*
2. *the subsets P of $\mathrm{Spec}(R)$ containing $\mathrm{Ass}(R)$ and closed under generalization.*

It is given by the inverse assignments $\mathcal{C} \mapsto \mathrm{Ass}(\mathcal{C})$ and $P \mapsto \{M \in \text{Mod-}R \mid \mathrm{Ass}(M) \subseteq P\}$.

The Auslander–Bridger transpose

Let $C \in \mathrm{mod}\text{–}R$ and $P_1 \xrightarrow{f} P_0 \to C \to 0$ be a projective presentation of C. The transpose of C, denoted by $\mathrm{Tr}(C)$, is the cokernel of f^+, where $(-)^+ = \mathrm{Hom}_R(-, R)$.
That is, we have an exact sequence

$$P_0^+ \xrightarrow{f^+} P_1^+ \to \mathrm{Tr}(C) \to 0.$$

$\mathrm{Tr}(C)$ is uniquely determined up to adding or splitting off a projective summand.

Lemma

Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be such that $\mathrm{Ass}(R) \cap V(\mathfrak{p}) = \emptyset$. Then

(i) $pd_R(\mathrm{Tr}(R/\mathfrak{p})) \leq 1$;

(ii) *$Hom_R(R/\mathfrak{p}, -)$ and $Tor_1^R(\mathrm{Tr}(R/\mathfrak{p}), -)$ are isomorphic functors.*

A classification of 1–tilting classes

Corollary

Let R be a commutative noetherian ring. Then all 1–cotilting classes are of cofinite type, so there is a bijection between right 1–tilting classes and the subsets P of $\mathrm{Spec}(R)$ containing $\mathrm{Ass}(R)$ and closed under generalization. For such P, the corresponding 1–tilting class is

$$\mathcal{T} = \bigcap_{\mathfrak{q} \in \mathrm{Spec}(R) \setminus P} \mathrm{Tr}(R/\mathfrak{q})^{\perp}.$$

Characteristic sequences

Definition

Let R be a commutative noetherian ring. A sequence $\mathcal{P} = (P_0, \dots, P_{n-1})$ of subsets of $\mathrm{Spec}(R)$ is called characteristic provided that

1. P_i is closed under generalization for all $i < n$,
2. $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{n-1}$, and
3. $\mathrm{Ass}(\Omega^{-i}(R)) \subseteq P_i$ for all $i < n$.

For each characteristic sequence $\mathcal{P}$, we define the class of modules

$$\mathcal{C}_{\mathcal{P}} = \{M \in \text{Mod-}R \mid \mathrm{Ass}(\Omega^{-i}(M)) \subseteq P_i \text{ for all } i < n\}$$

A classification of n–cotilting classes

Theorem

Let R be a commutative noetherian ring, $n \geq 1$, and $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a characteristic sequence. Then $\mathcal{C}_{\mathcal{P}}$ is an n–cotilting class, and the assignments

$$\mathcal{C} \mapsto (\mathrm{Ass}(\mathcal{C}_0), \ldots, \mathrm{Ass}(\mathcal{C}_{n-1}))$$

and

$$\mathcal{P} = (P_0, \ldots, P_{n-1}) \mapsto \mathcal{C}_{\mathcal{P}}$$

are inverse bijections.

Lemma

Let R be a ring and C be an n–cotilting module with the induced class $\mathcal{C}$.
For each $i \leq n$, let $\mathcal{C}_i = {}^{\perp}\Omega^{-i}(C)$.
Then $\mathcal{C}_i$ is an $(n-i)$–cotilting class.

The transpose revisited

Lemma

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ *and* $n \geq 1$ *such that* $\operatorname{Ass}(\Omega^{-i}(R)) \cap V(\mathfrak{p}) = \emptyset$ *for each* $i < n$. *Then*

(i) $pd_R(\operatorname{Tr}(R/\mathfrak{p})) \leq n$.

(ii) $Ext_R^{n-1}(R/\mathfrak{p}, -)$ *and* $Tor_1^R(\operatorname{Tr}(\Omega^{(n-1)}(R/\mathfrak{p})), -)$ *are isomorphic functors.*

(iii) $Ext_R^1(\operatorname{Tr}(\Omega^{(n-1)}(R/\mathfrak{p})), -)$ *and* $Tor_{n-1}^R(R/\mathfrak{p}, -)$ *are isomorphic functors.*

Full classification over commutative noetherian rings

Theorem

Let R be a commutative noetherian ring and $n \geq 1$. Then there are bijections between:

1. *the characteristic sequences $\mathcal{P}$ in $\mathrm{Spec}(R)$,*
2. *right n–tilting classes $\mathcal{T}$,*
3. *n–cotilting classes $\mathcal{C}$.*

A characteristic sequence $\mathcal{P}$ corresponds to the right n–tilting class

$$\begin{aligned}\mathcal{T} =& \{M \in \mathrm{Mod}\text{–}R \mid Tor_i^R(R/\mathfrak{p}, M) = 0 \,\forall i < n \,\forall \mathfrak{p} \notin P_i\} = \\ & \{M \in \mathrm{Mod}\text{–}R \mid Ext_R^1(\mathrm{Tr}(\Omega^{(i)}(R/\mathfrak{p})), M) = 0 \,\forall i < n \,\forall \mathfrak{p} \notin P_i\},\end{aligned}$$

and the n–cotilting class

$$\begin{aligned}\mathcal{C} =& \{M \in \mathrm{Mod}\text{–}R \mid Ext_R^i(R/\mathfrak{p}, M) = 0 \,\forall i < n \,\forall \mathfrak{p} \notin P_i\} = \\ & \{M \in \mathrm{Mod}\text{–}R \mid Tor_1^R(\mathrm{Tr}(\Omega^i(R/\mathfrak{p})), M) = 0 \,\forall i < n \,\forall \mathfrak{p} \notin P_i\}.\end{aligned}$$