MANIFOLDS, MAPPINGS AND GROUPS

0.1 Differentiable Manifolds, Fibre Bundles and Orientation

0.1.1 Manifolds, Vector Bundles

By a *manifold* M of dimension m we mean a Hausdorf separable topological space M admitting of a covering $\mathcal{U} = \{U, V, \cdots\}$ such that each $U, V, \cdots$ is homeomorphic to an open subset of $\mathbb{R}^m$. For $U \in \mathcal{U}$ such a homeomorphism will be denoted by φ_U. If $U, V \in \mathcal{U}$ and $U \cap V \neq \emptyset$, then $\varphi_{UV} = \varphi_U \varphi_V^{-1}$ is a homeomorphism of the open subset $\varphi_V(U \cap V) \subseteq \mathbb{R}^m$ onto $\varphi_U(U \cap V) \subseteq \mathbb{R}^m$. A pair (U, φ_U) (or simply U) will be called a *coordinate chart* or *coordinate system*. The functions φ_{UV} are called the *transition functions* for M. Let $(x_1^U, \cdots, x_m^U)$ denote the coordinates of a point in $\varphi_U(U)$, then the mapping φ_{UV} expresses x_i^U's in terms of x_i^V's, i.e.,

$$x_i^U = x_i^U(x_1^V, \cdots, x_m^V).$$

We also refer to $x = (x_j^U)$ or (x_j) as coordinates on M. If the transition functions φ_{UV} are $k \leq \infty$ times continuously differentiable (analytic), we say that M is of class C^k (analytic). A *smooth* manifold is one of class C^∞. Unless stated to the contrary, a manifold M is of class C^∞, and if no differentiability assumption is made on the transition functions, M is called a *topological manifold*. While most important examples of differentiable manifolds are C^∞ or even analytic, it is sometimes necesary to allow the greater generality of no differentiability assumption of the transition functions since a given topological manifold may have none or many differentiable structures. This subtle issue is not discussed here. Let $f : M \to N$ be a continuous map. Smoothness of a map is a local requirement and therefore smoothness of f means smoothness of $\varphi_N \cdot f \cdot \varphi_M^{-1}$ for all coordinate systems (U_M, φ_M) and (U_N, φ_N) for M and N.

If the covering $\mathcal{U}$ is given as an indexed family, e.g. $\{U_\alpha\}$, then the mappings φ_{U_α}, $\varphi_{U_\alpha U_\beta}$ etc. will be denoted by $\varphi_\alpha, \varphi_{\alpha\beta}$ etc. Similarly, we write x_i^α etc. for $x_i^{U_\alpha}$ etc. Unless stated to the contrary, all coverings are locally finite, i.e., every compact set intersects only finitely many open sets of the covering. By a *partition of unity* subordinate to a covering $\mathcal{U} = \{U_\alpha\}$ we mean C^∞ functions ϕ_α such that

1. $\text{supp}(\phi_\alpha) \subset U_\alpha$;

2. $0 \leq \phi_\alpha \leq 1$;

3. $\sum_\alpha \phi_\alpha \equiv 1$.

We omit the proof of the existence of partitions of unity which can be found in almost every elementary standard text on differentiable manifolds or differential topology. The significance of partition of unity is due to the fact that in some circumstances it allows one to patch together local data to obtain global ones on a manifold. This point becomes clear as we develop the theory.

The notion of a submanifold requires some elaboration. Let M and N be manifolds and $f : M \rightarrow N$ be a mapping such that $Df(x)$ is injective for all $x \in M$. Such a mapping is called an *immersion* and M or $f(M)$ is an *immersed submanifold* of N. In this case the topology on M may not be induced from that of N. The following example clarifies this point and has other applications:

Example 0.1.1.1 Let $M = \mathbb{R}$ $N = T^2 = \mathbb{R}^2/\mathbf{Z}^2$ be the two dimensional torus. We can represent N as a square with vertices at $(0,0),(1,0),(1,1)$ and $(0,1)$ and the points on the boundary whose coordinates differ by an integer identified. Alternatively, T^2 is the subset $\{(e^{2\pi i t_1}, e^{2\pi i t_2})$ of $\mathbb{C}^2$ as t_1, t_2 vary over $[0,1]$. Let $f : M \rightarrow N$ be given by $f(t) = (t, \gamma t)$ mod $\mathbf{Z}^2$, where γ is an irrational number. From elementary number theory or Fourier analysis we know that $\{\gamma m | m \in \mathbf{Z}\}$ mod 1 is dense in $[0,1]$. This implies that in the representation of T^2 as a square in the plane (with proper identifications of sides), the image of f intersects the interval $[0,1]$ on the vertical axis in a dense set of points. Since $f(M)$ consists of parallel line segments in the square, $f(M)$ is dense in N. Therefore the topology induced on $f(M)$ from N is not identical with the original topology of M. This example works in any dimension. For instance, consider the torus $T^m = S^1 \times \cdots \times S^1$. where S^1 is identified with complex numbers of norm 1 Let $x_1, \cdots, x_m$ be real numbers such that $1, x_1, \cdots, x_m$ are linearly independent over the rational numbers. Consider the mapping $f : \mathbb{R} \rightarrow T^m$ defined by

$$f(t) = (e^{itx_1}, \cdots, e^{itx_m}).$$

Then f is an injective analytic immersion. It is a classical result due to Kronecker that Imf is dense in T^m and therefore not closed (for a detailed proof see, e.g., [Ho]). The induced topology on Imf is distinct from that of $\mathbb{R}$. ♠

An immersion is not necessarily injective and the image can have self-intersections. For example, figure ∞ is the image of an immersion of the circle S^1 into the plane. If there is a covering $\mathcal{U} = \{U_\alpha\}$ of N such that $U_\alpha \cap f(M)$ is defined as the zero set of a smooth function

$F : U_\alpha \cap f(M) \to \mathbb{R}^{n-m}$ with $DF(x)$ of rank $n - m$ for all $x \in U_\alpha \cap f(M)$, then f is called an *embedding* and $f(M)$ or simply M a *submanifold* of N. It is not difficult to show that an injective immersion of a compact manifold M into a manifold N is an embedding.

The concept of a vector bundle on a manifold M (or more generally, on a topological space X) plays a fundamental role in geometry, topology and physics. A *vector bundle* with *fibre* F (= a vector space over $\mathbb{R}$ or $\mathbb{C}$) consists of topological spaces X (called *base*) and E (called *total space*) and a continuous map $\pi : E \to X$ such that

1. For every $x \in X$, $E_x = \pi^{-1}(x)$ (called *fibre over* x) is a vector space isomorphic to F;
2. There is a covering $\mathcal{U} = \{U_\alpha\}$ of X and homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \simeq U_\alpha \times F$ of the form $\varphi_\alpha(y) = (\pi(y), \varphi_{\alpha 1}(y))$ with $\varphi_{\alpha 1}$ linear on each fibre.

We normally denote a vector bundle by (E, π, X), $E \xrightarrow{\pi} X$, or $E \to X$. The *rank* of a vector bundle (E, π, X) is the dimension of a fibre. We may also refer to a vector bundle of rank k as a *k-plane* bundle. Should it be necessary to specify the underlying field of a fibre, we will refer to the vector bundle as real or complex.

The bundle (E, π, X) is *trivial* if $E \simeq X \times F$ with π projection on the first factor. Two bundles (E, π, X) and (E, π', X) are isomorphic if there is a homeomorphism $\theta : E' \to E$ such that $\pi\theta = \pi'$ (hence θ preserves fibres) with the restriction of θ to each fibre a linear isomorphism. By a *mapping of a vector bundle* (E, π, X) *to a vector bundle* (E', π', Y) we mean a pair of continuous (or smooth depending on the context) maps $f : X \to Y$ and $f' : E \to E'$ such that

1. $f\pi = \pi' f'$;
2. The restriction of f' to every fibre is a linear map of vector spaces.

If $X = Y$, $f = \mathrm{id.}$ and the restriction of f' to each fibre is injective, we say (E, π, X) or $(f'(E), \pi', X)$ is a *sub-bundle* of (E', π', X).

For a vector space F of dimension n over a field K, we define $GL(n, K)$ or $GL(F)$ (called the *general linear group of degree* n) as the group of $n \times n$ matrices with entries from K and determinant $\neq 0$, or invertible linear transformations of F. Similarly, $SL(n, K)$ (called the *special linear group of degree* n) is the subgroup of $GL(n, K)$ consisting of matrices of determinant 1. For $y \in U_\alpha \cap U_\beta$, we define $\rho_{\alpha\beta}$ by $\rho_{\alpha\beta}(\varphi_{\beta 1}(y)) = \varphi_{\alpha 1}(y)$ regarded as a linear operator on F. Thus we have continuous maps $\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(F)$ satisfying

$$\rho_{\alpha\gamma}\rho_{\gamma\beta}\rho_{\beta\alpha} = \mathrm{id.} \qquad (0.1.1.1)$$

The $GL(F)$-valued functions $\rho_{\alpha\beta}$ are called the *transition functions for the bundle* (E, π, X). In case the base is a manifold M, we generally assume smoothness of the transition functions of the bundle in consideration. It is not difficult to see that given a collection of $GL(F)$-valued functions $\{\rho_{\alpha\beta}\}$defined on non-empty intersections $U_\alpha \cap U_\beta$ for a covering $\mathcal{U} = \{U_\alpha\}$ of X, and such that $\rho_{\alpha\beta} = \rho_{\beta\alpha}^{-1}$ and (0.1.1.1) is satisfied, then there is a vector bundle with the transition functions $\{\rho_{\alpha\beta}\}$. In fact consider the disjoint union $\cup(U_\alpha \times F)$, and identify the points $(x, v) \in U_\alpha \times F$ and $(x, v') \in U_\beta \times F$ if $v' = \rho_{\beta\alpha}(x)(v)$. It is clear that the set of transition functions for a vector bundle is not unique. In fact by a change of bases for the fibres we obtain new transition functions $\{\rho'_{\alpha\beta}\}$ related to $\{\rho_{\alpha\beta}\}$ by

$$\rho'_{\alpha\beta}(x) = \psi_\alpha(x)\rho_{\alpha\beta}(x)\psi_\beta^{-1}(x), \tag{0.1.1.2}$$

where $\psi_\alpha : U_\alpha \to GL(F)$ describes the change of bases. It is straightforward to show that vector bundles defined by two sets transition $\{\rho_{\alpha\beta}\}$ and $\{\rho'_{\alpha\beta}\}$ are isomorphic if and only if they are related by (0.1.1.2) for some functions $\{\psi_\alpha\}$.

Given a vector bundle (E, π, X) one can construct other vector bundles through tensor operations. To be more precise, let (E, π, X) be defined by the transition functions $\rho_{\alpha\beta}$. Then the dual bundle $(E^\star, \pi, X)$ is defined by the transition functions $\rho'^{-1}_{\alpha\beta}$ where the superscript $'$ denote the transpose of the matrix. Now any linear transformation $T : F \to F$ induces linear transformations $\otimes^p T'^{-1} \otimes^q T$ on the vector spaces $\otimes^p F^\star \otimes^q F$ (tensor product of p copies of $F^\star$ and q copies of F). We denote by $\otimes^p E^\star \otimes^q E \to X$ the vector bundle with the transition function $\otimes\rho'^{-1}_{\alpha\beta} \otimes \rho_{\alpha\beta}$. The restriction of the transition functions to any $GL(F)$-invariant subspace of $\otimes^p F^\star \otimes^q F$ defines a sub-bundle of $\otimes^p E^\star \otimes^q E \to X$. For example, the p^{th} exterior power $\wedge^p F^\star$ of $F^\star$ and q^{th} symmetric power $\odot^q F$ are invariant under $GL(F)$, and the corresponding vector bundles are denoted by $\wedge^p E^\star \to M$ and $\odot^q E \to M$ respectively.

By a *section* of a vector bundle (E, π, X) we mean a mapping $s : X \to E$ such that $\pi s(x) = x$. In terms of transition functions this means we have mappings $s^\alpha : U_\alpha \to F$ for $U_\alpha \in \mathcal{U}$ such that $\rho_{\beta\alpha}(x)s^\alpha(x) = s^\beta(x)$. Note that the triviality of a vector bundle of rank k is equivalent to the existence of k sections $s_1, \cdots, s_k$ which are linearly independent at every point $x \in X$. A *frame* for a vector bundle (E, π, X) is a choice of a basis for a fibre $\pi^{-1}(x)$ where $x \in X$. By a *local frame* we mean a continuous (or smooth depending on the context) choice of bases for fibres $\pi^{-1}(x)$ where x ranges over an open subset of X. The set of frames for (E, π, X) is the set of all possible bases for all fibres $\pi^{-1}(x)$ as x ranges over X.

Example 0.1.1.2 The *cotangent bundle* $\mathcal{T}^\star M$ of a manifold M is defined by the transition functions $\rho_{\alpha\beta}(x) \in GL(m, \mathbb{R})$ given by

$$\rho_{\beta\alpha,ij}(x) = \frac{\partial x_j^\alpha}{\partial x_i^\beta}.$$

This is motivated by the fact that if the 1-form ω in $U \subseteq \mathbb{R}^m$ has the expression $\omega = \sum_j \omega_j^\alpha dx_j^\alpha$ relative to the coordinate system (x_j^α), then its expression relative to (x_i^β) is given by

$$\omega = \sum_i (\sum_j \frac{\partial x_j^\alpha}{\partial x_i^\beta}\omega_j^\alpha) dx_i^\beta.$$

The dual to the cotangent bundle of M is its *tangent bundle* $\mathcal{T}M \rightarrow M$, which is defined by the transition functions

$$r_{\beta\alpha,ij} = \frac{\partial x_i^\beta}{\partial x_j^\alpha},$$

i.e., the transition functions for $\mathcal{T}M \rightarrow M$ are the transpose inverse of the transition functions for the cotangent bundle. Since by the chain rule

$$\sum_j \xi_j^\alpha \frac{\partial}{\partial x_j^\alpha} = \sum_i (\sum_j \frac{\partial x_i^\beta}{\partial x_j^\alpha}\xi_j^\alpha)\frac{\partial}{\partial x_i^\beta},$$

we may regard the quantities $\frac{\partial}{\partial x_i^\alpha}$ as duals to the differentials dx_i^α, i.e., $dx_i^\alpha(\frac{\partial}{\partial x_j^\alpha}) = \delta_{ij}$. Historically, the use of the differentials dx_i^α preceded that of the differentiation operators $\frac{\partial}{\partial x_i^\alpha}$ in differential geometry[1]. Naturally 1-forms are sections of the cotangent bundle $\mathcal{T}^\star M$ and sections of the tangent bundle are called *vector fields*. The fibres of the tangent and cotangent bundles $x \in M$ are usually denoted as $\mathcal{T}_x M$ and $\mathcal{T}_x^\star M$. A manifold M whose (co)tangent bundle is trivial is called *parallelizable*. As indicated above one also considers the tensors powers $\otimes^p \mathcal{T}^\star M \otimes^q \mathcal{T}M$. Sections of $\otimes^p \mathcal{T}^\star M$ (resp. $\otimes^p \mathcal{T}M$) are often called *cotravariant* (resp. *covariant*) tensors in the mathematics literature while the opposite convention is prevalent in physics. Sections of $\otimes^p \mathcal{T}^\star M \otimes^q \mathcal{T}M$ are called *mixed* tensors of type (p, q). ♠

Given a 1-form ω on N and a map $f : M \rightarrow N$, the *pull-back* $f^\star(\omega)$ is a 1-form on M. For a local expression $\omega = \sum \omega_i(y_1, \cdots, y_n)dy_i$ and the representation of the map f as $y_i = y_i(x_1, \cdots, x_m)$, $f^\star(\omega)$ is obtained by substituting the expressions for y_i's and $dy_i = \sum \frac{\partial y_i}{\partial x_j}dx_j$ in ω. That this is well-defined is a simple exercise. The mapping $f^\star$ extends to sections of the tensor powers $\otimes^p \mathcal{T}^\star N$, and in particular if τ is an exterior p-form or

[1]Throughout this work we emphasize the use of forms rather than vector fields in accordance with the historical development. While it is important to keep in mind both points of view, this author considers contravariant vectors, rather than covariant ones, the essential technical tool in understanding geometric structures. Equation (0.1.1.3) which has no reasonable analogue for vector fields, is evidence of the greater technical and geometric significance of forms.

a symmetric p-tensor on N, then $f^\star(\tau)$ is a p-form or a symmetric p-tensor on M. An important property of $f^\star(\omega)$, for an exterior p-form ω, is

$$f^\star(d\omega) = df^\star(\omega); \tag{0.1.1.3}$$

sometimes called *functoriality* of d. Note that the linear mapping $f^\star$ has the property that $f^\star(\omega)(x)$ depends only on the value of ω at $f(x)$ and the derivatives of f at x. Therefore $f^\star$ defines a linear map $f^\star : T^\star_{f(x)}N \to T^\star_x M$. Dually, there is a linear map $f_\star : T_x M \to T_{f(x)}N$ which, in local coordinates, is simply the derivative of the map f regarded as the linear map $Df(x) : \mathbb{R}^m \to \mathbb{R}^n$. It is again a simple matter to verify that $f_\star$ is well-defined on a manifold.

Example 0.1.1.3 Let $F : U \to \mathbb{R}^n$, where $U \subseteq \mathbb{R}^{m+n}$, be a smooth map, and set $M = Z_F = \{x \in U | F(x) = 0\}$. Assume that the derivative $DF(x) : \mathbb{R}^{m+n} \to \mathbb{R}^n$ has rank n for all $x \in M$. For $x \in M$, consider a splitting $\mathbb{R}^{m+n} \simeq \mathbb{R}^m \times \mathbb{R}^n$ with $\ker(DF)(x) = \mathbb{R}^m$. By the implicit function theorem there is open set $V \subseteq \mathbb{R}^m$, and a neighborhood W of x in $\mathbb{R}^{m+n}$ such that $M \cap W$ is parametrized by a smooth map $\phi : V \to M \cap W$. The inverse of ϕ defines a coordinate system in a neighborhood of $x \in M$. Let $y = (y_1, \cdots, y_m)$ denote the standard coordinates in $\mathbb{R}^m$. The derivative of ϕ is

$$D\phi(y) = -D_2F^{-1} \cdot D_1F(\phi(y)), \tag{0.1.1.4}$$

where D_1 and D_2 denote the partial derivatives relative to $\mathbb{R}^m$ and $\mathbb{R}^n$ in the above splitting of $\mathbb{R}^{m+n}$. The tangent space T_xM to M at $x = \phi(y)$, with the zero vector translated to the origin in $\mathbb{R}^{m+n}$, is the image of $D\phi(y) = \ker DF(x)$. Let $\psi : V \to V$ be a diffeomorphism given symbolically by $y' = (y'_1, \cdots, y'_m) \to (y_1(y'), \cdots, y_m(y'))$. Let $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}^m$, then $D\phi(y)(\eta)$ and $D(\phi\psi)(\psi^{-1}(y))(D\psi^{-1}(y)(\eta))$ are the same tangent vector in T_xM. Therefore the components of a tangent vector with respect to the coordinates $\{y_j\}$ are obtained from those relative to $\{y'_j\}$ via multiplication on the left by the matrix

$$r'_{ij} = \frac{\partial y'_i}{\partial y_j}.$$

A comparison of r'_{ij} and r_{ij} of example 1.1 shows that the two descriptions of the tangent bundle have identical transition functions and are therefore the same. Let ϵ_i denote the column vector with j^{th} component δ_{ij}. (Unless stated to the contrary, δ_{ij} denotes the *Kronecker delta* which is 1 if $i = j$ and zero otherwise.) Then the above analysis also shows that $D\phi(y)(\epsilon_i)$ can be identified with the differentiation operator $\frac{\partial}{\partial y_i}$. ♠

Example 0.1.1.4 The tangent bundle of $\mathbb{R}^N$ is the trivial bundle of rank N on $\mathbb{R}^N$. Let $N = m + n$ and $M \subset \mathbb{R}^{n+m}$ be as in example 1.2. The $\mathcal{T}M$ is a sub-bundle of the trivial bundle $M\times\mathbb{R}^{m+n}$ which is the restriction of the tangent bundle of $\mathbb{R}^{m+n}$ to M. The quotient $M\times\mathbb{R}^{m+n}/\mathcal{T}M$ is called the *normal bundle* to M and is denoted by $\mathcal{N}M$. Using the standard inner product on $\mathbb{R}^{m+n}$ one may regard $\mathcal{N}M$ as the vector bundle consisting of vectors normal to M. Let us compute a set of transition functions for $\mathcal{N}M$. We have

$$\sum_{p\geq m+1}\xi_p^\alpha\frac{\partial}{\partial x_p^\alpha}=\sum_{p,q\geq m+1}\xi_p^\alpha\frac{\partial x_q^\beta}{\partial x_p^\alpha}\frac{\partial}{\partial x_q^\beta}+\sum_{\substack{p\geq m+1\\1\leq i\leq m}}\xi_p^\alpha\frac{\partial x_i^\beta}{\partial x_p^\alpha}\frac{\partial}{\partial x_i^\beta}$$

Therefore the $n\times n$ matrices $\varrho_{\beta\alpha}=(\varrho_{pq}^{\beta\alpha})$ where

$$\varrho_{pq}^{\beta\alpha}=\frac{\partial x_q^\beta}{\partial x_p^\alpha},$$

is a set of transition functions for the normal bundle $\mathcal{N}M$. ♠

We now describe a general construction associated with vector bundles which plays a fundamental role in understanding vector bundles over manifolds. To a continuous map $F:X\rightarrow Y$ of topological spaces and a vector bundle $\pi:E\rightarrow Y$ we assign a vector bundle $F^\star(E)\rightarrow X$, called the *pull-back* of E, as follows: Let the total space of the vector bundle be

$$F^\star(E)=\{(e,x)|\pi(e)=F(x)\}.$$

Now consider the diagram

$$\begin{array}{ccccc} F^\star(E) & \xrightarrow{\jmath} & E\times X & \xrightarrow{p_1} & E\\ \pi_F\downarrow & & p_2\downarrow & & \downarrow\pi\\ X & \xrightarrow{\text{id.}} & X & \xrightarrow{F} & Y, \end{array}$$

where p_j denotes projection on the j^{th} factor, $\jmath$ is the obvious inclusion, and π_F is the restriction of p_2 to $F^\star(E)$. It is immediate that if $E\rightarrow Y$ is a real (complex) k-plane bundle, then so is $F^\star(E)\rightarrow X$, and its transition functions are easy to describe. In fact, let $\{Y_j\}$ be a covering Y such that on each Y_j the vector bundle $E\rightarrow Y$ is trivial, and let ρ_{jk}'s be the corresponding transition functions. Then $F^\star(E)\rightarrow X$ is a trivial k-plane bundle on $X_j=F^{-1}(Y_j)$, and its transition functions are $\rho_{jk}(F(.))$ relative to the covering $\{X_j\}$ of X. Notice that the for an embedding $F:M\rightarrow N$ and a vector bundle $E\rightarrow N$, $F^\star(E)\rightarrow M$ is simply the restriction of the bundle E to the submanifold $F(M)$.

By restricting the class of transition functions $\varphi_{\alpha\beta}$, we specialize the class of manifolds under consideration. We have already noted that the requirement of smoothness (i.e., C^∞), as opposed to mere continuity, limits the class of manifolds. If furthermore we assume that the charts are open subsets of $\mathbb{C}^m \simeq \mathbb{R}^{2m}$ and the transition functions are complex analytic diffeomorphisms, then we obtain a more restricted class of manifolds, naturally called *complex manifolds*. The study of complex manifolds is postponed to another volume, however it is important to introduce those aspects of the theory which closely parallel the real case in this chapter. There are many examples which require only rudimentary knowledge of complex manifolds and complex vector bundles and are more appropriately treated in the present context. Example 0.1.1.3 extends to the complex case in the obvious manner in view of the complex version of the implicit function theorem. Of course here one assumes that all maps are complex analytic. In this manner one obtains many examples of complex manifolds.

Let $E \xrightarrow{\pi} M$ be a complex n-plane bundle over the complex manifold M. Let $\{U_j\}$ be a covering of M and assume that

1. $\pi^{-1}(U_j) \simeq U_j \times \mathbb{C}^n$ holomorphically;

2. The restriction of π to $\pi^{-1}(U_j)$ is projection on the first factor.

Then we say we have a *holomorphic vector bundle*. The description of holomorphic vector bundles in terms of transition functions ρ_{jk} is that ρ_{jk}'s, in addition to the usual requirements, are holomorphic functions on $U_j \cap U_k$ with values in the complex group $GL(n, \mathbb{C})$.

Obviously a complex manifold M of complex dimension m is also a real analytic manifold of dimension $2m$ which we denote by $M_{\mathbb{R}}$. There is an additional structure here which we now describe. Let $\mathcal{T}M_{\mathbb{R}}$ be the tangent bundle of the real manifold $M_{\mathbb{R}}$. There is a distinguished tensor field J of type (1,1) on M as follows: In view of the isomorphism $\mathrm{Hom}(V, V) \simeq V^\star \otimes V$ this amounts to having an endomorphism of $\mathcal{T}_z M_{\mathbb{R}}$ for every $z \in M_{\mathbb{R}}$. Let $(U_\alpha, \varphi_\alpha)$ be a coordinate chart, then we identify each $\mathcal{T}_z M_{\mathbb{R}}$ with $\mathbb{R}^{2m} \simeq \mathbb{C}^m$ which is the ambient space to $\mathrm{Im}\varphi_\alpha$, and let $J_z : \mathcal{T}_z M_{\mathbb{R}} \to \mathcal{T}_z M_{\mathbb{R}}$ be the operator of multiplication by $i = \sqrt{-1}$. Since the transition functions $\varphi_{\alpha\beta}$ for M are complex analytic, the induced maps on the tangent spaces are complex linear and therefore commute with the operator of multiplication by i. Consequently J_z is a well-defined operator on the tangent spaces. Clearly it has the property $J^2 = -I$. A real manifold N of dimension $n = 2k$ admitting of a type (1,1) tensor field J with the property $J^2 = -I$ is called an *almost complex manifold*. An almost complex manifold N may not have the structure of a complex manifold except when $k = 1$ (see subsection on Isothermal Coordinates in chapter xxx). The issue of when an almost complex manifold is in fact a complex manifold will be discussed in another volume.

0.1.2 Orientation and Volume Element

Two bases for a real vector space V define the same *orientation* if they differ by a linear transformation of positive determinant. Therefore the set of bases for a real vector space is partitioned into two classes corresponding to the determinant of change of bases relative to a fixed one, being positive or negative. One arbitrarily calls the choice of one class the *positive orientation* for V, and the bases in that class are *positively oriented.* Naturally the bases in other class are called *negatively oriented.* A vector bundle $E \xrightarrow{\pi} M$ is *orientable* if there is a covering $\mathcal{U} = \{U_\alpha\}$ and sections ξ_j^α of $\pi^{-1}(U_\alpha) \rightarrow U_\alpha$ such that (i) $(\xi_1^\alpha(x), \cdots, \xi_k^\alpha(x))$ is a basis for E_x for every $x \in U_\alpha$, and (ii) if $x \in U_\alpha \cap U_\beta$, then $(\xi_1^\alpha(x), \cdots, \xi_k^\alpha(x))$ and $(\xi_1^\beta(x), \cdots, \xi_k^\beta(x))$ differ by a linear transformation of positive determinant. Therefore if $E \rightarrow M$ is orientable, then we have an equivalence relation $\sim$ on the set of local frames for $E \rightarrow M$. In fact, if $(x; \xi_1(x), \cdots, \xi_k(x))$ and $(y; \eta_1(y), \cdots, \eta_k(y))$ are local frames on open subsets $U, V \subset M$ and $U \cap V \neq \emptyset$, then we say $(x; \xi_1(x), \cdots, \xi_k(x)) \sim (y; \eta_1(y), \cdots, \eta_k(y))$, if for all $x = y \in U \cap V$ the bases $\xi_1(x), \cdots, \xi_k(x)$ and $\eta_1(y), \cdots, \eta_k(y)$ differ by a linear transformation of positive determinant. If $U \cap V = \emptyset$ we define $(x; \xi_1(x), \cdots, \xi_k(x)) \sim (y; \eta_1(y), \cdots, \eta_k(y))$ if there is a sequence of open sets $U = U_\circ, U_1, \cdots, U_{l+1} = V$ and local frames $(x; \xi_1^j(x), \cdots, \xi_k^j(x))$ on U_j such that for $j = 0, \cdots, l$

1. $U_j \cap U_{j+1} \neq \emptyset$;

2. $(x; \xi_1^j(x), \cdots, \xi_k^j(x)) \sim (x; \xi_1^{j+1}(x), \cdots, \xi_k^{j+1}(x))$.

This breaks up the set of local frames for $E \rightarrow M$ into two classes, and the choice of either one of these two is an orientation for $E \rightarrow M$.

There is an alternative way of defining orientability and orientation. Let $E \rightarrow M$ be a real vector bundle of rank k, and $\wedge^k E^\star \xrightarrow{\pi_k} M$ the vector bundle of rank 1 (i.e., *line bundle*) constructed via the obvious tensor operation explained earlier. This means that if the $\rho_{\alpha\beta}$ are the transition functions for $E \rightarrow M$, then the transition functions for $\wedge^k E^\star \rightarrow M$ are $\det(\rho_{\alpha\beta}^{-1})$. If $\wedge^k E^\star \xrightarrow{\pi_k} M$ admits of a global nowhere vanishing section ω, then ω determines an orientation for every fibre E_x, $x \in M$, by the rule: A basis $(\xi_1(x), \cdots, \xi_k(x))$ for E_x is positively or negatively oriented as $\omega(\xi_1(x), \cdots, \xi_k(x))$ is positive or negative. Conversely, assume $E \rightarrow M$ is oriented, and let $\mathcal{U} = \{U_\alpha\}$ be a covering as in the preceding paragraph. Fix local sections $(\xi_1^\alpha(x), \cdots, \xi_k^\alpha(x))$ forming positively oriented bases at every $x \in U_\alpha$. Let ω_α be a section of $\pi_k^{-1}(U_\alpha) \rightarrow M$ with the property $\omega_\alpha((\xi_1^\alpha(x), \cdots, \xi_k^\alpha(x)) > 0$ for $x \in U_\alpha$. Then $\omega = \sum \phi_\alpha \omega_\alpha$ is a nowhere vanishing section of $\wedge^k E^\star \xrightarrow{\pi_k} M$ defining the orientation of $E \rightarrow M$.

A manifold M is *orientable* if its tangent bundle is orientable. By the analysis of the preceding paragraph, this is equivalent to the existence of a nowhere vanishing m-form ω on M which we call a *volume element.* We now describe how a volume element enables one to integrate functions on a manifold. The critical point is that the change of variable formula for an integral in several dimensions is

$$\int_U f(x_1, \cdots, x_m) dx_1 \cdots dx_m = \int_V f(x_1(y), \cdots, x_m(y)) \; |\frac{\partial(x)}{\partial(y)}| \; dy_1 \cdots dy_m,$$

where $x_i = x_i(y_1, \cdots, y_m)$ is the coordinate expression of a diffeomorphism $\varphi : V \to U$, and $\frac{\partial(x)}{\partial(y)} = \det(D\varphi)$ is its Jacobian. Now the transformation formula for an m-form under the diffeomorphism φ is

$$f(x_1, \cdots, x_m) dx_1 \wedge \cdots \wedge dx_m \longleftrightarrow f(x_1(y), \cdots, x_m(y)) \; \frac{\partial(x)}{\partial(y)} \; dy_1 \wedge \cdots \wedge dy_m.$$

Therefore for a positively oriented diffeomorphism, the change of variable formula is built into an m-form, and it is thus the integral of an m-form which is naturally defined on $\mathbb{R}^m$, i.e., is invariant under positively oriented diffeomorphisms. Invariance under a diffeomorphism is the property which makes a scalar quantity unambiguously defined on a manifold. Therefore we can say that given an orientable manifold and a volume element, then the notion of integration of a function is well-defined. Given a function f on M, let $\mathcal{U} = \{U_\alpha\}$ be a covering of M by coordinate charts, and $\{\phi_\alpha\}$ a partition of unity subordinate to $\mathcal{U}$. Then $\int_{U_\alpha} \phi_\alpha f \omega$ is the integral of an m-form on an open subset of $\mathbb{R}^m$ which is defined and is invariant under positively oriented diffeomorphisms. Therefore we can define

$$\int_M f = \sum_\alpha \int_{U_\alpha} \phi_\alpha f \omega.$$

To summarize, the notion of the integral of a function on a manifold M is well-defined once we fix a nowhere vanishing m-form on M. One may loosely rephrase the above analysis by saying that an m-form on an orientable manifold M with a fixed orientation can be integrated.

0.2 Homogeneous Spaces and Invariant Elements

0.2.1 Lie Groups

Groups appear in the study of geometry in various ways. We review some aspects of group theory which are relevant to later chapters. A *Lie group* is a real analytic manifold G together with a group operation $G \times G \rightarrow G$, such that the mapping $(g', g) \rightarrow g'g^{-1}$ is analytic. By a homomorphism of Lie groups we mean an analytic homomorphism[2]. The identity of the group will be generally denoted by e. A connected Lie group is called an *analytic group*. For a Lie group G, let G° denote the connected component of G containing the identity. Let $g, h \in G^\circ$ and γ and δ be curves joining e to g and h. The curves $t \rightarrow (\gamma(t))^{-1}$ and $t \rightarrow \gamma(t)\delta(t)$ join e to g^{-1} and gh. Therefore G° is a group; $g^{-1}eg = e$ and the continuity of the group operations imply that G° is a normal subgroup of G. It is useful to note that a discrete normal subgroup H of an analytic group G necessarily lies in the center $Z(G)$ of G. In particular, if the kernel of a homomorphism $\rho : G \rightarrow G'$ is discrete and G is connected, then $\ker\rho \subset Z(G)$. A Lie group whose underlying manifold is a complex manifold and the mapping $(g', g) \rightarrow g'g^{-1}$ is complex analytic is called a *complex (Lie) group*. However, unless stated to the contrary, it is the real analytic structure that we work with and not the complex analytic one even if it exists. General notions of compactness, closedness etc. are applicable to groups in the obvious manner. The tangent and cotangent bundles of a Lie group are parallelizable since any basis for $\mathcal{T}_eG$ or $\mathcal{T}_e^\star G$ gives bases for all tangent and cotangent spaces by applications of left or right translation mappings $L_h(g) = hg$ or $R_h(g) = gh^{-1}$. Therefore a tangent vector at $e \in G$ gives rise by left translation to a vector field on G which is naturally called a *left invariant* vector field. The tangent space (to a Lie group G) at the identity or equivalently the set of left invariant will be denoted by the corresponding script letter such as $\mathcal{G}$. Similarly, one defines the notions of *right invariant* vector field and left and right invariant forms. In all applications of interest to us, the groups that appear are either finite or matrix groups, although sometimes they may not be directly be given as such. It is essential to look at specific matrix groups and understand the relevant concepts in their context.

Example 0.2.1.1 Let K be the field of real or complex numbers. In this example we clarify the Lie group structure of $GL(m, K)$ and $SL(m, K)$. Now $GL(m, K)$ is defined by the single inequality $\det g \neq 0$ and is an open subset of the space $M_n(K)$ of all $n \times n$ matrices with entries from K. Therefore the tangent space to $GL(m, K)$ at $e = I \in GL(m, K)$ is denoted

[2]It is no gain of generality to relax the analyticity condition since by the positive solution to Hilbert's fifth problem, any group whose underlying space is a manifold is necessarily analytic. Every continuous homomorphism of Lie groups is necessarily analytic (see e.g., [Ho] for a proof.)

by $\mathcal{GL}(m,K) = M_m(K) \simeq K^{m^2}$. On the other hand, $SL(m,K)$ is defined by a single polynomial equation $\det g = 1$. First we have to verify that the equation $\det g = 1$ defines a Lie group. Let $g^{(i)}$ denote the i^{th} column of the matrix g, and so $e^{(i)}$ denotes the i^{th} column of the identity matrix. Then using the fact that determinant is an n-linear function of the columns we obtain

$$D(\det)(e)(\xi^{(1)},\cdots,\xi^{(m)}) = \det((\xi^{(1)},e^{(2)},\cdots,e^{(m)})) + \cdots + \det((e^{(1)},\cdots,e^{(m-1)},\xi^{(m)})).$$

It follows that for $\xi \in M_n(K)$ we have

$$D(\det)(e)(\xi) = \mathrm{Tr}(\xi).$$

Therefore the hypothesis of the implicit function theorem are fulfilled and $SL(m,K)$ has the structure of an analytic manifold near e. Since $\det(hg) = \det(h)\det(g)$, the same argument is applicable to show that for every fixed $h \in GL(m,K)$, the set of matrices $g \in GL(m,K)$ with $\det(hg) = \det(h)$ is an analytic submanifold near h. In particular, $SL(m,K)$ is an analytic manifold and the tangent space at $e \in SL(m,K)$ can be naturally identified with

$$\mathcal{SL}(m,K) \simeq \{\xi \in M_n(K) | \mathrm{Tr}(\xi) = 0\}. \tag{0.2.1.1}$$

Of course, $SL(m,\mathbb{C})$ and $GL(m,\mathbb{C})$ are complex Lie groups. ♠

Example 0.2.1.2 The *orthogonal* group $O(m)$ is the closed subgroup of $GL(m,\mathbb{R})$ consisting of matrices $A = (A_{ij})$ such that $A'A = I$ where the superscript $'$ denotes transpose. Since the defining equations for the orthogonal group, in long hand notation, are

$$\sum_{i=1}^{m} A_{ij}^2 = 1, \quad \sum_{i=1}^{m} A_{ij}A_{ik} = 0 \ \text{ for all } j \neq k,$$

$O(m)$ is compact. These conditions clearly establish a one to one correspondence between the orthonormal bases for $\mathbb{R}^m$ (relative to the standard inner product) and $O(m)$, with any fixed orthonormal basis (e.g. the standard one) corresponding to the identity matrix. To prove that $O(m)$ is a Lie group we proceed as in the case of $SL(m,K)$ by invoking the implicit function theorem. In fact, we differentiate the defining equations for $O(m)$ to obtain the linear map

$$\alpha_A : \xi \longrightarrow \xi' A + A'\xi, \quad \text{for } \xi \in M_n(\mathbb{R}).$$

Thus to prove $O(m)$ is a manifold it suffices to show that this map has rank $\frac{m(m+1)}{2}$ for every fixed $A \in O(m)$. Since for every fixed nonsingular matrix A, the set $\{\xi' A + A'\xi\}$ as ξ

ranges over all $n \times n$ matrices, is precisely set of symmetric matrices, the rank requirement of the implicit function theorem is fulfilled and $O(m)$ is an analytic manifold. In particular, the tangent space at $e = I \in O(m)$ is the linear space of the skew symmetric matrices. It is sometimes necessary to consider more general orthogonal groups. Given a symmetric nondegenerate bilinear form Q on a real vector space of dimension m, one considers the group $O(Q)$ of invertible matrices g such that

$$Q(g.x, g.y) = Q(x, y)$$

for all $x, y \in \mathbb{R}^m$. Recall that a real symmetric nondegenerate bilinear form Q on $\mathbb{R}^m$ is determined up to change of basis by its *signature* (p, q) (i.e., Q has p negative and q positive eigenvalues). We denote by $O(p, q)$, $p \leq q$, the orthogonal group corresponding to the diagonal matrix $Q = (Q_{ij})$ with $Q_{ii} = -1$ for $i \leq p$ and $Q_{jj} = 1$ for $p + 1 \leq j \leq p + q = m$. The dimension of $O(p, q)$ is $\frac{1}{2}m(m-1)$, $p + q = m$, independently of the signature. If Q and Q' have the same signature, then the orthogonal groups $O(Q)$ and $O(Q')$ are conjugate in $GL(m, \mathbb{R})$. Note that $SO(p, q)$ is compact if and only if $p = 0$ or $q = 0$. ♠

Example 0.2.1.3 Similarly, the *unitary* group $U(m)$ is the subset of $GL(m, \mathbb{C})$ consisting of matrices $U = (U_{ij})$ such that $\bar{U}'U = I$ where $^-$denotes complex conjugate. By an argument similar to that for $O(m)$, $U(m)$ is a compact real analytic manifold and therefore a compact Lie group. Notice that because of the complex conjugation in the defining equations for $U(m)$, the unitary group is a real group and does not have the structure of a complex manifold. Just as in the case of $O(m)$, one shows that the tangent space to $U(m)$ at $e = I$ is the the linear space of skew hermitian matrices. Clearly $U(m)$ can be identified with the set of orthonormal bases for $\mathbb{C}^m$ relative to a fixed hermitian inner product. Just as in the case of the orthogonal group one defines the unitary group $U(p, q)$, $p + q = m$, as the subgroup of $GL(m, \mathbb{C})$ consisting of matrices U such that $\bar{U}'QU = Q$ where Q is a diagonal matrix with p eigenvalues 1 and q eigenvalues -1. $U(p, q)$ is compact if and only if $p = 0$ or $q = 0$. ♠

The *special orthogonal* groups $SO(m)$, $SO(p, q)$ and *special unitary* groups $SU(m)$, $SU(p, q)$ are the subgroups of the orthogonal and unitary groups $O(m)$, $O(p, q)$, $U(m)$ and $U(p, q)$ consisting of matrices of determinant 1. Since an orthogonal matrix has determinant ± 1, $SO(m)$ is a subgroup of index two in $O(m)$. Note that $SO(m)$ and $SU(m)$ are normal subgroups of $O(m)$ and $U(m)$ respectively. For a subgroup $G \subset GL(n, K)$, the notation SG often signifies the subgroup of G consisting of elements of determinant 1. For example, for $p + q = n$, $S(U(p) \times U(q))$ means $U(p) \times U(q)$ is embedded in $U(n)$ as

$$\begin{pmatrix} U(p) & 0 \\ 0 & U(q) \end{pmatrix} \subset GL(n, \mathbb{C}),$$

and $S(U(p) \times U(q))$ is its subgroup of elements of determinant 1.

Example 0.2.1.4 From linear algebra we know that every unitary matrix U can be diagonalized by a unitary transformation A, i.e., $AUA^{-1} = D$ with D a diagonal matrix. Denote the diagonal entries of D by $e^{i\theta_j}$, $j = 1, \cdots, m$, U can be connected to e by unitary diagonal matrices D_ϵ with diagonal entries $e^{i\epsilon\theta_j}$ where $0 \leq \epsilon \leq 1$. Then the path $\epsilon \rightarrow A^{-1}D_\epsilon A$ connectes the identity to the matrix U, and consequently $U(m)$ is connected. If $U \in SU(m)$ then $\sum \theta_j = 0$ and $D_\epsilon \in SU(m)$ proving connectedness of $SU(m)$. Connectedness of $SO(m)$ is proven by a similar argument. In fact, from linear algebra we know that every orthogonal matrix of determinant 1 in dimension $m = 2k$ is conjugate in $SO(m)$ to one of the block diagonal form with 2×2 block diagonals

$$\begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}, \quad j = 1, \cdots, k.$$

For $m = 2k + 1$, an orthogonal matrix of determinant 1 can be put in the form $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ where $A \in SO(2k)$. Since every such matrix can be connected to the identity, $SO(m)$ is connected. ♠

Exercise 0.2.1.1 *Using the Jordan decomposition or otherwise, show that* $SL(m, \mathbb{C})$ *and* $GL(m, \mathbb{C})$ *are connected.*

Example 0.2.1.5 A consequence of connectedness of $GL(m, \mathbb{C})$ (exercise 0.2.1.2) is that every complex vector bundle is orientable. The key point is to show that every complex vector space $V = \mathbb{C}^n$ regarded as $\mathbb{R}^{2n}$ has a natural orientation. Let $\{h_1, \cdots, h_n\}$ be the standard basis for $\mathbb{C}^n$, and set $e_j = h_j$, $e_{n+j} = ih_j$ for $j \leq n$. Then $\{e_1, \cdots, e_{2n}\}$ is a basis for $\mathbb{R}^{2n}$. Any other basis $\{h'_1, \cdots, h'_n\}$ for $\mathbb{C}^n$ differs from $\{h_1, \cdots, h_n\}$ by an element $(A_{jk} + iB_{jk}) \in GL(n, \mathbb{C})$, with A_{jk} and B_{jk} real matrices. Then the basis $\{e'_1, \cdots, e'_{2n}\}$ differs from $\{e_1, \cdots, e_{2n}\}$ by the $2n \times 2n$ real matrix $g = (g_{jk})$ where

$$g_{jk} = A_{jk} = g_{n+j\ n+k}, \quad g_{n+j\ k} = B_{jk} = -g_{j\ n+k}.$$

Since $A + iB \rightarrow g$ is a continuous homomorphism of $GL(n, \mathbb{C})$ into $GL(2n, \mathbb{R})$, and $GL(n, \mathbb{C})$ is connected, $\det(g) > 0$. Therefore all basis for $\mathbb{R}^{2n}$ obtained in this fashion from bases for $\mathbb{C}^n$ define the same orientation. This implies that all complex vector bundles are orientable. ♠

Example 0.2.1.6 With a little linear algebra one can gain some insight into the structure of the groups $GL(m, K)$ and $SL(m, K)$ as analytic manifolds. It is clear that $GL(m, \mathbb{R})$ has at least two connected components $GL_\pm(m, \mathbb{R})$ corresponding to $\det(g)$ being positive or negative. Let $K = \mathbb{R}$ and $\mathcal{P}_m$ be the space of positive definite symmetric matrices. $GL(m,\mathbb{R})$ acts on $\mathcal{P}_m$ by

$$P \longrightarrow g'Pg, \quad \text{for } g \in GL(m,\mathbb{R}),\ P \in \mathcal{P}_m.$$

From linear algebra (using the Gram-Schmidt process) we know that for every $P \in \mathcal{P}_m$ there is a unique upper triangular matrix $T = T(P)$ with positive eigenvalues such that $T'PT = I$. It follows that the action of $GL(m,\mathbb{R})$ on $\mathcal{P}_m$ is transitive. For $g \in GL_+(m,\mathbb{R})$, let $P = g'g$ and $T = T(P)$ to obtain $(gT)'(gT) = I$ so that $gT \in SO(m)$. Therefore every $g \in GL_+(m,\mathbb{R})$ has a unique decomposition in the form $g = kT$ where $k \in SO(m)$ and T is an upper triangular with positive eigenvalues. If $\det(g) = 1$ then $\det(g'g) = 1$ and necessarily $\det(T) = 1$ which shows that the decomposition is valid for $SL(m,\mathbb{R})$ as well. The operations $g \to T \to k$ are rational and therefore the decomposition $g = kT$ is a real analytic diffeomorphism. It is customary to further decompose $T = au$ with a a diagonal matrix with positive eigenvalues and u an upper triangular matrix with 1's along the diagonal. Let A (resp. U) denote the group of diagonal matrices with positive eigenvalues (resp. upper triangular matrices with 1's along the diagonal), and $A_1 \subset A$ the subgroup of matrices of determinant 1. The above analysis gives analytic diffeomorphisms

$$GL_+(m,\mathbb{R}) \simeq SO(m)AU, \quad SL(m,\mathbb{R}) \simeq SO(m)A_1U. \tag{0.2.1.2}$$

Notice that these decompositions are only analytic manifold decompositions and not as product groups, i.e., $(kau)(k'a'u') \neq (kk')(aa')(uu')$. The decomposition (0.2.1.2) is a special case of the *Iwasawa decomposition*. An immediate consequence of the Iwasawa decomposition is that the groups $SL(m,\mathbb{R})$ and $GL_+(m,\mathbb{R})$ are connected. ♠

Exercise 0.2.1.2 *Using the space of positive definite hermitian (rather than symmetric) matrices and repeating the argument of the preceding example, prove the following decompositions:*

$$GL(m,\mathbb{C}) \simeq U(m)AU_{\mathbb{C}}, \quad SL(m,\mathbb{C}) \simeq SU(m)A_1U_{\mathbb{C}},$$

where $U_{\mathbb{C}}$ is the group of upper triangular complex matrices with 1's along the diagonal. Deduce that $GL(m,\mathbb{C})$ and $SL(m,\mathbb{C})$ are connected. (The above decompositions are examples of the Iwasawa decompositions for $GL(m,\mathbb{C})$ and $SL(m,\mathbb{C})$.)

Let J be a nondegenerate skew symmetric bilinear form on K^m, where K is any field of characteristic zero. From elementary linear algebra we know that $m = 2n$ and by a suitable choice of basis for K^m the matrix of J takes the form

$$J \longleftrightarrow \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{0.2.1.3}$$

where I is the identity $n \times n$ matrix. A subspace $V \subset K^m$ such that for all $v, w \in V$ we have $J(v, w) = 0$ is called *isotropic*. An isotropic subspace necessary has dimension $\leq n$. It is a standard theorem in elementary linear algebra that any basis for an isotropic subspace V can be extended to a basis for $\mathbb{R}^m$ so that the matrix representation (0.2.1.3) is valid. A maximal isotropic subspace is called a *Lagrangian* subspace. Every Lagrangian subspace has dimension n and every isotropic subspace of dimension n is Lagrangian. Non-degenerate skew-symmetric forms arise naturally in different contexts. For example, let $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ with the standard (or in fact any) hermitian inner product $< ., . >$. Then the imaginary part of the hermitian inner product is

$$\Im < z, w >= \sum_{j=1}^{n} (y_j u_j - x_j v_j),$$

where $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_n)$, $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$, which is visibly a nondegenerate skew-symmetric form.

Example 0.2.1.7 The set of $m \times m$ matrices g such that $g' J g = J$ is a Lie group of dimension $2n^2 + n$ assuming that $K = \mathbb{R}$ or $\mathbb{C}$. This group is denoted by $Sp(n, K)$ and is called the *symplectic group* modified by real or complex should it be necessary to specify the field K. The group $U(m) \cap Sp(n, \mathbb{C})$ will be denoted by $USp(n)$ and is called the *compact symplectic group*. Writing $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as $n \times n$ blocks, the condition $g \in Sp(n, K)$ becomes

$$A'C - C'A = 0, \quad B'D - D'B = 0, \quad A'D - C'B = I. \tag{0.2.1.4}$$

Thus for $n = 1$ we obtain the isomorphism $Sp(1, K) = SL(2, K)$. Let $V, W \subset K^m$ be Lagrangian subspaces. It is a standard fact from linear algebra that a basis for a Lagrangian subspace can be extended to a basis for K^m such that representation (0.2.1.3) remains valid. It follows that there is $g \in Sp(n, K)$ such that $g(V) = W$, i.e., the symplectic group acts transitively on the set of Lagrangian subspaces. Clearly the image of a Lagrangian subspace under a symplectic transformation is Lagrangian.

Now assume $K = \mathbb{R}$, $e_1, \cdots, e_m$ be the standard basis and we assume that J has the standard form (0.2.1.3) relative to this basis. Let $V_\circ$ denote the span of $e_1, \cdots, e_n$ which is a Lagrangian subspace, and V be an arbitrary Lagrangian subspace. Let $< ., . >$ denote the standard inner product on $\mathbb{R}^m$ so that $< e_j, e_k >= \delta_{jk}$. For a non-singular $n \times n$ matrix g, the linear transformation

$$\begin{pmatrix} A' & 0 \\ 0 & A^{-1} \end{pmatrix}$$

is a symplectic transformation. By taking orthonormal bases for $V_\circ$ and V we deduce that there is a transformation $g \in Sp(n,\mathbb{R}) \cap O(m)$ mapping $V_\circ$ to V. Note also that $g \in Sp(n,\mathbb{R}) \cap O(m)$ if and only if the equations (0.2.1.4) and

$$A'A + C'C = I, \quad B'B + D'D = I, \quad A'B + C'B = 0, \tag{0.2.1.5}$$

are satisfied. It follows from (0.2.1.4) and (0.2.1.5) that $g \in Sp(n,\mathbb{R}) \cap O(m)$ if and only if $A + iC$ and $B + iD$ are $n \times n$ unitary matrices and furthermore

$$(A' + iC')(B - iD) = A'B + C'D + i(C'B - A'D) = iI.$$

This equation implies that $D' + iB'$ is the inverse to the unitary matrix $A + iC$ and consequently

$$A = D, \quad \text{and} \quad B = -C. \tag{0.2.1.6}$$

Thus we have shown that $Sp(n,\mathbb{R}) \cap O(m)$ is isomorphic to the unitary group and the isomorphism is given by $g \to A + iC$ which is easily checked to be a homomorphism. ♠

Exercise 0.2.1.3 *Construct the analogue of the Iwasawa decomposition $Sp(n,\mathbb{R}) \simeq KAU$, where $K \simeq U(n)$ and identify explicity the subgroups A and U. Deduce that $Sp(n,\mathbb{R})$ is connected and in particular* $\det(g) = 1$ *for all $g \in Sp(n,\mathbb{R})$.*

Example 0.2.1.8 In this and the following example we discuss an application of the Iwasawa decomposition to the space of lattices in $\mathbb{R}^m$. This application will be useful in chapter 5 in connection with the topological uniformization theorem and the construction of certain three dimensional manifolds. Fix a lattice $L_\circ \subset \mathbb{R}^m$ and let $f_1, \cdots, f_m$ be a basis for $L_\circ$. Let $L \subset \mathbb{R}^m$ be another lattice with basis $f'_1, \cdots, f'_m$. Expressing each f'_j as a linear combination of f_j's and writing the coefficients as the j^{th} column of a matrix, we obtain a matrix $g_L \in GL(m,\mathbb{R})$. A change of basis for the lattice L has the effect of multiplying the matrix g_L on the right by matrix $h \in GL(m,\mathbf{Z})$, where we recall that $GL(m,\mathbf{Z})$ is the the group of $m \times m$ matrices with integer entries and determinant ± 1. Therefore the set of lattices in $\mathbb{R}^m$ is identified with $GL(m,\mathbb{R})/GL(m,\mathbf{Z})$ and is accordingly topologized. Let U be the subgroup of upper (or lower) triangular matrices with 1's along the diagonal, Let $U_{\mathbf{Z}} \subset U$ be the subgroup consisting of matrices with integer entries. Looking at the expression for the product uh with $u \in U$ and $h \in U_{\mathbf{Z}}$ we see easily that there is a compact subset $C \subset U$ such that every $u \in U$ has an expression of the form

$$u = u_C h, \quad \text{with} \quad u_C \in C, \quad \text{and} \quad h \in U_{\mathbf{Z}}. \tag{0.2.1.7}$$

Therefore every lattice L has a representation of the form $g_L = k_L a_L u_L$, following the Iwasawa decomposition, with u_L in a fixed compact subset $C \subset U$. Since permutation matrices are in $GL(m, \mathbf{Z})$, we can also assume that the diagonal entries $a_{11}, a_{22}, \cdots, a_{mm}$ of a_L are in decreasing order:

$$a_{11} \geq a_{22} \geq \cdots \geq a_{mm} > 0. \tag{0.2.1.8}$$

Summarizing, every lattice $L \subset \mathbb{R}^m$ can be represented as $g_L = k_L a_L u_L$, following the Iwasawa decomposition, with $u_L \in C$ and the diagonal entries of a_L satifying the inequalities (0.2.1.8). ♠

Example 0.2.1.9 Assume we have fixed a lattice $L_\circ \subset \mathbb{R}^m$. Let $\mathcal{X}$ be a set of lattices in $\mathbb{R}^m$, so that $\mathcal{X} \subset GL(m, \mathbb{R})/GL(m, \mathbf{Z})$. We want to obtain conditions for compactness of the set $\mathcal{X}$. From the discussion in example 0.2.1.8, we see that $\mathcal{X}$ is relatively compact in $GL(m, \mathbb{R})/GL(m, \mathbf{Z})$ if and only if there are constants $R > r > 0$ such that for all $L \in \mathcal{X}$

$$R \geq a_{11} \geq a_{22} \geq \cdots \geq a_{mm} \geq r. \tag{0.2.1.9}$$

This condition may be expressed in a more geometric language. To do so, we define the volume of a lattice as

$$\mathrm{vol}(L) = |\det(g_L)|.$$

It is straightforward to see that compactness criterion (0.2.1.9) may be re-stated as:

- (*Mahler's Compactness Criterion*) $\mathcal{X}$ is relatively compact in $GL(m, \mathbb{R})/GL(m, \mathbf{Z})$ if and only if the following two conditions are satisfied:
    1. There is a neighborhood U of $\mathbf{0}$ in $\mathbb{R}^m$ such that $L \cap U = \mathbf{0}$ for all $L \in \mathcal{X}$;
    2. vol is a bounded function on $\mathcal{X}$.

In chapter 5 we will make use of this criterion for $m = 3$ and $m = 6$. In the former case, $\mathbb{R}^3$ is identified with the Lie algebra $\mathcal{SL}(2, \mathbb{R})$ and $\mathcal{X}$ is the orbit of a certain lattice $L_\circ$ under the adjoint action of $SL(2, \mathbb{R})$. Similarly in the six dimensional case, we will consider the lattices obtained from a fixed lattice in $\mathcal{SL}(2, \mathbb{C})$ via the adjoint action of $SL(2, \mathbb{C})$. ♠

We can now introduce the important notion of a principal bundle. A *principal bundle* is a quadruple (G, P, π, X) (or simply $P \to X$) where $\pi : P \to X$, G is a Lie group acting on the right on P and such that

1. There is a covering $\mathcal{U} = \{U_\alpha\}$ of X such that $\pi^{-1}(U_\alpha) \simeq U_\alpha \times G$ with the restriction of π to $\pi^{-1}(U_\alpha)$ being projection on the first factor;
2. G acts (simply transitively) on the fibres of π, i.e., on the sets $\pi^{-1}(x)$ for all $x \in X$ according to the rule $(x, h) \xrightarrow{g} (x, hg^{-1})$.

Just as in the case of vector bundles, a principal bundle is trivial if it is the product of the base and the fibre ($\simeq G$) with the obvious projection map. Note that a principal bundle is trivial if and only if it admits of a section. In fact, if $P \xrightarrow{\pi} X$ admits of a section $s : X \to P$, then we have the trivialization $P \simeq X \times G$ given by $p \to (\pi(p), g(p))$ where $g(p)$ is the unique element of G such that $s(\pi(p)).g(p) = p$. From a vector bundle one can construct certain principal bundles. For example, given a vector bundle of rank k, $E \xrightarrow{\pi} M$, we let P_E be the set of all bases (or frames) for all the fibres of π. This means $P_E = \{(x; \xi_1, \cdots, \xi_k) | x \in M, \ (\xi_1, \cdots, \xi_k) \text{ a basis for } \pi^{-1}(x)\}$. Since bases for $\mathbb{R}^k$ are parametrized by $GL(k, \mathbb{R})$, we have the structure of a principal bundle with $G = GL(k, \mathbb{R})$.

Let P be a manifold and G a compact Lie group acting on P freely on the right, i.e., for all $x \in P$ and $e \neq g \in G$, $x.g \neq x$. Assume $M = P/G$ is a manifold, then it is trivial to show that $P \to M$ is a principal fibre bundle. It is clear that all principal fibre bundles on manifolds are of this form (regardless of the compactness of the group G).

Given a principal bundle (G, P, π, X), one can construct associated vector bundles as follows: By a real or complex *representation* of a group G we mean a continuous (and therefore real analytic for Lie groups) homomorphism $\rho : G \to GL(n, K)$, where $K = \mathbb{R}$, or $\mathbb{C}$. Let $F = K^n$, and consider the action of G on $P \times F$ by $(p, v) \xrightarrow{g} (p \cdot g^{-1}, \rho(g)v)$. Let E_ρ be the orbit space of the action of G on $P \times F$, i.e., the quotient of $P \times F$ under the equivalence relation $(p, v) \sim (p \cdot g^{-1}, \rho(g)v)$. Therefore we have a map $P \times F \to E_\rho$. Then the projection on the first factor $P \times F \to P$ followed by $P \to M$ factors through E_ρ, and defines the structure of a vector bundle on $E_\rho \to M$ which is diagramatically represented as follows:

$$\begin{array}{ccc} P \times F & \longrightarrow & E_\rho \\ \downarrow & & \downarrow \\ P & \xrightarrow{\pi} & X \end{array}$$

As an example note that if $E \to M$ is a rank k vector bundle, $P_E \to M$ the associated bundle of frames, and ρ the natural representation of $GL(k, \mathbb{R})$, then $E_\rho \to M$ is the original vector bundle $E \to M$. Similarly, to various tensor powers of the representation ρ are associated tensor products of the bundle $E \to M$ as described earlier.

Example 0.2.1.10 Since a subgroup $G \subset GL(m, K)$ is given a a group of $m \times m$ matrices, we may call the inclusion map of G in $GL(m, K)$ as its *natural representation.* Given any representation $\rho : G \to GL(m, K)$ of a group G, one can construct many other representations from it. In fact, every $A \in GL(m, K)$ induces linear transformations $A \otimes \cdots \otimes A \otimes A'^{-1} \otimes \cdots \otimes A'^{-1}$ acting on $V \otimes \cdots \otimes V \otimes V^\star \otimes \cdots \otimes V^\star$ where $V = K^m$. Therefore ρ gives rise to representations on tensor product of V and $V^\star$ any number of times. Representations of groups are discussed later in this chapter. ♠

Remark 0.2.1.1 It becomes necessary to consider bundles whose fibres spaces other than a group or a vector space. For example, we will consider *sphere bundles* or *ball bundles* in which case the transition functions $\rho_{\alpha\beta}$ take values in some group of transformations of the sphere or the ball. Unless stated to the contrary, the sphere or ball bundles we consider are subspaces of a vector bundle and transitions functions $\rho_{\alpha\beta}$ of the vector bundle are such that they leave the sphere or the ball invariant. This will become more clear when we consider metrics on vector bundles. Given a principal bundle (G, P, π, X) and homomorphism ρ of G into the group of diffeomorphisms of a manifold F, we can construct, just as in the case of the vector bundle $E_\rho \to X$, a bundle on X with fibre F. ♡

0.2.2 Homogeneous Spaces

Let G be a Lie group acting on the manifold M on the right. It is interesting to know under what conditions the orbit space $N = M/G$ is a manifold with the projection $p : M \to N$ a *submersion*, i.e., $Dp(x)$ is surjective for all $x \in M$. The answer to this question is based on the following observation: If M/G is a manifold with the above proviso, then $p \times p : M \times M \to N \times N$ is also a submersion. Let $\Delta = \{(q, q) | q \in N\}$. Then $R = (p \times p)^{-1}(\Delta) \subset M \times M$ is a closed submanifold. We have

Proposition 0.2.2.1 *With the above notation, N has the structure of a manifold with $M \to N$ a submersion if and only if R is a closed submanifold. If M and R are analytic, then so is N.*

Proof - We have already established the necessity. Closure of R implies that $R \cap \{x\} \times M$ is closed for all $x \in M$, i.e., every orbit of G in M is closed. Consequently N is a normal topological space. To complete the proof we have to show that for every $x \in M$ there is a neighborhood V of x and a submanifold T such that for every $y \in V$, $T \cap y.G$ is a single point $t(y) \in T$. Then the coordinate charts in T's describe the manifold structure of M/G. Since R is a submanifold for every $(x, y) \in R$ there is a neighborhood $U \times U \subset M \times M$ and a smooth function $H : U \times U \to \mathbb{R}^k$ such that $R \cap U \times U = \{(w, z) \in U \times U | H(w, z) = 0\}$, and DH has rank k everywhere. Let F be the restriction of H to $\{x\} \times M$. Since the kernel of $DH(x, x) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ contains the diagonal $\{(\xi, \xi) | \xi \in \mathbb{R}^m\}$, $DF(y)$ has rank k for all y sufficiently close to x. Then $T = \{y \in M | F(y) = 0\}$ is the required submanifold by the implicit function theorem. The analyticity assertion is a consequence of the analytic implicit function theorem. *clubsuit*

By a *homogeneous space* we mean a manifold of the form G/H where G is a Lie group and H is a closed subgroup. An immediate consequence of proposition 0.2.2.1 is

Corollary 0.2.2.1 *Let G be a Lie group and H a closed subgroup, then G/H is an analytic manifold.*

We now use corollary 0.2.2.1 to give some important examples of manifolds.

Example 0.2.2.1 Let $\mathbf{G}_{k,n}(\mathbb{R})$ be the set of k-dimensional subspaces of $\mathbb{R}^{k+n}$. Let us show that $\mathbf{G}_{k,n}(\mathbb{R})$ is an analytic manifold. Clearly the orthogonal group $O(k+n)$ acts transitively on $\mathbf{G}_{k,n}(\mathbb{R})$, and the isotropy subgroup of the subspace where the first k coordinates are arbitrary and the remaining are zero, is $O(k)\times O(n)$. Therefore $\mathbf{G}_{k,n}(\mathbb{R})\simeq O(k+n)/O(k)\times O(n)$ which is a compact analytic manifold. Note that we can also write $\mathbf{G}_{k,n}(\mathbb{R})\simeq SO(k+n)/S(O(k)\times O(n))$ which shows that $\mathbf{G}_{k,n}(\mathbb{R})$ is connected. Similarly, one can consider the set $\mathbf{G}^{\circ}_{k,n}(\mathbb{R})$ of oriented k-dimensional subspaces of $\mathbb{R}^{k+n}$, i.e., every point of $\mathbf{G}^{\circ}_{k,n}(\mathbb{R})$ is a k-dimensional subspace together with an orientation of the subspace. Then $SO(k+n)$ acts transitively on $\mathbf{G}^{\circ}_{k,n}(\mathbb{R})$ and $\mathbf{G}^{\circ}_{k,n}(\mathbb{R})\simeq SO(k+n)/SO(k)\times SO(n)$ which is a compact analytic manifold. $\mathbf{G}_{k,n}(\mathbb{R})$ (resp. $\mathbf{G}^{\circ}_{k,n}(\mathbb{R})$) is called the *real Grassmann manifold of* (resp. *oriented*) k-planes in $\mathbb{R}^{k+n}$. $\mathbf{G}_{1,n}(\mathbb{R})$ is called the *real projective space* of dimension n and is also denoted by $\mathbb{R}P(n)$. Note that $\mathbf{G}^{\circ}_{1,n}(\mathbb{R})\simeq S^n$ and $\mathbb{R}P(n)=\mathbf{G}_{1,n}(\mathbb{R})$ is the quotient of S^n where the points $x\in S^n$ and $-x$ (*anti-podal points*) are identified. ♠

Example 0.2.2.2 Let $\mathbf{s}: 0<s_1<\cdots<s_r<n$ be a sequence of positive integers. By a *flag* in the complex vector space $V\simeq\mathbb{C}^n$ we mean a sequence $f: 0=\subset V_1\subset\cdots\subset V_r\subset V$ of subspaces with $\dim(V_j)=s_j$. If $r=n-1$ then necessarily $\mathrm{dimV}_j=s_j=j$ and we call a sequence $f: 0=\subset V_1\subset\cdots\subset V_{n-1}\subset V$ a *complete flag*. The set of flags for a given fixed sequence $\mathbf{s}$ is homogeneous space $\mathbf{F}_{\mathbf{s}}$ for the unitary group $U(n)$. In fact, let $\{e_1,\cdots,e_n\}$ be the standard basis for $V=\mathbb{C}^n$ with the standard Hermitian inner product $<,>$, and E_j be the span of $\{e_1,\cdots,e_{s_j}\}$. Then it is easy to see that

$$\mathbf{F}_{\mathbf{s}}=U(n)/U_{t_1}\times\cdots\times U_{t_{r+1}},$$

where U_{t_j} is the unitary group in the orthogonal complement of $E_{s_{j-1}}$ in E_{s_j}, $t_r=s_r-s_{r-1}$ with $s_\circ=0$ and $s_{r+1}=n$. Thus $\mathbf{F}_{\mathbf{s}}$ is a compact analytic manifold. The manifold of complete flags will be denoted by $\mathbf{F}_n$. In this case all t_j's are 1 and each $U_j\simeq U(1)=\{e^{i\theta}\}$. Complex Grassmann manifold $\mathbf{G}_{k,n}$ of k dimensional linear subspaces of $\mathbb{C}^{n+k}$ is the flag manifold $\mathbf{F}_{\mathbf{s}}$ with $r=1$ and $s_1=k$ (and n replaced by $n+k$). ♠

Exercise 0.2.2.1 *Combine the ideas of examples 0.2.2.1 0.2.2.2 to define real flag manifolds and realize them as homogeneous spaces of compact groups. Distinguish between the cases where an orientation requirement is or is not imposed.*

Exercise 0.2.2.2 *Let $\mathbf{LG}_n$ denote the set of Lagrangian subspaces of $\mathbb{R}^{2n}$ with the symplectic structure J. Show that $\mathbf{LG}_n \simeq U(n)/O(n)$.* (See example 0.2.1.7. $\mathbf{LG}_n$ is called the *Lagrangian Grassmanian.*)

Example 0.2.2.3 In example 0.2.2.2 we showed that the complex flag manifolds are homogeneous spaces of the unitary group and are therefore compact real analytic manifolds. We now show that they are actually complex manifolds as well. Let $G = GL(n;\mathbb{C})$ and $B \subset G$ be the closed subgroup of upper triangular matrices. Then G and B are complex groups and corollary 0.2.2.1 is applicable to show that the manifold of complete flags $\mathbf{F}_n = G/B$ is a complex manifold. Similar considerations apply to the flag manifolds $\mathbf{F}_{\mathbf{s}}$. ♠

Example 0.2.2.4 The complex projective space $\mathbb{C}P(n)$ is probably the most common example of a compact complex manifold. It is defined as the set of lines through the origin (i.e., one dimensional linear subspaces) of $\mathbb{C}^{n+1}$ and is therefore the complex manifold $\mathbf{G}_{1,n}$. It is useful to give it the following equivalent description: Let $\mathbb{C}^\times$ be the multiplicative group of non-zero complex numbers. Then $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ by multiplication, $[z_\circ, \cdots, z_n] \rightarrow [\lambda z_\circ, \cdots, \lambda z_n]$, for $\lambda \in \mathbb{C}^\times$. Proposition 0.2.2.1 is applicable (the same proof works for the complex analytic case provided all data are complex analytic), and the quotient space is $\mathbb{C}P(n)$. The complex manifold structure of $\mathbb{C}P(n)$ is easy to describe. A point of $\mathbb{C}P(n)$ is specified by its *homogeneous coordinates* $[z_\circ, \cdots, z_n]$ (not all z_i's zero) which is defined up to multiplication by a non-zero complex number. Consider the covering $\{U_i\}, i = 0, \cdots, n$ of $\mathbb{C}P(n)$ defined by $U_i = \{z = [z_\circ, \cdots, z_n] | z_i \neq 0\}$. Dividing by z_i we obtain a homeomorphism of U_i with $V_i = \mathbb{C}^n$ given by $[z_\circ, \cdots, z_{i-1}, 1, z_{i+1}, \cdots, z_n] \rightarrow (z_\circ, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n)$. The transition functions $\varphi_{ij} : V_i \cap V_j \rightarrow V_i \cap V_j$ are simply multiplication by $\frac{z_i}{z_j}$. From the definition of the complex projective space as the quotient of $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ under the action of the multiplicative group $\mathbb{C}^\times$, it follows that $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{C}P(n)$ is a principal bundle with fibre $\mathbb{C}^\times$. This bundle is called the *tautological principal bundle.* The associated complex line bundle corresponding to the natural representation $\mathbb{C}^\times \rightarrow GL(1, \mathbb{C}) \simeq \mathbb{C}^\times$ given by $\lambda \rightarrow \lambda$ is called the *tautological bundle* and denoted by $\mathcal{L} \rightarrow \mathbb{C}P(n)$. The complex line bundle corresponding to the representation $\lambda \rightarrow \lambda^{-1}$ is called the *hyperplane section* bundle and denoted by $\mathcal{L}^{-1} \rightarrow \mathbb{C}P(n)$. We denote $\mathcal{L} \otimes \cdots \otimes \mathcal{L} \rightarrow \mathbb{C}P(n)$ simply as $\mathcal{L}^k \rightarrow \mathbb{C}P(n)$ and similarly $\mathcal{L}^{-k} \rightarrow \mathbb{C}P(n)$ is defined. Notice that homogeneous polynomials of degree k are sections of $\mathcal{L}^{-k} \rightarrow \mathbb{C}P(n)$. ♠

Exercise 0.2.2.3 *Show that homogeneous polynomials of degree k in $n+1$ variables are sections of $\mathcal{L}^{-k} \rightarrow \mathbb{C}P(n)$ and conversely.*

Example 0.2.2.5 Just as in the case of the projective space $\mathbb{C}P(n)$, every Grassmann manifold $\mathbf{G}_{k,n}$, $\mathbf{G}_{k,n}(\mathbb{R})$, $\mathbf{G}^\circ_{k,n}(\mathbb{R})$ carries a tautological bundle which we denote by $\mathcal{E}_k \rightarrow \mathbf{G}_{k,n}$,

$\mathcal{E}_k \to \mathbf{G}_{k,n}(\mathbb{R})$, or $\mathcal{E}_k \to \mathbf{G}^{\circ}_{k,n}(\mathbb{R})$ respectively. Often we drop the subscript k from the total space when there is no concern for confusion. The fibre over a point of the Grassmann manifold is the subspace of the Euclidean space that it represents. In case of $\mathcal{E}_k \to \mathbf{G}^{\circ}_{k,n}(\mathbb{R})$, the fibre carries an orientation as well. ♠

Exercise 0.2.2.4 *Compute the transition functions for the vector bundles in example 0.2.2.5.*

Example 0.2.2.6 Let $F(z_\circ, \cdots, z_n)$ be a homogeneous polynomial, and set

$$Z_F = \{[z_\circ, \cdots, z_n] | F(z_\circ, \cdots, z_n) = 0\}.$$

In view of homogeneity of F, Z_F is a well-defined subset of $\mathbb{C}P(n)$. For Z_F to be a submanifold, it suffices to verify the hypothesis of the implicit function theorem on each coordinate chart U_i. This means we set $z_i = 1$ to obtain a polynomial F^i in n variables and check non-vanishing of at least one partial derivative $\partial F^i/\partial z_j$ for $j \neq i$. The extension to the common zeros of k homogeneous polynomials is in the obvious manner. Thus one constructs many compact complex manifolds. ♠

0.2.3 Invariant Forms on Lie Groups and Homogeneous Spaces

It is very useful to have an understanding of what left invariant forms are like. For a closed subgroup $G \subset GL(m, K)$ (therefore a Lie group), let dg denote the (exterior) derivative of $g \in G$ relative to any local parametrization of the group G. Then the matrix of 1-forms $g^{-1}dg$ is clearly invariant under left translations by any fixed $h \in G$. Therefore the entries of the matrix $g^{-1}dg$ are left invariant 1-forms on G. For $G = GL(m, K)$ (we can let the parametrization be the identity mapping) and then the 1-forms that appear as entries of $g^{-1}dg$ are linearly independent and form bases for cotangent spaces. For $G = SL(m, K)$ the situation is a little different in view of (0.2.1.1). In fact, it is a simple exercise to see that the same computation leading to (0.2.1.1) shows for $(\omega_{ij}) = g^{-1}dg$ we have

$$\mathrm{Tr}(g^{-1}dg) = \sum \omega_{ii} = 0. \qquad (0.2.3.1)$$

Notice that this equation is valid regardless of what parametrization of $SL(m, K)$ we use.

Example 0.2.3.1 Let us make some calculations on $SL(2, \mathbb{R})$. Consider the parametrizations

$$g = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \qquad (0.2.3.2)$$

Note that the first parametrization (ϕ, t, x) is a special case of the Iwasawa decomposition and the second (y, t, x) is only valid in an open (and dense) subset as is familiar from elementary linear algebra for engineers (see also chapter 3, and the discussion of the Bruhat decomposition). Computing $g^{-1}dg$ we obtain matrices of 1-forms

$$\begin{pmatrix} -e^{2t}xd\phi + dt & (e^{2t}x^2 - e^{-2t})d\phi + 2xdt + dx \\ e^{2t}d\phi & e^{2t}xd\phi - dt \end{pmatrix}, \quad \begin{pmatrix} -e^{2t}xdy + dt & e^{2t}x^2dy + 2xdt + dx \\ e^{2t}dy & e^{2t}xdy - dt \end{pmatrix}.$$

$(1,1), (1,2)$ and $(2,1)$ entries of the above matrices form bases for left invariant 1-forms on $SL(2,\mathbb{R})$. While these expressions look complicated, we shall see that one can extract interesting information from these and similar expressions for other groups. ♠

A left invariant volume element together with an orientation defines a left invariant measure on a Lie group which we may assume to be positive. Every locally compact group admits of a unique up to scalar multiplication left (or right) invariant measure (called *Haar measure*), and in the case of Lie groups this can be proven very easily:

Proposition 0.2.3.1 *On a Lie group there is a unique, up to scalar multiplication by a scalar, left invariant volume element.*

Proof - Let $\dim(G) = N$ and $\omega_1, \cdots, \omega_N$ be linearly independent 1-forms appearing as matrix entries of $\omega = g^{-1}dg$. Notice that if these 1-forms are linearly independent at one point, then they are linearly independent everywhere by left invariance. Clearly $dv = \omega_1 \wedge \cdots \wedge \omega_N$ is a left invariant nowhere vanishing N-form on G. Any other volume element dv' will differ from dv at one point by multiplication by a constant c and by left invariance $dv' = cdv$ everywhere. ♣

Example 0.2.3.2 Let us explicitly compute the invariant volume element on $SL(2,\mathbb{R})$. In the (ϕ, t, x) and (y, t, x) parametrizations, the $(1,1)$, $(2,1)$ and $(1,2)$ entries of the matrix ω given in example 0.2.1.1 are linearly independent. Taking their wedge product we obtain the expressions

$$e^{2t}d\phi \wedge dt \wedge dx, \quad \text{and} \quad e^{2t}dy \wedge dt \wedge dx,$$

for the volume element on $SL(2,\mathbb{R})$. ♠

Exercise 0.2.3.1 *Compute the matrix $\omega = g^{-1}dg$ and the left invariant volume element for $GL(2,\mathbb{R})$ relative to the parametrizations*

$$g = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{t_2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{t_2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Exercise 0.2.3.2 *Show that in the parametrization*

$$g = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

the left invariant volume element is $dy \wedge dx \wedge dt$*. What is the corresponding expression for the volume element on* $GL(2, \mathbb{R})$*.*

It is clear that by looking at $dg.g^{-1}$ rather than $g^{-1}dg$ we obtain right invariant 1-forms and consequently a right invariant volume element. In general, for nonabelian groups, right invariant and left invariant volume elements, and *a fortriori* 1-forms, are not identical. In many cases of interest the left and right invariant volume elements turn out to be identical. Groups for which right invariant and left invariant (Haar) measures are identical are called *unimodular*. Since $dg^{-1} = -g^{-1}(dg)g^{-1}$, the transformation $g \rightarrow g^{-1}$ maps left invariant 1-forms to right invariant 1-forms, and unimodular groups have the property that for an integrable function F and every $h \in G$

$$\int_G F(g)dv_G = \int_G F(hg)dv_G(g) = \int_G F(gh)dv_G(g) = \int_G F(g^{-1})dv_G, \qquad (0.2.3.3)$$

where the notation $dv_G(g)$ is intended to emphasize that integration is with respect to the variable g. The simplest non-unimodular group is

Exercise 0.2.3.3 *Let* G *be the group matrices of the form* $\begin{pmatrix} s & x \\ 0 & 1 \end{pmatrix}$ *where* $s > 0$ *and* $x \in \mathbb{R}$ *(the connected component of the group of affine transformations of the line). Show that the left invariant volume element is* $s^{-2}ds \wedge dx$ *while the right invariant volume element is* $s^{-1}ds \wedge dx$*.*

Let us try to understand when a group is unimodular. Let dv_G denote the left invariant volume element on the Lie group G. Then right translation of ω by an element $h \in G$ is obtained from the matrix of 1-forms $(gh)^{-1}d(gh)) = h^{-1}(g^{-1}dg)h$ by taking wedge product of the appropriate entries of this matrix as explained earlier. Denoting this new left invariant volume element by dv_{G^h}, we obtain

$$dv_{G^h} = \Delta(h)dv_G,$$

where $\Delta(h)$ is the determinant of the linear tranformation $\alpha \rightarrow h^{-1}\alpha h$ of $\mathcal{G}^\star$ and therefore Δ is a continuous homomorphism of G to the multiplicative group of real numbers $\mathbb{R}^\times$. Let $[G, G]$ denote the closed normal subgroup generated by the commutators $xyx^{-1}y^{-1}$, then then homomorphism Δ maps $[G, G]$ to 1 and so $\Delta : G/[G, G] \rightarrow \mathbb{R}^\times$. This observation implies

Corollary 0.2.3.1 *Every compact Lie group is unimodular.*

Corollary 0.2.3.2 *The groups $GL(m, K)$ and $SL(m, K)$ are unimodular.*

Proof of corollary 0.2.3.1 - Clearly the only continuous homomorphism of a compact Lie group G to $\mathbb{R}^\times$ takes values in $\{\pm 1\}$ and consequently G is unimodular. ♣
Proof of corollary 0.2.3.2 - Using the fact that for $GL(m, K)$ the left invariant volume element is the wedge product of all the entries of the matrix $g^{-1}dg$, we obtain after a simple calculation

$$\Delta(h) = \det(h)^m \det(h)^{-m} = 1.$$

It is an exercise in linear algebra that all normal subgroups of $SL(m, K)$ are contained in $\{\zeta I\}$ where ζ runs over all m^{th} roots of unity in K. Therefore $[SL(m, K), SL(m, K)] = SL(m, K)$, and consequently $\Delta(h) = 1$. ♣

Let G and N be Lie groups and $\rho : G \rightarrow \text{Aut}(N)$ be a homomorphism where $\text{Aut}(N)$ is the group[3] of continuous automorphisms of N. The *semi-direct product* $N.G$ is the Lie group whose underlying manifold is $N \times G$ and the group operation given by

$$(n, g)(n', g') = (n \star \rho(g)(n), gg'),$$

where $\star$ denotes the group operation on N. Note that if ρ is the trivial homomorphism then semi-direct product becomes direct product. Clearly both N and G are embedded in $N.G$ as (closed) subgroups $\{(n, e_G)\}$ and $\{e_N, g)\}$ respectively. We simply write N and G for these subgroups. N is a normal subgroup of $N.G$ and unless ρ is the trivial homomorphism, G is not a normal subgroup. Note that

$$gng^{-1} = (e_N, g)(n, e_G)(e_N, g^{-1}) = (\rho(g)(n), e_G) = \rho(g)(n).$$

Semi-direct products occur naturally in group theory. For example, for $\mathcal{S}_n$ denoting the symmetric group on n letters we have

Exercise 0.2.3.4 *Show that $\mathcal{S}_3$ is isomorphic to the semi-direct product $N.G$ with $N = \mathbf{Z}/3 = \{0, 1, 2\}$, $G = \mathbf{Z}/2 = \{1, \epsilon\}$, and $\rho(\epsilon)$ the automorphism of $\mathbf{Z}/3$ mapping $2 \rightarrow 1$ and $1 \rightarrow 2$.*

Exercise 0.2.3.5 *Show that $N = \{e, (12)(34), (13)(24), (14)(23)\} \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ is a normal subgroup of $\mathcal{S}_4$ and we have the semi-direct product decomposition $\mathcal{S}_4 \simeq N.\mathcal{S}_3$. Describe explicitly the homomorphism ρ. Show also that $\mathcal{S}_n \simeq \mathsf{A}_n.\mathbf{Z}/2$ where $\mathsf{A}_n \subset \mathcal{S}_n$ is the alternating group.*

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[3] $\text{Aut}(N)$ is itself a Lie group with the compact open topology, but we omit the proof of this fact since in cases of interest to us its validity will be almost immediate.

Example 0.2.3.3 Let $G \subset GL(m, K)$ be any closed subgroup and $N = V = K^m$. Then G acts on N as a group of automorphisms (linear transformations of $V = K^m$) and so the semi-direct product $V.G$ is defined. Note that $V.G$ can be represented as $(m+1) \times (m+1)$ matrices $\begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix}$, where $v \in V = K^m$ is a column vector and $g \in G$ an $m \times m$ matrix since

$$\begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g' & v' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} gg' & v + g(v') \\ 0 & 1 \end{pmatrix}.$$

For $G = GL(m, \mathbb{R})$, $V.G$ is called the group of *affine transformations* of $\mathbb{R}^m$. The group of *Euclidean motions* of $\mathbb{R}^m$ is $V.G$ with $G = O(m)$. Replacing $O(m)$ with $SO(m)$ gives the group of *proper Euclidean* or *rigid motions* of $\mathbb{R}^m$. We use the notation $E(m) = \mathbb{R}^m.O(m)$ and $SE(m) = \mathbb{R}^m.SO(m)$. Clearly, $(v, g)^{-1} = (-g^{-1}(v), g^{-1})$. Denoting a typical element of $V.G$ by h we obtain, in matrix notation,

$$h^{-1}dh = \begin{pmatrix} g^{-1} & -g^{-1}(v) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dg & dv \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g^{-1}dg & g^{-1}(dv) \\ 0 & 0 \end{pmatrix}. \tag{0.2.3.4}$$

This equation enables one to effectively reduce the computation of left invariant 1-forms on $V.G$ to those of G and V. The left invariant volume element $dv_{SE(m)}$ on $SE(m)$ is called the *kinematic density*. Since $\det(g) = 1$ for $g \in SO(m)$ we obtain

$$dv_{SE(m)} = dv_{SO(m)} \wedge dv_1 \wedge \cdots \wedge dv_m, \tag{0.2.3.5}$$

where v_i's are the components of the vector v and $dv_{SO(m)}$ is the left invariant volume element on $SO(m)$. For an arbitrary closed subgroup $G \subset GL(m, \mathbb{R})$ we clearly have

$$dv_{V.G} = \det(g)^{-1} dv_G \wedge dv_1 \wedge \cdots \wedge dv_m. \tag{0.2.3.6}$$

The significance of kinematic density in geometry will become clear in the next chapter. ♠

Exercise 0.2.3.6 *Let $G \subset GL(m, \mathbb{R})$ be a closed subgroup and assume that the left and right invariant volume elements on G are identical (resp. G is unimodular). Compute the right invariant volume element on $V.G$. Show that if* $\det(g) = 1$ *(resp.* $\det(g) = \pm 1$*) then left and right invariant volume elements are identical (resp. $V.G$ is unimodular).*

Exercise 0.2.3.7 *Let U_m be the group of (real) $m \times m$ upper triangular matrices with 1's along the diagonal, $V = \mathbb{R}^{m-1}$. Then U_{m-1} acts on V as group of linear transformations in the natural fashion. Show that $U_m \simeq V.U_{m-1}$ (semi-direct product) and deduce that the left and right invariant volume elements on U_m are identical and given by*

$$dv_{U_m} = \bigwedge_{i<j} du_{ij},$$

where $(u_{ij}) \in U_m$ and du_{ij} denotes the standard 1-form (or Lebesgue measure) on $\mathbb{R}$.

Exercise 0.2.3.8 *Show that the 1-forms* $\frac{dz_j}{z_j}$ *are invariant on the multiplicative abelian group* $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ *and an invariant volume element is* $\frac{1}{(-2i)^m}\frac{dz_1\wedge d\bar{z}_1}{|z_1|^2}\wedge\cdots\wedge\frac{dz_m\wedge d\bar{z}_m}{|z_m|^2}$. *Let* T *be the group of complex* $m\times m$ *upper triangular matrices. Prove that*

$$T \simeq (\mathbb{C}^\times)^m.U_{\mathbb{C}} \quad (\text{semi}-\text{direct product}).$$

Specify the action of $(\mathbb{C}^\times)^m$ *on* $U_{\mathbb{C}}$, *and deduce that a left invariant volume element on* T *is*

$$dv_T = \frac{1}{(-2i)^{\frac{m(m+1)}{2}}}\frac{dz_1\wedge d\bar{z}_1}{|z_1|^2}\wedge\cdots\wedge\frac{dz_m\wedge d\bar{z}_m}{|z_m|^2}\bigwedge_{j<k}(dz_{jk}\wedge d\bar{z}_{jk}),$$

where we have used the parametrization representing a matrix in T *as* $a.u$ *with* a *an invertible complex diagonal matrix with eigenvalues* z_j, *and* $u=(z_{jk})$ *an upper triangular matrix all whose eigenvalues are 1. Derive a similar formula for the real case.*

Example 0.2.3.4 Let us compute a (left) invariant volume element for $GL(m,\mathbb{R})$ using the paramerization given by the Iwasawa decomposition. Writing $g = kau \in SO(m)AU$ we obtain

$$g^{-1}dg = u^{-1}a^{-1}(k^{-1}dk)au + u^{-1}(a^{-1}da)u + u^{-1}du. \qquad (0.2.3.7)$$

From the defining relation $k'k = I$ for $SO(m)$ it follows that $k'dk + (dk')k = 0$, i.e., $k^{-1}dk$ is a skew-symmetric matrix of 1-forms. Consequently, the wedge product of its entries below the diagonal is an invariant volume element for $SO(m)$. Denoting the diagonal entries of a by $e^{t_1},\cdots,e^{t_m}$, the first term on the right hand side of (0.2.3.7) becomes (the not necessarily skew-symmetric matrix)

$$u^{-1}a^{-1}(k^{-1}dk)au = \begin{pmatrix} \star & \star & \cdots & \star \\ \gamma_{21}+e^{t_1-t_2}\omega_{21} & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \star \\ e^{t_1-t_m}\omega_{m1} & \gamma_{m2}+e^{t_2-t_m}\omega_{m2} & \cdots & \star \end{pmatrix},$$

where $\star$'s are unspecified linear combinations of ω_{ij}'s, and γ_{ij}, $(i>j)$, is a linear combination of only those ω_{kl}'s for which $k-l > i-j$. (Compute the matrix $u^{-1}Xu$ where $u\in U$ and X is arbitrary.) In particular, $\gamma_{m1}=0$. Similarly, the matrix $u^{-1}(a^{-1}da)u$ is upper triangular with diagonal entries $dt_1,\cdots,dt_m$ and entries above the diagonal certain linear combinations of $dt_1,\cdots,dt_m$. Finally $u^{-1}du$ is upper triangular with zeros along the diagonal and the wedge product of its entries above the diagonal is the invariant volume element on U. An invariant volume element on $GL(m,\mathbb{R})$ is the wedge product of all the entries of $g^{-1}dg$. Therefore, starting with ω_{m1} taking wedge product with $\gamma_{m-11}+\omega_{m-11}$ and $\gamma_{m2}+\omega_{m2}$ etc. we see that γ_{ij}'s do not contribute to the product, and we easily obtain

$$dv_{GL(m,\mathbb{R})} = (\prod_{i<j} e^{t_i-t_j})dv_{SO(m)} \wedge dv_A \wedge dv_U. \qquad (0.2.3.8)$$

It is easy to see that (0.2.3.8) is valid for $SL(m,\mathbb{R})$ as well. ♠

Exercise 0.2.3.9 *Show that the invariant volume element on $GL(m,\mathbb{C})$ or $SL(m,\mathbb{C})$ is given by*

$$(\prod_{i<j} e^{2(t_i-t_j)})dv_K \wedge dv_A \wedge dv_{U_{\mathbb{C}}},$$

where $K = U(m)$ or $SU(m)$.

Exercise 0.2.3.10 *Consider the parametrization of an open dense subset of $GL(m,\mathbb{R})$ as $U'AU$ where U' is the group of lower triangular matrices with 1's along the diagonal. Show that relative to this parametrization an invariant volume element is*

$$dv_{GL(m,\mathbb{R})} = (\prod_{i<j} e^{t_i-t_j})dv_{U'} \wedge dv_A \wedge dv_U.$$

Show that the same formula is valid for $SL(m,\mathbb{R})$. Derive similar expressions for $GL(m,\mathbb{C})$ and $SL(m,\mathbb{C})$.

Example 0.2.3.5 There is a classical parametrization of the group $SO(3)$ known as Euler angles which was motivated by the study of the rotational motion of a rigid body. Let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$, $R \in SO(3)$ and $f_i = R(e_i)$. Denote by θ the angle between e_3 and f_3 which we take it to be in $[0,\pi]$. The plane $f_1 \wedge f_2$ intersects the plane $e_1 \wedge e_2$ along a line L (called *line of nodes* in physics literature). We fix a unique unit vector l along L by the requirement that $l, f_3 \times l$ defines the same orientation as f_1, f_2 and $f_3, l, f_3 \times l$ is positively oriented basis for $\mathbb{R}^3$. Let ϕ be the angle between e_1 and l and ψ the angle between l and f_1. Then $\phi, \psi \in [0, 2\pi]$. The quantities θ, ϕ, ψ are called the *Euler angles.* To see how $SO(3)$ is parametrized by the Euler angles, first rotate space by θ in the $e_1 \wedge e_3$ plane fixing e_2. Next rotate space in $e_1 \wedge e_2$ plane (fixing e_3) through $\phi - \frac{\pi}{2}$. Let f'_i be the position of e_i after application of these two rotations. Finally rotate space through angle $\frac{\pi}{2} + \psi$ in $f'_1 \wedge f'_2$ plane (fixing f'_3). The result of these three transformations is given by

$$\begin{pmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Denoting the above product by g, we obtain after a simple calculation

$$g^{-1}dg = \begin{pmatrix} 0 & \cos\theta d\phi - d\psi & -\cos\psi\sin\theta d\phi - \sin\psi d\theta \\ -\cos\theta d\phi + d\psi & 0 & \sin\psi\sin\theta d\phi - \cos\psi d\theta \\ \cos\psi\sin\theta d\phi + \sin\psi d\theta & -\sin\psi\sin\theta d\phi + \cos\psi d\theta & 0 \end{pmatrix}$$

Taking the wedge product of entries above the diagonal we obtain the expression

$$dv_{SO(3)} = \sin\theta d\theta \wedge d\phi \wedge d\psi.$$

for the volume element of $SO(3)$ relative to Euler angles. This formula is of interest in the study of dynamics of rigid bodies. ♠

Example 0.2.3.6 In practice it becomes essential to integrate *class functions* ψ (i.e., functions ψ satisfying $\psi(hgh^{-1}) = \psi(g)$ for all $h, g \in G$) on the group G. The expression for Haar measure simplifies in this case. To see how this simplification comes about let us consider the case of the unitary group $G = U(n)$. Since every unitary matrix is diagonalizable, we consider the parametrization of $U(n)$ given by

$$G/T \times T \longrightarrow U(n), \quad (uT, t) \to utu^{-1},$$

where $T \subset U(n)$ is the subgroup of diagonal matrices. We note that mapping is not injective. On the set of *regular* elements (i.e., unitary matrices with distinct eigenvalues) the map in $n!$ to 1 since permutation matrices[4] are unitary. Since the complement of this set has measure zero, we can ignore the complement for the purpose of this calculation. Let u denote a variable point on the flag manifold G/T, then

$$(utu^{-1})^{-1}d(utu^{-1}) = u[t^{-1}(u^{-1}du)t + (t^{-1}dt) - u^{-1}du]u^{-1}.$$

To express the Haar measure relative to this parametrization, we simply take the wedge product of the entries of the matrix $(utu^{-1})^{-1}d(utu^{-1})$ which are the left invariant 1-forms on $U(n)$. The effect of conjugation by u on the volume element is by multiplication by

[4]Consider $\mathbb{R}^n$ with the (standard) basis $e_1, \cdots, e_n$. The symmetric group $\mathcal{S}_n$ acts on $\mathbb{R}^n$ by permuting the basis vectors. Matrices representing these transformations are called *permutation matrices* and are characterized by the properties

1. All entries are 0 or 1;
2. Every row or column contains exactly one 1.

Conjugation of a diagonal matrix by a permutation matrix has the effect of permuting the digonal entries. In particular, permutation matrices lie in the normalizer of digonal matrices.

$\det(u^n)\det(u^{-n}) = 1$, and therefore to compute the volume element it suffices to take wedge product of the entries of the matrix $t^{-1}(u^{-1}du)t - u^{-1}du + (t^{-1}dt)$. Let $u^{-1}du = (\omega_{jk})$ which is a skew hermitian matrix of left invariant 1-forms. Let t denote the diagonal matrix with diagonal entries $e^{i\varphi_k}$, for $k = 1, \cdots, n$. The differentials $d\varphi_j$ appear only in $t^{-1}dt$. Then it is a simple matter to see that the wedge product of entries of $t^{-1}(u^{-1}du)t - u^{-1}du + (t^{-1}dt)$ yields the expression

$$[\prod_{j<k}(e^{i(\varphi_j-\varphi_k)} - 1)(e^{i(\varphi_k-\varphi_j)} - 1)\omega_{jk} \wedge \bar{\omega}_{jk}] \wedge d\varphi_1 \wedge \cdots \wedge d\varphi_n,$$

where c is a constant depending only on n. Now if a function ψ on G is invariant under conjugation then the $(n^2 - n)$-form $\prod \omega_{jk} \wedge \bar{\omega}_{jk}$, which depends only on the variable u, can be integrated out to obtain the important formula

$$\int_{U(n)} \psi(g)dg = c\int_{\circ}^{2\pi} \cdots \int_{\circ}^{2\pi} \prod_{j<k} |e^{i\varphi_j} - e^{i\varphi_k}|^2 \psi(t) d\varphi_1 \wedge \cdots \wedge d\varphi_n, \qquad (0.2.3.9)$$

for some constant $c > 0$ which depends on the normalization of measures. The constant c will be determined explicitly in the subsection on Characters. One can similarly derive analogous formulae for othe compact groups. ♠

0.2.4 Finite Subgroups of $O(3)$

Finite subgroups of $O(2)$ are easy to determine. In fact we have:

Exercise 0.2.4.1 *Show that a finite subgroup of order n of $SO(2)$ is cyclic and is generated by the rotation through angle $\frac{2\pi}{n}$. Deduce that a finite subgroup of $O(2)$ is either cyclic, or is the dihedral group of order $2n$ generated by rotation through angle $\frac{2\pi}{n}$ and a reflection conjugate to* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The determination of finite subgroups of $SO(3)$ and $O(3)$ is more difficult. The characteristic roots of an orthogonal matrix are roots of unity and occur in conjugate pairs. Therefore the eigenvalues of a 3×3 orthogonal matrix A are $e^{\pm i\theta}, \pm 1$ where ± 1 occurs as a root according as $\det A = \pm 1$. Therefore if $\det A = 1$, then A fixes a line (called *axis of rotation*) and is a rotation in the plane orthogonal to the axis (called *plane of rotation*). For $I \neq A \in SO(3)$, the unit vectors $\pm v$ which are eigenvectors for eigenvalue 1 are called the *poles* of $A \in SO(3)$. For a finite subgroup $W \subset O(3)$, we denote by P_W the set of poles of non-identity elements of $W' = W \cap SO(3)$. The key point which makes the determination of finite subgroups of $O(3)$ possible is the following observation:

Lemma 0.2.4.1 *The action of W on $\mathbb{R}^3$ preserves the set of poles P_W.*

Proof - Follows from the fact that for $A, B \in W$, $v \in \mathsf{P}_W$, and $Av = v$, Bv is fixed by BAB^{-1}. ♣

Let $X = \{(A, v) \mid I \neq A \in W \cap SO(3),\ v \text{ pole for } A\}$. Then W acts on the finite set X by

$$(A, v) \xrightarrow{B} (BAB^{-1}, Bv).$$

Under the action of W, P_W splits as a union of l orbits and we choose representatives $v_1, \cdots, v_l$ one for each orbit. Let $W' = W \cap SO(3)$, and W'_v denote the isotropy subgroup of v in W'. Denoting the order of W' by w, we see immediately that $|X| = 2(w-1)$. On the other hand, we can calculate this number by looking at the action of W' on P_W. Note that

$$|X| \ = \sum_{v \in \mathsf{P}_W} (|W'_v| - 1). \tag{0.2.4.1}$$

The right hand side of (0.2.4.1) can be reorganized in terms of the orbits of the action of W' on P_W. To do so let $w_i = |W'_{v_i}|$ and $n_i = \frac{w}{w_i}$ be the cardinality of the orbit of v_i. Then right hand side of (0.2.4.1) becomes

$$\sum_{i=1}^{l} n_i(w_i - 1) = wl - \sum_{i=1}^{l} n_i.$$

Since $|X| = 2(w-1)$, this equation together with (0.2.4.1) yield

$$2 - \frac{2}{w} = l - \sum_{i=1}^{l} \frac{1}{w_i}. \tag{0.2.4.2}$$

This is the basic equation which makes the determination of finite subgroups of $O(3)$ and $SO(3)$ possible. Since $w_i \geq 2$, (0.2.4.2) implies

$$l \ = \ 2 \text{ or } 3.$$

For $l = 2$, (0.2.4.2) becomes

$$\frac{2}{w} = \frac{1}{w_1} + \frac{1}{w_2}. \tag{0.2.4.3}$$

Since $w_i | w$, the only solution is

$$w_1 = w_2 = w.$$

This implies that for $l = 2$, W' is a cyclic group of rotations in the plane orthogonal to the line through the poles $\pm v$.

Consider the case $l = 3$. Recalling that $n_i = \frac{w}{w_i}$ is an integer, one easily obtains the complete set of solutions to (0.2.4.2). Arranging the solutions such that $w_1 \leq w_2 \leq w_3$ we obtain four sets of solutions which are exhibited in the following table:

Solution Set	w	w_1	w_2	w_3	n_1	n_2	n_3
1	$2m$	2	2	m	m	m	2
2	12	2	3	3	6	4	4
3	24	2	3	4	12	8	6
4	60	2	3	5	30	20	12

It is not difficult to give geometric meaning to the solutions. We analyze sets 1 and 4 and leave cases 2 and 3 to exercises 0.2.4.2 and 0.2.4.3 below. In case one consider the pole v_3 whose isotropy subgroup W'_{v_3} is a group of order m. The line through $\pm v_3$ is fixed by W'_{v_3}, so W_{v_3} is a cyclic group of rotations of the plane Π orthogonal to v_3. Thus W'_{v_3} is realized as the group of rotations of a regular m-gon Δ_m in $\mathbb{R}^2$. There is only one orbit of poles whose isotropy subgroups are cyclic of order m, namely, $\pm v_3$. Thus if $\sigma \in W' \backslash W'_{v_3}$, then $\sigma \in W'_{v_1} \cup W'_{v_2}$, and σ has order 2 since $w_1 = w_2 = 2$ in case 1. It follows that σ is a rotation through angle π. Let $\zeta \in W'_{v_3}$ be a generator, then $\sigma\zeta \notin W'_{v_3}$ and is also a rotation through π. By looking at the regular m-gon Δ_m we see that the rotations σ and $\sigma\zeta$ have their axes in the plane Π and that they make angle $\frac{\pi}{m}$. This geometric construction gives an injective homomorphism of the dihedral group of order $2m$ into $SO(3)$. Analytically it can be described as the mapping

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $\theta = \frac{2\pi}{m}$.

Next we consider the fourth solution set from the table. Here the subgroup W'_{v_3} has order 5 and therefore the orbit of v_3 under W' has cardinality 12. There are ten poles other than $\pm v_3$ which are permuted by W'_{v_3}. These ten poles cannot lie on an equator. In fact consider a rotation through angle $\frac{2\pi}{5}$ whose axis contains one (and therefore necessarily two) of these poles. This generates more than twelve poles with isotropy subgroup cyclic of order 5 which is not possible. These ten poles are permuted by W'_{v_3}, and are split up into two orbits of cardinality five. Each set of five is equidistant from v_3 and $-v_3$ since $\pm v_3$ is fixed W'_{v_3}. The representative v_3 was arbitrary, the same is true for every pole with isotropy subgroup of order 5. Thus we can realize the twelve poles as the vertices of icosahedron and W' is a group of proper symmetries of it. The remaining poles are the mid-points of the edges and the centroids of faces. Since $w = 60$, W' is the group of proper symmetries of the icosahedron which is isomorphic to $\mathcal{A}_5$.

Exercise 0.2.4.2 *Show that the second solution set of the table gives the group of proper symmetries of the regular tetrahedron.*

Exercise 0.2.4.3 *Show that the third solution set of the table gives the group of proper symmetries of the cube or the regular octahedron.*

We summarize the above analysis as

Proposition 0.2.4.1 *A finite subgroup of $SO(3)$ is conjugate to one of the following:*

1. *A cyclic group of order m of rotations.*
2. *A dihedral group of order $2m \geq 4$.*
3. *The group of order twelve ($\simeq \mathcal{A}_4$) of proper symmetries of the regular tetrahedron.*
4. *The group B_3' of order twenty four of proper symmetries of the cube or the regular octahedron.*
5. *The simple group of order sixty ($\simeq \mathcal{A}_5$) of proper symmetries of the icosahedron.*

A finite subgroup $W \subset O(3)$ is either one of the above or contains one of the above as a subgroup of index two.

0.3 Special Tensors and Geometric Structures

0.3.1 Metrics and Volume Elements

Let $E \odot E \to M$ be the vector bundle associated to the principal bundle $P_E \to M$ via the second symmetric power representation $GL(k, \mathbb{R})$. A section g of $E^\star \odot E^\star \to M$ defines a (possibly indefinite and degenerate) inner product on the fibres of $E \to M$ in the obvious manner. If furthermore the inner product $g(x)$ is positive definite for every $x \in M$, we say g is a *metric*. Metrics always exist. In fact, let $\mathcal{U} = \{U_\alpha\}$ be a covering of M such that $E_\alpha = \pi^{-1}(U_\alpha) \simeq U_\alpha \times F$. Let g_α be a metric for the trivial vector bundle $\pi^{-1}(U_\alpha) \to U_\alpha$, i.e., a section of the trivial bundle $E^\star_\alpha \odot E^\star_\alpha \to M$. Let $\{\phi_\alpha\}$ a partition of unity subordinate to $\mathcal{U}$. Then $\sum \phi_\alpha g_\alpha$ is a section of the bundle $E^\star \odot E^\star \to M$, and is positive definite on every fibre of $E \to M$. Similarly, for a complex vector bundle we have the notion and existence of an hermitian metric. Now given a real rank k vector bundle $E \to M$ with a fixed metric g, there is the associated bundle $P_g \to M$ of orthonormal frames. This is a principal bundle with group $G = O(k)$. If $E \to M$ is orientable, we can fix an orientation and only consider positively oriented orthonormal frames. This gives us a new principal bundle with group $G = SO(k)$. A metric for the tangent bundle of M is called a *Riemannian metric*. A manifold M together with a Riemannian metric (generally denoted by g or ds^2) is called a *Riemannian manifold*.

Example 0.3.1.1 Once a metric is fixed on a vector bundle $E \to M$, then it makes sense to consider orthonormal frames for the bundle. The set of orthonormal frames for the vector bundle $E \to M$ is a principal bundle $(O(k), P, \pi, M)$. By considering change of orthonormal frames, we obtain a set transition functions $\rho_{\alpha\beta}$ for $E \to M$ which take values in the orthogonal group $O(k)$. If in addition an orientation is fixed for $E \to M$, then we can restrict ourselves to positively oriented frames and consequently the transition functions take values in $SO(k)$. Therefore a metric allows one to obtain transition functions taking values in the orthogonal group, and an orientation makes it possible to choose transitions functions with positive determinant. This process of choosing transition functions in a way that they take values in a subgroup of $GL(k, K)$ is called *reduction of the structure group.* The subset of E consisting of points

$$\{(x, v) \mid v \in E_x, \text{ and } < v, v >_x = 1\}$$

is meaningful since we have a metric $< ., . >_x$ on each fibre E_x. This is a called the associated *unit sphere bundle*. Similarly we can replace the condition $< v, v >= 1$ with $< v, v > \leq h(x)$, where h is a positive function (possibly identically 1) on M to obtain a *ball bundle*. Notice that the specification of a metric is the tool which enables one to obtain a sphere or a ball bundle from a vector bundle. ♠

Exercise 0.3.1.1 *Let $E \rightarrow M$ be a k-plane bundle and let ω be a nowhere vanishing section of $\wedge^k E^\star \rightarrow M$ (see subsection Orientation and Volume Element). Show that ω enables one to choose transition functions for $E \rightarrow M$ taking values in $SL(k, \mathbb{R})$.*

The space of all possible Riemannian metrics on a manifold is very large, however in specific circumstances there are preferred metrics. For example, for submanifolds of $\mathbb{R}^N$ it is natural to use the induced metric from the ambient space. Let us clarify this point. The metric on $\mathbb{R}^N$ is given by $ds^2 = \sum_{A=1}^N dx_A^2$. Note that this is a section of the second symmetric power of the cotangent bundle of $\mathbb{R}^N$. Its significance is that given a C^1 curve $\gamma : I \rightarrow \mathbb{R}^N$, where $I = [0, 1]$, the arc length of γ is computed by the formula

$$\int_0^1 \sqrt{\sum_{A=0}^N \left(\frac{d\gamma_A(t)}{dt}\right)^2} dt,$$

where $\gamma(t) = (\gamma_1(t), \cdots, \gamma_N(t))$. Note that $\sum_{i=0}^N (\frac{d\gamma_A(t)}{dt})^2$ is the result of the evaluation of the quadratic form ds^2 on the tangent vector to the curve γ, i.e. $ds^2(\dot{\gamma})$. Let $f : M \rightarrow \mathbb{R}^N$ be a submanifold. Then, as described earlier, $f^\star(ds^2) = ds_M^2$ is a positive definite symmetric covariant 2-tensor on M, i.e., a Riemannian metric. If f is given by the expression $x_A = x_A(u_1, \cdots, u_m)$, relative to the standard coordinates in $\mathbb{R}^m$ and $\mathbb{R}^N$, for $A = 1, \cdots, N$, then ds_M^2 is obtained by substituting

$$dx_A = \sum_{i=0}^m \frac{\partial x_A}{\partial u_i} du_i,$$

in $ds^2 = \sum_{A=1}^N dx_A^2$. This is the induced metric on M, or more precisely $f(M)$, which depends on the embedding f. Therefore it has the general form $ds_M^2 = \sum_{i=1}^m g_{ij} du_i du_j$ where $g_{ij} = g_{ji}$, and the symmetric positive definite matrix $g = (g_{ij}) = (Df)' Df$. Here Df is the derivative of f (relative to $u_1, \cdots, u_m$) which is an $N \times m$ matrix, and superscript $'$ denotes the transposed matrix. Thus if we represent a curve $\gamma : I \rightarrow M$ in terms of the coordinate system $(u_1, \cdots, u_m)$, i.e., $\gamma(t) = (\gamma_1(t), \cdots, \gamma_m(t))$, then its arc length is

$$\int_0^1 \sqrt{\sum_{i,j=1}^m g_{ij}(\gamma(t)) \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt}} dt. \qquad (0.3.1.1)$$

A Riemannian metric on a manifold M (not necessarily embedded in $\mathbb{R}^N$) is generally written as

$$ds^2 \sum_{i,j} g_{ij} du_i du_j$$

relative to a coordinate system $u_1, \cdots, u_m$, where $g = (g_{ij})$ is a symmetric positive definite matrix. Its relation to the computation of arc length is given by (0.3.1.1).

Since a non-degenerate bilinear form defines an isomorphism between a vector space and its dual, a Riemannian metric defines an isomorphism between the tangent and cotangent bundles of a manifold. Furthermore, an orientation for M (i.e., for $\mathcal{T}M$) gives an orientation for $\mathcal{T}^\star M$ for example by requiring the basis dual to a positively oriented basis for $\mathcal{T}_x M$ to be positively oriented. There is a volume element associated to a Riemannian metric on an oriented manifold M. Let us fix an orientation for M, and a let $\mathcal{U} = \{U_\alpha\}$ be a covering of M such that the (co)tangent bundle of M is trivial on each U_α. Then for each α we can choose a basis of 1-forms $(\omega_1^\alpha, \cdots, \omega_m^\alpha)$ such that $(\omega_1^\alpha(x), \cdots, \omega_m^\alpha(x))$ is a positively oriented orthonormal basis for each $\mathcal{T}_x^\star M$, $x \in U_\alpha$. Thus $ds_M^2 = \sum(\omega_i^\alpha)^2$. Set $dv^\alpha = \omega_1^\alpha \wedge \cdots \wedge \omega_m^\alpha$. Two choices of such basis of 1-forms differ by a special orthogonal transformation $A(x) \in SO(m)$ at each point $x \in U_\alpha$. Therefore if $x \in U_\alpha \cap U_\beta$ then $dv^\alpha = \det(A) dv^\beta = dv^\beta$. Hence we can simply state

$$dv = \omega_1 \wedge \cdots \wedge \omega_m \tag{0.3.1.2}$$

is a volume element on M. This is the *volume element associated to a Riemannian metric* on an oriented manifold. In terms of the matrix $g = (g_{ij})$ it has the expression

$$dv = \sqrt{\det g}\, du_1 \wedge \cdots \wedge du_m. \tag{0.3.1.3}$$

Example 0.3.1.2 Consider the sphere $S_r^{n-1} \subset \mathbb{R}^n$ of radius $r > 0$. In spherical polar coordinates it is described by

$$\begin{array}{ll} x_1 = r\cos\varphi_1 & 0 \le r < \infty \\ x_2 = r\sin\varphi_1\cos\varphi_2 & 0 \le \varphi_k \le \pi,\ 1 \le k \le n-2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & 0 \le \varphi_{n-1} < 2\pi \\ x_n = r\sin\varphi_1\cdots\sin\varphi_{n-1} & \end{array}$$

Then the Riemannian metric on S_r^{n-1} has the expression

$$ds^2 = r^2(d\varphi_1^2 + \sin^2\varphi_1 d\varphi_2^2 + \sin^2\varphi_1\sin^2\varphi_2 d\varphi_3^2 + \cdots + \sin^2\varphi_1\cdots\sin^2\varphi_{n-2}d\varphi_{n-1}^2);$$

and the corresponding volume element is

$$dv = r^{n-1}\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\cdots\sin\varphi_{n-2}d\varphi_1\wedge\cdots\wedge d\varphi_{n-1}.$$

Integrating dv we obtain the volume of S^{n-1}

$$\mathrm{vol}(S_r^{n-1}) = r^{n-1}\frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Taking wedge product of dv with dr and integrating we obtain

$$\mathrm{vol}(B_r^n) = r^n \frac{2\pi^{n/2}}{n\Gamma(n/2)} = r^n \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)},$$

as the volume of the ball of radius $r > 0$. ♠

The following exercises demostrate some rather surprising properties of the volumes of the balls B_r^n for n large.

Exercise 0.3.1.2 *Use Stirling's formula to show that the volume of the ball of radius 1, for n large, is approximately*

$$(\frac{2\pi e}{n})^{n/2}.$$

Deduce that, for n large, the radius of the ball of unit volume is approximately

$$\sqrt{\frac{n}{2\pi e}}.$$

(Notice that this means that the ball of unit volume has arbitrarily large radius as $n \to \infty$, and the the volume of the ball of radius $r = 1$ goes to zero very fast as $n \to \infty$.)

Exercise 0.3.1.3 *Let $B^n \subset \mathbb{R}^n$ denote the ball of unit volume centered at the origin, and H_t denote the half space $x_1 \leq t$. Define $\Phi(t) = \mathrm{vol}(B^n \cap H_t)$. Show that as $n \to \infty$, $\Phi(t)$ approaches*

$$(\frac{e}{2\pi})^{1/4} \int_{-\infty}^{t} e^{-\pi e s^2} ds.$$

Deduce that for every $\epsilon > 0$ the volume of the portion of the ball B^n which lies in the region $-\epsilon \leq x_1 \leq \epsilon$ approaches a positive limit as $n \to \infty$. In particular, for $\epsilon = \frac{1}{2}$, this limit is about .96. (This exercise can be re-stated as the volumes of infinitesimally thin slabs of B_n bounded by affine hyperplanes intersecting B_n is approximately given by the normal curve with mean 0 and variance $\frac{1}{2\pi e}$ as $n \to \infty$.)

Example 0.3.1.3 Let F be a C^2 real valued function on $\mathbb{R}^{m+1}$ with rank$DF = 1$ everywhere on the zero set $F(x_1, \cdots, x_{m+1}) = 0$ which consequently defines a C^1 hypersurface $M \subset \mathbb{R}^{m+1}$. If $\frac{\partial F}{\partial x_{m+1}}(\mathbf{x}) \neq 0$, then in a neighborhood of $\mathbf{x}$ we can represent the set M as the graph of a function $x_{m+1} = x_{m+1}(x_1, \cdots, x_m)$ (see example 0.1.1.3), and the derivative of x_{m+1} as a function $(x_1, \cdots, x_m)$ is expressible in terms of partial derivatives of F as given in example 0.1.1.3. It follows that the Riemannian metric on M (i.e., the restriction of $dx_1^2 + \cdots + dx_{m+1}^2$ to M) is given by the matrix $g = (g_{ij})$ where

$$g_{ij} = \delta_{ij} + \frac{1}{(\frac{\partial F}{\partial x_{m+1}})^2} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j}.$$

For $m = 2$, $\sqrt{\det g} dx_1 \wedge \cdots \wedge dx_m$ simplifies to

$$\frac{\sqrt{(\frac{\partial F}{\partial x_1})^2 + (\frac{\partial F}{\partial x_2})^2 + (\frac{\partial F}{\partial x_3})^2}}{|\frac{\partial F}{\partial x_3}|} dx_1 \wedge dx_2.$$

as the element of area on the surface $F = 0$. The maximality of rank condition on DF implies that a normal direction

$$\mathrm{grad} F = (\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_{m+1}})$$

is defined everywhere on M. Consequently M is orientable and the normal bundle of M (which is a line bundle) is trivial since it has a nowhere vanishing section $\mathrm{grad} F$. ♠

Example 0.3.1.4 Let $Q = I^m \subset \mathbb{R}^m$ denote the unit cube, and ω be a nonwhere vanishing m-form on Q. A diffeomorphism $\varphi : Q \to Q$ transforms the Euclidean volume element $dv = dx_1 \wedge \cdots \wedge dx_m$ to $\varphi^\star(dv)$. One may ask under what conditions on ω there is a diffeomorphism φ such that

$$\omega = \varphi^\star(dv). \tag{0.3.1.4}$$

An obvious necessary condition is

$$\int_Q \omega = \int_Q dv = 1. \tag{0.3.1.5}$$

By a simple argument we show that (0.3.1.5) is also sufficient[5]. We set $\omega = f(t_1, \cdots, t_m) dt_1 \wedge \cdots \wedge dt_m$ where f is a positive function on Q. Define m positive functions $h_1(t_1), h_2(t_1, t_2), \cdots, h_m(t_1, \cdots, t_m)$, $0 \leq t_j \leq 1$, as follows: Set

$$h_1(t) = \int_\circ^1 \cdots \int_\circ^1 f(t, x_2, \cdots, x_m) dx_2 \wedge \cdots \wedge dx_m,$$

[5]In essence this example has long been known to statisticians who routinely use it for transforming a distribution to the uniform measure on $[0, 1]^n$.

and inductively define h_j's by

$$h_1(t_1)h_2(t_1,t_2)\cdots h_j(t_1,\cdots,t_j)=\int_\circ^1\cdots\int_\circ^1 f(t_1,\cdots,t_j,x_{j+1},\cdots,x_m)dx_{j+1}\wedge\cdots\wedge dx_m. \tag{0.3.1.6}$$

Now set $\varphi(t_1,\cdots,t_m)=(x_1,\cdots,x_m)$ where

$$x_j(t_1,\cdots,t_m)=\int_\circ^{t_j} h(t_1,\cdots,t_{j-1},t)dt.$$

Since x_j is a function of $(t_1,\cdots,t_j)$ only, the derivative of φ is a triangular matrix and its determinant (Jacobian) is

$$\frac{\partial x_1}{\partial t_1}\frac{\partial x_2}{\partial t_2}\cdots\frac{\partial x_m}{\partial t_m}=h_1(t_1)h_2(t_1,t_2)\cdots h_m(t_1,\cdots,t_m).$$

In other words, (0.3.1.4) is valid. Integrating (0.3.1.6) on $[0,1]$ with respect to t_j and using the definition of $h_1,\cdots,h_{j-1}$ we obtain

$$x_j(t_1,\cdots,t_{j-1},1)=1,\quad j=1,\cdots,m.$$

Now it follows easily that φ is a diffeomorphism of Q mapping the boundary onto itself. Our analysis implies that if ω and ω' are volume elements on Q such that

$$\int_Q\omega=\int_Q\omega',$$

then there is a diffeomorphism φ of Q, mapping the boundary onto the boundary, such that $\varphi^\star(\omega')=\omega$. ♠

The notion of Lie dervative is quite useful in differential geometry. Let ξ be a vector field on a manifold M, φ_t the corresponding one parameter family of diffeomorphisms, defined for $t\in(-\epsilon,\epsilon)$, and ω a contravariant tensor field. Define the *Lie derivative* of ω relative to ξ as

$$\mathsf{L}_\xi(\omega)=\lim_{t\to\circ}\frac{\varphi_{-t}^\star(\omega)-\omega}{t}.$$

To compute the effect of Lie derivative on forms we introduce the notion of *interior multiplication* $\imath_X$ mapping p-forms to $(p-1)$-forms defined as

$$\imath_\xi(\omega)(\eta_1,\cdots,\eta_{p-1})=\omega(\xi,\eta_1,\cdots,\eta_{p-1}).$$

Then we have the formal identity (known as *H. Cartan's formula*) for Lie derivative of p-forms

$$\mathsf{L}_\xi(\omega) = (d\imath_\xi + \imath_\xi d)(\omega). \tag{0.3.1.7}$$

The proof of this important relation is a formal calculation and is omitted (see, e.g. [KN]). The operator of interior differentiation is an anti-derivation on forms in the sense that

$$|\imath_\xi(\omega \wedge \omega') = (\imath_\xi \omega) \wedge \omega') + (-1)^{\deg \omega} \omega \wedge \omega'. \tag{0.3.1.8}$$

Example 0.3.1.5 The vanishing of the Lie derivative of a 1-form relative to a vector field has an interpretation in terms of integrating factors in elementary differential equations. Let $\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ be a vector field on an open subset $U \subset \mathbb{R}^2$ which may be written as the differential equation $bdx - ady = 0$. Let α be a 1-form such that $\alpha(\xi)$ is nowhere vanishing on U. Assume that $\mathsf{L}_\xi(\alpha) = 0$, then in view of (0.3.1.7) we have $\imath_\xi d\alpha = -d(\alpha(\xi))$. Now $d\alpha \wedge \alpha$, being a 3-form, vanishes identically on $\mathbb{R}^2$ and in view of (0.3.1.8) we have $\imath_\xi d\alpha \wedge \alpha = -\alpha(\xi) d\alpha$. Therefore

$$d(\frac{1}{\alpha(\xi)}\alpha) = \frac{1}{\alpha(\xi)^2}(\imath_\xi d\alpha) \wedge \alpha + \frac{1}{\alpha(\xi)} d\alpha = 0. \tag{0.3.1.9}$$

Hence $\frac{1}{\alpha(\xi)}$ is an integrating factor for the differential equation $\alpha = Mdx + Ndy = 0$ if $\mathsf{L}_\xi(\alpha) = 0$. ♠

The notion of Lie derivative allows us to generalize the notion of divergence from advanced calculus to manifolds with a given volume element ω. For a vector field ξ on M define $\mathrm{div}(\xi)$ by

$$\mathsf{L}_\xi(\omega) = (\mathrm{div}(\xi))\omega.$$

Note that $\mathrm{div}(\xi)$ is a function on M. It is clear that the volume element ω is invariant under the one parameter family φ_t if and only if $\mathsf{L}_\xi(\omega) = 0$.

Exercise 0.3.1.4 *Let $M \subset \mathbb{R}^m$ be an open set and $\omega = e^\rho dx_1 \wedge \cdots \wedge dx_m$ where ρ is a function on M. For a vector field $\xi = \sum \xi_j \frac{\partial}{\partial x_j}$, use H. Cartan's formula to prove*

$$\mathrm{div}(X) = e^{-\rho} \sum_{j=1}^m \frac{\partial(e^\rho \xi_j)}{\partial x_j}.$$

Example 0.3.1.6 Consider the two dimensional torus $T^2 = \mathbb{R}^2/\mathbf{Z}^2$ with the volume element $\omega = dx_1 \wedge dx_2$. The one parameter group φ_t generated by a vector field $\xi = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2}$, leaves the volume element ω invariant if and only if

$$\frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} = 0. \tag{0.3.1.10}$$

The solutions to this partial differential equation are easy to obtain. Given a pair of smooth periodic functions ξ_1 and ξ_2 on $\mathbb{R}^2$ with vanishing constant term (in their Fourier expansion) and satisfying (0.3.1.10), one easily constructs a smooth periodic function H such that

$$\frac{\partial H}{\partial x_2} = -\xi_1, \quad \frac{\partial H}{\partial x_1} = \xi_2.$$

Then the system of ordinary differential equations on the torus defined by ξ is identical with the system

$$\frac{dx_1}{dt} = -\frac{\partial H}{\partial x_2}, \quad \frac{dx_2}{dt} = \frac{\partial H}{\partial x_1}.$$

Let φ_t denote the one parameter group of diffeomorphisms of T^2 associated with ξ. Then the above analysis shows that every smooth periodic function H gives a one parameter group of volume element preserving diffeomorphisms of the torus T^2 equipped with the volume element $dx_1 \wedge dx_2$. Therefore the group of $dx_1 \wedge dx_2$ preserving diffeomorphisms of T^2 is an infinite dimensional group. The effect of the one parameter group φ_t can more or less be described geometrically. The essential point is that the curves $H(x_1, x_2) = \text{const.}$ (called *level curves* or *sets*) are invariant under the flow φ_t. This follows from[6]:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial H}{\partial x_2}\frac{dx_2}{dt} = 0.$$

Thus the level curves for H are invariant under the flow φ_t which also preserves $dx_1 \wedge dx_2$. ♠

0.3.2 Manifolds with Boundary

The generalization of the notion of a manifold to that of a manifold with boundary is very important. By a *manifold with boundary* we mean a Hausdorff separable topological space M such that every point $y \in M$ has a neighborhood U homeomorphic to either an open

[6]See also the subsection on Contact and Symplectic Structures.

subset of $\mathbb{R}^m$ or the set $\{x = (x_1, \cdots, x_m) | x_m \geq 0\}$ with y mapped to the origin. Points for which $x_m = 0$ are called *boundary points*. We assume that the transition functions for M are smooth up to the boundary. There are some technical issues regarding well-definedness of the notion of boundary which we simply ignore. The idea of the boundary is sufficiently intuitive that no confusion should arise. Non-boundary points are called *interior* points. We normally denote the set of interior points of M by $\breve{M}$, and its boundary by ∂M. Clearly $\breve{M}$ and ∂M are manifolds. Notice that at the boundary points we have (co)tangent spaces and (co)tangent bundles to the boundary and also the restriction of the (co)tangent spaces and (co)tangent bundles of M to the boundary. The transition functions for the latter bundles are obtained by the restriction of the transition functions to $U_\alpha \cap \partial M$. Smoothness up to the boundary implies that the transition functions for the (co)tangent bundles exist and are smooth up to the boundary, and therefore their restrictions to the boundary make sense. We denote the restriction of the tangent and cotangent bundles to the boundary by $\mathcal{T}_{\partial M} M$ and $\mathcal{T}^\star_{\partial M} M$ respectively. The fibre of $\mathcal{T}_{\partial M} M$ at $x \in \partial M$ will be denoted by $\mathcal{T}_{\partial M,x} M$. In practice it often helpful and harmless to think of a manifold with boundary M as the subset of another manifold M' of the same dimension, and M defined by single inequality of the form $f(x) \geq 0$ near each *boundary component* (i.e., connected component of the boundary) N and df is non-vanishing in a neighborhood of N. The boundary component N is the set $\{x \in M' | f(x) = 0\}$. The meaning of the notion of inward normal is immediately clear in this context. In fact, $\xi \in \mathcal{T}_{\partial M,x} M$ *points inward* if $df(x)(\xi) > 0$. Unless stated to the contrary, a Riemannian metric on M is assumed to continue smoothly to a section of $\mathcal{T}^\star_{\partial M} M \odot \mathcal{T}^\star_{\partial M} M$ on the boundary. Given a Riemannian metric g on M, the *gradient* of a function ψ is the vector field $\mathrm{grad}(\psi)$ defined by the requirement

$$d\psi(\xi) = g(\mathrm{grad}(\psi), \xi). \qquad (0.3.2.1)$$

Clearly $\mathrm{grad}(f)$ always points inward regardless of the choice of the Riemannian metric. An orientation for M induces an orientation on ∂M. In fact, we fix an inward pointing vector field η, e.g. $\eta = \mathrm{grad}(f)$, on ∂M. Then a basis $(\xi_1, \cdots, \xi_{m-1})$ for $T_x \partial M$ is positively oriented if $(-\eta(x), \xi_1, \cdots, \xi_{m-1})$ is a positively oriented basis for $\mathcal{T}_{\partial M,x} M$. With this orientation of the boundary we can state the fundamental result

Theorem 0.3.2.1 *(*Stokes*) Let ω be an $(m-1)$-form on the oriented manifold M with possibly non-empty boundary ∂M and oriented as specified above. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

The proof of this theorem is not difficult and reduces to integration by parts. It can also be found in many standard texts.

Example 0.3.2.1 Theorem 0.3.2.1 contains the theorems of Green and Stokes given in advanced calculus texts. For example, Stokes theorem is often stated as

$$\int_U (\mathrm{div}\xi) dx_1 dx_2 dx_3 = \int_{\partial U} < \xi, \mathbf{n} > d\sigma, \tag{0.3.2.2}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is a vector field on the relatively compact region $U \subset \mathbb{R}^3$ which we assume is defined by the inequality $F(x_1, x_2, x_3) < 0$, $dx_1 dx_2 dx_3$ denotes the element of volume on $\mathbb{R}^3$, $< ., . >$ is the standard inner product on $\mathbb{R}^3$, $\mathbf{n}$ is the unit outward normal to ∂U and the differential $d\sigma$ is the element of area on ∂U. Since ∂U is defined by the equation $F(x_1, x_2, x_3) = 0$, under the assumption of $\frac{\partial F}{\partial x_3} \neq 0$, we can represent the surface ∂U as the graph of a function $x_3 = x_3(x_1, x_2)$ by the implicit function theorem. Then by example 0.3.1.3 we have

$$d\sigma = \frac{\sqrt{(\frac{\partial F}{\partial x_1})^2 + (\frac{\partial F}{\partial x_2})^2 + (\frac{\partial F}{\partial x_3})^2}}{|\frac{\partial F}{\partial x_3}|} dx_1 dx_2,$$

and similar equations in terms of $dx_2 \wedge dx_3$ and $dx_3 \wedge dx_1$. Since $\mathbf{n} = \frac{\mathrm{grad} F}{||\mathrm{grad} F||}$, we have

$$< e_1, \mathbf{n} > d\sigma = dx_2 \wedge dx_3, \quad < e_2, \mathbf{n} > d\sigma = dx_3 \wedge dx_1, \quad < e_3, \mathbf{n} > d\sigma = dx_1 \wedge dx_2,$$

where e_1, e_2, e_3 is the standard basis for $\mathbb{R}^3$. By using the *wedge* notation we have incorporated the orientation which has to be taken into account. To derive (0.3.2.2) from theorem 0.3.2.1 we set

$$\omega = \xi_1 dx_2 \wedge dx_3 + \xi_2 dx_3 \wedge dx_1 + \xi_3 dx_1 \wedge dx_2.$$

This clearly gives (0.3.2.2). This argument generalizes almost verbatim to any number of dimensions to yield

$$\int_U (\mathrm{div}\xi) dx_1 \wedge \cdots \wedge dx_{m+1} = \int_{\partial U} < \xi, \mathbf{n} > d\sigma, \tag{0.3.2.3}$$

where $U \imath \mathbb{R}^{m+1}$ is an open relatively compact subset with smooth boundary ∂U. In particular, setting $\xi = x = (x_1, \cdots, x_{m+1})$ we obtain

$$(m+1)\mathrm{vol}(U) = (m+1) \int_U dx_1 \wedge \cdots \wedge dx_{m+1} = \int_{\partial U} < x, \mathbf{n} > dv_{\partial U}, \tag{0.3.2.4}$$

which is a useful special case of Stokes' theorem. ♠

In practice it becomes necessary to integrate forms on objects more general than manifolds or manifolds with boundary. For example, a triangle T with its interior is a manifold with corners which is more general than a manifold with boundary. It is clear that integration of 2-forms is permissible on T. There are technical results describing the level of permissible generality in Stokes' theorem. We shall not dwell on such results, and in many applications the validity of Stokes' theorem is quite clear.

The simplest manifolds with boundary are open subsets of $\mathbb{R}^m$ with smooth boundary. Even for $m = 2$ there are interesting geometric problems which have inspired exciting developments in geometry and analysis. For instance, one may ask among simple curves in the plane enclosing a region of a given fixed area, which one has minimal length. By looking at arbitrarily thin rectangles and smoothing out the corners we obtain curves of arbitrarily large length enclosing a region of fixed area. It has been conjecturally known since antiquity that the circle is the unique solution to this problem, however, the first completely satisfactory proof appeared only late in the nineteenth century. More precisely, we know that if Γ is a simple closed curve in the plane of length L enclosing a region of area A then

$$L^2 - 4\pi A \geq 0, \tag{0.3.2.5}$$

with equality if and only if Γ is a circle. The inequality (0.3.2.5) is known as the *isoperimetric inquality* in the plane. This problem has an obvious generalization to higher dimensions and the inequality analogous to (0.3.2.5) is

$$A^m \geq m^m c_m V^{m-1}, \tag{0.3.2.6}$$

where c_m is the volume of the unit ball in $\mathbb{R}^m$, $A = \mathrm{vol}(\partial U)$, $V = \mathrm{vol}(U)$ and $U \subset \mathbb{R}^m$ is a relatively compact open subset with smooth boundary ∂U. Naturally A is calculated relative to the volume element corresponding to the Riemannian metric induced from $\mathbb{R}^m$. We also refer to (0.3.2.6) as the *isoperimetric inequality.* There are a number of proofs of (0.3.2.5) which often do not generalize to higher dimensions. Here we give a proof of (0.3.2.6) based on the Brunn-Minkowski inequality. To state this inequality we need to introduce a definition. Given subsets $K_\circ, K_1 \subset \mathbb{R}^m$ and $\lambda \in \mathbb{R}$ we define

$$K_\circ + K_1 = \{x + y \mid x \in K_\circ,\ y \in K_1\}, \quad \lambda K_i = \{\lambda x \mid x \in K_i\}.$$

We refer to $K_\circ + K_1$ as the *sum of* $K_\circ$ *and* K_1. For a compact set $K \subset \mathbb{R}^m$ let

$$v_m(K) = \int_K dx_1 \wedge \cdots \wedge dx_m.$$

The notion of volume of a compact subset of $\mathbb{R}^m$ relative to the Lebesgue measure is defined, and in particular, if K is contained in a hyperplane then $v_m(K) = 0$. The fundamental inequality of interest to us is given in the following proposition:

Proposition 0.3.2.1 (Brunn-Minkowski Inequality) *Let $K_\circ, K_1 \subset \mathbb{R}^m$ be closures of relatively compact open subsets and $\lambda \in [0,1]$ we have*

$$\Big(v_m((1-\lambda)K_\circ + \lambda K_1)\Big)^{\frac{1}{m}} \geq (1-\lambda)v_m(K_\circ)^{\frac{1}{m}} + \lambda v_m(K_1)^{\frac{1}{m}}.$$

In order to appreciate the significance of this proposition, we show that it implies the isoperimetric inequality. Let B_ϵ be a ball of radius $\epsilon > 0$ centered at the origin and $U_\epsilon = U + B_\epsilon$. It follows from proposition 0.3.2.1 that

$$v_m(U_\epsilon) \geq \Big(v_m(U)^{\frac{1}{m}} + v_m(B_\epsilon)^{\frac{1}{m}})^m > v_m(U) + mc_m^{\frac{1}{m}} v_m(U)^{\frac{m-1}{m}}.$$

Therefore

$$\frac{v_m(U_\epsilon) - v_m(U)}{\epsilon} \geq mc_m^{\frac{1}{m}} v_m(U)^{\frac{m-1}{m}} \tag{0.3.2.7}$$

It is not difficult to show that the left hand side of (0.3.2.7) tends to $\mathrm{vol}(\partial U)$ as $\epsilon \longrightarrow 0$ (see also example **??** of chapter 2). This yields $\mathrm{vol}(\partial U) \geq mc_m^{\frac{1}{m}} v_m(U)^{\frac{m-1}{m}}$ which is precisely the isoperimetric inequality (0.3.2.6).

Remark 0.3.2.1 There are many versions of the isoperimetric dependiong on the class of objects under consideration. For instance one may limit oneself to triangles in the Euclidean plane. Let a, b and c denote the lengths of the sides of a triangle, and $p = \frac{1}{2}(a+b+c)$. Then the arithmetic geometric mean inequality implies

$$(p-a)(p-b)(p-c) \leq \frac{p^3}{27},$$

with equality if and only if $a = b = c$. Since the area of a triangle is $A = \sqrt{p(p-a)(p-b)(p-c)}$ we obtain

$$\frac{p^2}{3\sqrt{3}} \geq A, \tag{0.3.2.8}$$

with equality for the equilateral triangle. This in particular implies that among all triangles with the same perimeter, the equilateral triangle has maximum area. Other inequalities of this type are given in [Had]. The three dimensional situation is considerably more complex. F. Tóth has shown in [Tot] that if K is a *polytope* (i.e., the convex closure of a finite set of points in $\mathbb{R}^3$) with non-empty interior, A its surface area and V its volume, then

$$\frac{A^3}{V^2} \geq 54(n-2)\tan\alpha(4\sin^2\alpha - 1), \tag{0.3.2.9}$$

where n is the number of 2-faces of K and $\alpha = \frac{\pi n}{6(k-2)}$. This inequality is sharp in the sense that equality holds for the regular tetrahedron, the cube and the regular dodecahedron. ♡

Before giving the proof of this proposition it is useful to make some observations. The quantities $v_m(K)$ have the homogeneity property $v_m(\alpha K) = \alpha^m v_m(K)$. If $v_m(K) = 0$, then the obvious inclusion $(1-\lambda)x + \lambda K_1 \subset (1-\lambda)K_\circ + \lambda K_1$, for all $x \in K_\circ$ implies

$$v_m((1-\lambda)K_\circ + \lambda K_1) \geq v_m((1-\lambda)x + \lambda K_1) = \lambda^m v_m(K_1),$$

and the validity of Brunn-Minkowski inequality follows. So we limit ourselves to the case $v_m(K_i) > 0$. Now if we replace K_i with $\frac{1}{v_m(K_i)^{\frac{1}{m}}} K_i$ and λ with

$$\lambda' = \frac{\lambda v_m(K_1)^{\frac{1}{m}}}{(1-\lambda)v_m(K_\circ)^{\frac{1}{m}} + \lambda v_m(K_1)^{\frac{1}{m}}},$$

the proof of the proposition reduces to the case where $v_m(K_i) = 1$ in which case it will suffice to prove

$$[v_m((1-\lambda)K_\circ + \lambda K_1)]^{\frac{1}{m}} \geq 1. \tag{0.3.2.10}$$

Given a vector $\xi \in \mathbb{R}^m$, $c \in \mathbb{R}$ we let $H_{\xi,c}$ (resp. $H^-_{\xi,c}$) be the hyperplane (resp. the half-space) defined by the $< x, \xi >= c$ (resp. $< x, \xi > \leq c$) where $< .,. >$ denotes the standard inner product on $\mathbb{R}^m$. Having fixed ξ, for a compact set $K \subset \mathbb{R}^m$ we set

$$v_{m-1}(K,c) = v_{m-1}(K \cap H_{\xi,c}), \quad v_m(K,c) = v_m(K \cap H^-_{\xi,c}) = \int_{-\infty}^{c} v_{m-1}(K,t)dt.$$

If α is the supremum of all real numbers α' such that $v_{m-1}(K,\gamma) = 0$ for all $\gamma \leq \alpha'$, then we can change the lower limit of the above integeral from $-\infty$ to α. Let β be the infimum of all β' such that for all $\gamma \geq \beta'$ we have $v_{m-1}(K,\gamma) = 0$. In view of the convexity assumption and $v_m(K) > 0$, $v_m(K,c)$ is a strictly increasing function of c on the interval (α, β). Therefore it has an inverse which we denote by $y_K(s)$, i.e., $y_K(s) = t$ means $v_m(K,t) = s$. If we furthermore assume $v_m(K) = 1$ then the domain of y_K is $(0,1)$. Clearly $v_m(K,t)$ is a differentiable function of t since it is defined as an integral of a continuous function, and

$$y'_K(s) = \frac{1}{v_{m-1}(K, y_k(s))}. \tag{0.3.2.11}$$

We also need the following elementary lemma:

Lemma 0.3.2.1 *Let α, λ, a, b be positive real numbers with $\lambda \in (0,1)$. Then*

$$\Big((1-\lambda)a^\alpha + \lambda b^\alpha\Big)^{\frac{1}{\alpha}} \Big(\frac{1-\lambda}{a} + \frac{\lambda}{b}\Big) \geq 1.$$

Proof - Taking logarithms from both sides the inequality reduces to

$$\frac{1}{\alpha}\log\Big((1-\lambda)a^\alpha+\lambda b^\alpha\Big)+\log\Big(\frac{1-\lambda}{a}+\frac{\lambda}{b}\Big)\geq 0.$$

The convexity property

$$\log((1-\lambda)a+\lambda b)\geq(1-\lambda)\log a+\lambda\log b \tag{0.3.2.12}$$

implies the required result. ♣

With these preliminaries out of the way we can prove the Brunn-Minkowski inequality.

Proof of proposition 0.3.2.1 - The proof is by induction on the dimension m. To understand the key point of the argument we first assume the sets K_i are convex. For $m=1$, K_i's are unit intervals and the proof is immediate. We may assume $v_m(K_i)=1$ and that it suffices to establish (0.3.2.10). It is convenient to set $K_\lambda=(1-\lambda)K_\circ+\lambda K_1$, and

$$y_{K,\lambda}(s)=(1-\lambda)y_{K_\circ}(s)+\lambda y_{K_1}(s).$$

Define the numbers $\alpha_\lambda,\beta_\lambda$ as α,β with K_λ replacing K. We have

$$v_m(K_\lambda)=\int_{\alpha_\lambda}^{\beta_\lambda}v_{m-1}(K_\lambda,t)dt=\int_\circ^1 v_{m-1}(K_\lambda\cap H_{\xi,y_K(s)})y'_{K_\lambda}(s)ds.$$

Using (0.3.2.11) and the obvious inclusion $(1-\lambda)(K_\circ\cap H_{\xi,y_{K_\lambda}(s)})+\lambda(K_1\cap H_{\xi,y_{K_\lambda}(s)})\subset K_\lambda\cap H(\xi,y_{K_\lambda}(s))$ we obtain

$$v_m(K_\lambda)\geq\int_\circ^1 v_{m-1}((1-\lambda)(K_\circ\cap H_{\xi,y_{K_\circ}}(s))+\lambda(K_1\cap H_{\xi,y_{K_1}}(s)))\Big(\frac{1-\lambda}{a}+\frac{\lambda}{b}\Big)ds,$$

where $a=v_{m-1}(K_\circ,y_{K_\circ}(s))$ and $b=v_{m-1}(K_1,y_{K_1}(s))$. In view of the induction hypothesis this yields

$$v_m(K_\lambda)\geq\int_\circ^1\Big((1-\lambda)a^{\frac{1}{m-1}}+\lambda b^{\frac{1}{m-1}}\Big)^{m-1}\Big(\frac{1-\lambda}{a}+\frac{\lambda}{b}\Big)ds. \tag{0.3.2.13}$$

Applying lemma 0.3.2.1 to (0.3.2.13) we obtain $v_m(K_\lambda)\geq 1$ proving the proposition for K_i convex. The convexity assumption was used to ensure that $v_m(K,t)$ is a strictly increasing function of t. For finite unions of convex sets we can replace the interval (α,β) with finitely many intervals (α_j,β_j) on each of which the function $v_M(K,t)$ is strictly increasing. We then approximate the sets K_i with such finite unions. This requires a technical modification of the proof and will not be pursued here. ♣

Remark 0.3.2.2 The Brunn-Minkowski inequality is valid for Borel sets K_i of finite volume in $\mathbb{R}^m$. Minkowski established a variety of isoperimetric inequalities for convex sets in $\mathbb{R}^m$. His work led to the introduction of the concept of mixed volumes by Alexandrov and Fenchel-Jessen. For an account this subject see [Schn] which also contains extensive references. ♡

0.3.3 Vector Fields

A vector field ξ on a manifold determines a system of first order ordinary differential equations and vice versa. The correspondence is easily described in local coordinates. In fact, in local coordinates $x = (x_1, \cdots, x_m)$ a vector field has representation $\xi = \sum_j \xi_j \frac{\partial}{\partial x_j}$ and the corresponding system of differential equations is

$$\frac{dx_j}{dt} = \xi_j, \quad \text{for } j = 1, 2, \cdots, m. \tag{0.3.3.1}$$

Note that the right hand side of (0.3.3.1), i.e., the functions ξ_j, are independent of t (time independence) so that at every $x \in M$ we have a unique tangent vector $\xi(x) \in \mathcal{T}_x M$. The transition functions for the tangent bundle or equivalently the chain rule shows that this coorespondence is well defined. A system of first order ordinary differential equations admits of a unique solution once the initial data $x_j(0) = x_j^\circ$ are specified. A system of the form

$$\frac{du}{dt} = F(t, u), \tag{0.3.3.2}$$

with explicit time dependence on right hand side, can be converted to one of the form (0.3.3.1) by replacing the vectors $u = (u_1, \cdots, u_n)$ and F with the vectors $x = (t, u_1, \cdots, u_n)$ and $f = (1, F_1, \cdots, F_n)$.

It is customary to denote the solution x of (0.3.3.1) subject to the initial condition $x(0) = x^\circ$ by $\varphi_t(x^\circ)$ or $\varphi_t^\xi(x^\circ)$, so that in local coordinates

$$\varphi_t(x^\circ) = (x_1(t), x_2(t), \cdots, x_m(t)),$$

and $x_j(.)$ is the unique solution to (0.3.3.1) for initial data $x_j(0) = x_j^\circ$. The maps $t \rightarrow \varphi_t(x)$, defines a curve on M whose tangent vector fields are ξ (along the curve). From time independence of the functions ξ_j and the uniqueness of the solution to (0.3.3.1) once the initial conditions are specified, it follows that $\varphi_{t+s}(x^\circ) = \varphi_t(\varphi_s(x^\circ))$. For this reason φ_t is sometimes called the *one parameter group* of the vector field ξ or the differential equation (0.3.3.1), and the mappings $t \rightarrow \varphi_t(x^\circ)$ its *trajectories*. It is important to note that since the existence theorem for ordinary differential equations is only a local result, the one parameter groups $\varphi_t(.)$ only exist for t in a neighborhood of $0 \in \mathbb{R}$. The mappings $x \rightarrow \varphi_t(x)$, for fixed $t \in \mathbb{R}$, are diffeomorphisms of M (of course assuming $\varphi_t(x)$ exists) and the inverse is given by $x \rightarrow \varphi_{-t}(x)$. The one parameter group φ_t^ξ is *complete* if it exists for all $t \in \mathbb{R}$. By removing points from a manifold we may have situations where $\varphi_t^\xi(x)$ exists for all $t \in \mathbb{R}$ but $\varphi_t^\xi(y)$ may exist only for t in a bounded interval where $x, y \in M$. However,

Lemma 0.3.3.1 *Let M be a compact manifold and ξ a vector field on M. Then φ_t^ξ is complete.*

Proof - Let $x \in M$ and T be the supremum of all s such that the solution $\varphi_t(x)$ exists for $t \in [0, s)$, and assume $T < \infty$. Let $t_j \in [0, T)$ be such that $\lim_j t_j = T$ and $\lim_{j \to \infty} \varphi^\xi_{t_j}(x) = y$ exists (compactness of M). The solution $\varphi^\xi_t(y)$, $t \in (-\epsilon, \epsilon)$ shows that $\varphi^\xi_t(x)$ extends beyond T and so φ^ξ_t is complete. ♣

If A is an $m \times m$ real matrix then the system of differential equations

$$\frac{dx}{dt} = Ax \tag{0.3.3.3}$$

on $\mathbb{R}^m$ is called a *linear* system and can be explicitly solved. In fact, its solution is given by

$$x(t) = e^{At}x^\circ = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j x^\circ.$$

Exercise 0.3.3.1 *Consider the linear system (0.3.3.3) for $m = 2$. Draw the trajectories for this system (near the origin) for the following cases:*

1. *Both eigenvalues λ_1, λ_2 of A have norm > 1 (resp. < 1);*
2. *A has one positive and one negative eigenvalue;*
3. *$|\lambda_1| = 1 = |\lambda_2|$.*

A *singularity* or *singular point* of a vector field is a point where it vanishes. Clearly this condition is independent of the choice of coordinate system and is meaningful on a manifold. Choosing local coordinates $x_1, \cdots, x_m$ in $U \subset M$, we represent a vector field as $\xi = (\xi_1(x), \cdots, \xi_m(x))$ or more precisely $(x_1, \cdots, x_m, \xi_1(x), \cdots, \xi_m(x))$. Let D denote differentiation with respect to the variable $x \in U$. A change of variables $y_i = y_i(x_1, \cdots, x_m)$ will transform the representation ξ to η given by

$$\eta = J^{-1}\xi, \qquad \text{where} \quad J = (\frac{\partial y_i}{\partial x_j}),$$

and ξ denotes the column vector of $\xi_j(x)$'s representing the vector field ξ. Note that $\det J^{-1} \neq 0$. We have

$$D\eta = -J^{-1}(DJ)J^{-1}\xi + J^{-1}D\xi.$$

Since at a singular point $\xi(x) = 0$, non-vanishing of $\det(D\xi)$ is independent of the choice of the coordinate system. A singularity of the vector field ξ is called *simple* or *nondegenerate* if it satisfies the condition $\det(D\xi) \neq 0$.

Recall that the gradient of a function ψ on a Riemannian manifold M is defined by the (0.3.2.1). Since the tangent spaces to the submanifolds $\psi =$ constant are defined by $d\psi = 0$, the gradient vector field gradψ is orthogonal to the submanifolds $\psi =$ constant. One easily verifies that in terms of local coordinates $(x_1, \cdots, x_m)$ the gradient vector field is given by the (row) vector

$$\mathrm{grad}\psi \longleftrightarrow \Psi' g^{-1}, \quad \text{where } \Psi' = (\frac{\partial\psi}{\partial x_1}, \cdots, \frac{\partial\psi}{\partial x_m}). \tag{0.3.3.4}$$

It is clear that the singularities of gradψ are precisely the *critical points* of the function ψ, i.e., points where $d\psi$ vanishes. From the local representation of the gradient vector field it follows that nondegeneracy of a singular point of gradψ is equivalent to nonsingularity of the *Hessian* of ψ which is defined as

$$\mathsf{H}(\psi) = (\frac{\partial^2\psi}{\partial x_j \partial x_k}). \tag{0.3.3.5}$$

While the above argument establishes independence of nonsingularity of the Hessian at a critical point from the choice of coordinate system, it is useful to calculate the transformation formula for the Hessian. Clearly we have

$$\frac{\partial^2\psi}{\partial x_j \partial x_i} = \sum_{k,l} \frac{\partial^2\psi}{\partial y_k \partial y_l}\frac{\partial y_k}{\partial x_j}\frac{\partial y_l}{\partial x_i} + \sum_k \frac{\partial\psi}{\partial y_k}\frac{\partial^2 y_k}{\partial x_j \partial x_i} \tag{0.3.3.6}$$

At a critical point the second sum in (0.3.3.6) vanishes. Therefore the Hessian at a critical point transforms as a quadratic form under a linear change of coordinates. Consequently, the non-vanishing of $\det \mathsf{H}(\psi)$ and the number of positive and negative eigenvalues of $\mathsf{H}(\psi)$ at a critical point are independent of the choice of the coordinate chart. The *Morse index* of ψ at a nondegenerate critical point $x \in M$ is the number of negative eigenvalues of the symmetric matrix $\mathsf{H}(\psi)$. A function $\psi : M \to \mathbb{R}$ all whose critical points are nondegenerate is called a *Morse function*. The set of critical points of a Morse function is necessarily discrete. The most basic example of a Morse function $f : \mathbb{R}^m \to \mathbb{R}$ with a unique critical point of Morse index p is:

$$f(x) = -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_m^2. \tag{0.3.3.7}$$

We shall return to the discussion of Morse index and its geometric implications later.

Example 0.3.3.1 Let $R > r > 0$ and consider the surface defined by the equation

$$F(x_1, x_2, x_3) \equiv (x_1^2 + x_2^2 - R^2)^2 + x_3^2 - r^2 = 0.$$

It is not difficult to see that this equation defines a compact submanifold M diffeomorphic to a torus. Let ψ be the restriction of the linear map

$$\Psi : (x_1, x_2, x_3) \longrightarrow ax + cz$$

where a, c are real numbers, to M. The critical points of ψ are computed by solving the equations

$$(\frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial x_3}) = \lambda(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}),\ F(x_1, x_2, x_3) = 0.$$

These equations mean that we are on M and the derivative of the function Ψ is normal to the surface M. In other words, $d\Psi$ vanishes on the tangent space to M. Examining these equations in detail one sees easily that if $a \neq 0$ then ψ has four critical points. If $a = 0$, then the subsets

$$x_1^2 + x_2^2 - 4 = 0,\ x_3 = \pm 1$$

consist of critical points of the function ψ. It is a simple calculation to see that if $a \neq 0$, then the four critical points of ψ are nondegenerate. There is one maximum and one minimum which have indices two and zero, and the other two critical points have index one. Figure (XXX) explains the situation intuitively. For $a = 0$, ψ is not a Morse function. ♠

0.3.4 Poincaré Lemma and the Theorem of Frobenius

A p-form ω is *closed* if $d\omega = 0$, and is *exact* if there is η such that $d\eta = \omega$. Since $dd = 0$, every exact form is closed. The local converse to this fact is the Poincaré lemma.

Theorem 0.3.4.1 (Poincaré Lemma) *Let ω be an closed p-form on a star shaped region U in $\mathbb{R}^n$. Then ω is exact, i.e., there is a $p-1$-form η defined in U such that $d\eta = \omega$.*

The reader is referred to [Ca] for the standard proof of this theorem. Poincaré lemma and corollary 0.3.4.1 below are valid for forms with values in a vector or even a Banach space. The proof carries over without change. The special case of the Poincaré Lemma for 1-forms may be restated as follows:

Corollary 0.3.4.1 *Let $f_1, \cdots, f_n$ be C^1 functions on a star-shaped region in $\mathbb{R}^n$. Then there is a function f such that $\frac{\partial f}{\partial x_i} = f_i$ if and only if $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all i, j.*

A remarkable application of the Poincaré Lemma (and the implicit function theorem) is the celebrated integrability theorem of Frobenius which we now describe. Let $I = (-1, 1)$, I^n denote the product of n copies of I, $E = \mathbb{R}^k$ (or even a Banach space), $U \subset E$ an open ball centered the origin, and

$$f : I^n \times U \to \mathcal{L}(\mathbb{R}^n, E),$$

where $\mathcal{L}(\mathbb{R}^n, E)$ denotes the set of linear maps of $\mathbb{R}^n$ into E. Consider the system of partial differential equations

$$D_t u = f(t, u), \tag{0.3.4.1}$$

where D_t denotes the derivative with respect to the variable $t = (t_1, \cdots, t_n) \in I^n$, and $u : I^n \to E$ is the unknown function. The theorem of Frobenius gives necessary and sufficient conditions for integrability of (0.3.4.1). By *integrability* of (0.3.4.1) we mean the existence, for every $(t^\circ, x^\circ) \in I^n \times U$, of a unique (local) solution $u(.)$ such that $u(t^\circ) = x^\circ$. Note that for $n = 1$, (0.3.4.1) reduces to a system of ordinary differential equations for the (time dependent) system (0.3.3.2). The proof given below is an adoptation of the argument in [R] for the existence theorem of ordinary differential equations. For $n > 1$ there is an obvious necessary condition, namely

$$D_1 f(t,x)(\tau_1)(\tau_2) + D_2 f(t,x)(f(t,x)(\tau_1))(\tau_2) = D_1 f(t,x)(\tau_2)(\tau_1) + D_2 f(t,x)(f(t,x)(\tau_2))(\tau_1) \tag{0.3.4.2}$$

This condition is simply the symmetry of the second derivative, and is derived from (0.3.4.1) by applying D_t to both sides of (0.3.4.1), and using the chain rule. To show sufficiency we use Corollary 0.3.4.1 and the implicit function theorem. Let $k \geq 1$, $C^k(I^n; E)$ denote the space of k times continuously differentiable mappings of I^n to E, and $C^{k,\circ}(I^n; E)$ the subspace consisting of maps v with $v(0) = 0$. We also let $C^{k,\circ}(I^n; U) \subset C^{k,\circ}(I^n; E)$ be the open subset consisting of those maps with take values in U. Let $C^k_s(I^n; \mathcal{L}(\mathbb{R}^n, E))$ be the subspace of $C^k(I^n; \mathcal{L}(\mathbb{R}^n, E))$ consisting of maps $v : I^n \to \mathcal{L}(\mathbb{R}^n, E)$ such that

$$D_t v(\tau_1, \tau_2) = D_t v(\tau_2, \tau_1),$$

for all $\tau_i \in \mathbb{R}^n$. Consider the map $F : \mathbb{R} \times U \times C^{k,\circ}(I^n; E) \to C^{k-1}_s(I^n; \mathcal{L}(\mathbb{R}^n, E))$ given by

$$F(a, x, v)(t) = D_t v(t) - af(at, x + v(t, x))$$

In view of (0.3.4.2) the image of F lies in $C^{k-1}_s(I^n; \mathcal{L}(\mathbb{R}^n, E))$. The derivative of F relative to the third variable $C^{k,\circ}(I^n; E)$ is

$$D_3 F(0, x, \mathbf{0})(\delta)(t) = D_t \delta(t) : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, E).$$

From corollary 0.3.4.1 to the Poincaré Lemma we see that $D_3F(0,x,0)$ is a topological isomorphism onto the linear space $C_s^{k-1}(I^n;\mathcal{L}(\mathbb{R}^n,E))$. Therefore the implicit function theorem is applicable and we obtain an open set $(-2\epsilon,2\epsilon)\times V\subset\mathbb{R}\times U$ and a mapping $\varphi:(-2\epsilon,2\epsilon)\times V\to C^{k,\circ}(I^n;E)$ such that

$$F(a,x,\varphi(a,x))=0.$$

Define $u:(-\epsilon,\epsilon)^n\times V\to U$ by

$$u(t;x)=\varphi(\epsilon,x)(t/\epsilon)+x.$$

It follows that u is the desired (local) solution to (0.3.4.1). Uniqueness follows from the uniqueness assertion of the implicit function theorem. Therefore we have shown

Corollary 0.3.4.2 (Frobenius Integrability Theorem) *The system (0.3.4.1) is integrable if and only if f satisfies the integrability condition (0.3.4.2).*

The above proof of the integrability theorem of Frobenius demonstrates the general principle that sometimes a nonlinear problem (e.g. (0.3.4.1) under the assumption (0.3.4.2)) may be solved by linearizing the problem which is more easily solvable (in this case via the Poincaré Lemma or corollary 0.3.4.1) and then using the implicit function theorem to obtain the solution to the original nonlinear problem.

It is customary to restate corollary 0.3.4.2 in a form more suitable for applications. To this end let $\omega_1,\cdots,\omega_k$ be smooth 1-forms on the manifold M, and assume that at every point $x\in M$, $\omega_1(x),\cdots,\omega_k(x)$ are linearly independent. Let $E_x=\{\tau_x\in\mathcal{T}_xM|\omega_j(x)(\tau_x)=0\}$. Then $E=\cup_{x\in M}E_x$ is a sub-bundle of rank $m-k$ the tangent bundle $\mathcal{T}M$. A sub-bundle of the tangent bundle of a manifold is often called a *distribution*. A system of the form $\omega_1=0,\cdots,\omega_k=0$, where ω_j's are 1-forms linearly independent at every point of a manifold M is called a *Pfaffian system*.

Corollary 0.3.4.3 (Frobenius) *With the above notation and hypotheses, for every $x\in M$ there is an immersed submanifold $N\subset M$ through x such that the restriction E_N of E to N is the tangent bundle of N, if and only if any one of the following three equivalent conditions is satisfied:*

1. *For every pair of sections ξ,η of $E\to M$, the bracket $[\xi,\eta]$ is also a section of $E\to M$;*

2. *For every j, $d\omega_j$ lies in the ideal $\mathcal{I}$ generated by $\omega_1,\cdots\omega_k$ in the ring of all smooth differential forms on M, (this means there are 1-forms α_{jl} such that*

$$d\omega_j=\sum_{l=1}^{m}\alpha_{jl}\wedge\omega_l$$

for every j;)

3. *For every* j

$$d\omega_j \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0.$$

If N exists we say the distribution $E \rightarrow M$ is *integrable.*

Proof - The equivalence of (a) and (b) follows easily from the exterior differentiation formula for a p-form ω

$$\begin{aligned} d\omega(\xi_\circ, \cdots, \xi_p) &= \tfrac{1}{p+1}\sum_{i=0}^p (-1)^i \xi_i(\omega(\xi_\circ, \cdots, \hat{\xi}_i, \cdots, \xi_p)) + \\ & \tfrac{1}{p+1}\sum_{0\leq i<j\leq p}(-1)^{i+j}\omega([\xi_i,\xi_j], \xi_\circ, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_p), \end{aligned} \quad (0.3.4.3)$$

where $\hat{\xi}_i$ means ξ_i is omitted. In this formula $\xi_i(\omega(\xi_\circ, \cdots, \hat{\xi}_i, \cdots, \xi_p))$ means ξ_i is applied as a differential operator to the function $\omega(\xi_\circ, \cdots, \hat{\xi}_i, \cdots, \xi_p)$. The equivalence of (b) and (c) is a simple exercise. To see the equivalence of this formulation and the statement of corollary 0.3.4.2, let (U, φ_U) be a coordinate chart and $(x_1, \cdots, x_m)$ be the coordinates of a point in $\varphi_U(U)$. Let $n = m - k$ and identify U with $\varphi_U(U)$. After a possible permutation of the coordinates we may assume that $dx_1, \cdots, dx_n, \omega_1, \cdots, \omega_k$ is a basis at every point of U. To relate condition (b) to the integrability condition of corollary 0.3.4.2, i.e. (0.3.4.2), it is convenient to introduce some notation. Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \quad x' = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_m \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \end{pmatrix}.$$

Then there is an invertible $k \times k$ matrix $A = (\alpha_{ij})$ of smooth functions such that

$$du = A\omega + f dx', \quad (0.3.4.4)$$

where $f = (f_{ij})$ is a $k \times n$ matrix of smooth functions of x. Taking exterior derivative of (0.3.4.4) we obtain

$$dA \wedge \omega + A d\omega = -df \wedge dx'. \quad (0.3.4.5)$$

Now by (0.3.4.4)

$$df_{ij} = \sum_{k=1}^n \frac{\partial f_{ij}}{\partial x_k} dx_k + \sum_{s=n+1}^m \sum_{l=1}^n \frac{\partial f_{ij}}{\partial x_s} f_{sl} dx_l \quad \mod \mathcal{I}. \quad (0.3.4.6)$$

In view of (0.3.4.5) the condition $d\omega \in \mathcal{I}$ is equivalent to the vanishing of the coefficient of $dx_i \wedge dx_j$, for $1 \leq i < j \leq n$, for all the entries of $df \wedge dx'$. Substituting (0.3.4.6) in $df \wedge dx'$, this condition becomes

$$\frac{\partial f_{ij}}{\partial x_k} + \sum_{s=n+1}^{m} \frac{\partial f_{ij}}{\partial x_s} f_{sk} = \frac{\partial f_{ik}}{\partial x_j} + \sum_{s=n+1}^{m} \frac{\partial f_{ik}}{\partial x_s} f_{sj}. \tag{0.3.4.7}$$

This is precisely the integrability condition of corollary 0.3.4.2 for the solvability of the system of partial differential equations $A\omega \equiv du - f dx' = 0$, where u is the unknown function of the independent variables x'. Since A is invertible, solvability of this system is equivalent to the integrability assertion of the corollary as desired. ♣

Remark 0.3.4.1 Note that the integral manifold for the Pfaffian system $\omega_j = 0$, $j = 1, \cdots, k$, is represented as the graph of an $\mathbb{R}^k$-valued function u defined on an open subset of $\mathbb{R}^{m-k}$. It is in this form that we have uniqueness of local solution to the system (0.3.4.1), that is, there is a unique integral manifold of the Pfaffian system passing through a pre-assigned point $(t^\circ, x^\circ) \in \mathbb{R}^{m-k} \times \mathbb{R}^k$. This is similar to converting the differential equation (0.3.3.2) to the time independent form (0.3.3.1). Since in geometric applications of Pfaffian systems, the variables $t = (t_1, \cdots, t_{m-k})$ are not *a priori* separated, it is judicious to directly treat (0.3.4.1) rather than the form analogous to time idependent form (0.3.3.1) of a system of ordinary differential equations. ♡

Let $E \to M$ be an integrable distribution defined by linearly independent 1-forms $\omega_1, \cdots, \omega_k$. By maximally extending a local solution N to the integrable distribution $E \to M$ we obtain an immersed submanifold of M. The fact that local solutions can be patched together to obtain maximal global solutions which are immersed submanifolds uses the uniqueness part of the the definition of integrability. The maximal or global solutions are, in general, not embedded submanifolds. For example on the torus $T^2 = S^1 \times S^1$ the 1-form $\omega = a_1 dx_1 + a_2 dx_2$ where a_1 and a_2 are real numbers linearly independent over the rational numbers, defines an integrable distribution with every maximal integral manifold dense in T^2.

By a *foliation* of a manifold N we mean a decomposition $N = \cup N_\alpha$ into disjoint union of submanifolds N_α of the same dimension; $\dim N - \dim N_\alpha$ is called the *codimension* of the foliation, and each N_α is a *leaf* of the foliation. An integrable Pfaffian system defines a foliation of the underlying manifold.

Remark 0.3.4.2 One often encounters the situation where a submanifold $M \subset \mathbb{R}^N$ is given and we want to study the geometry of M. We shall see in the next chapter that it is generally convenient to specify M as an integral manifold of a system of 1-forms $\omega_p = 0$

where $p = m + 1, \cdots, N$. This means that we are considering a family of submanifolds of $\mathbb{R}^N$ which more or less look like M. It is no loss of generality to regard the manifold M as the zero set of an $\mathbb{R}^{N-m}$-valued function $F = (F_1, \cdots, F_{N-m})$ of N variables. To embed M in a family we consider for every fixed value of $t = (t_1, \cdots, t_{N-m})$, the submanifold defined by $F(x_1, \cdots, x_N) = t$, provided the hypotheses of the implicit function theorem are fulfilled relative to the variables $x_1, \cdots, x_N$. The submanifolds M_t fill out an open set in $\mathbb{R}^N$. These submanifolds are the integral manifolds for the Pfaffian system $\omega_p = 0$ where $p = m + 1, \cdots, N$ and $\omega_{m+j} = dF_j$. Of course some mild assumptions on F are necessary. For example we want to avoid equations such as $F(x_1, x_2) = (x_1^2 + x_2^2 - 1)^2$ since $F(x_1, x_2) = t < 0$ makes no sense for real quantities. It is unnecessary for our purposes to elaborate on this issue. (This should not be construed as the assertion that points where the implicit function theorem fails are not of interest, since significant information is often encoded in singularities. The point is that it is generally clear from the context whether we want to examine the nature of a singularity or dealing with the generic nonsingular case.) The exact form of the vector valued function F is generally inconsequential. It should be pointed out that there are many situations where it is convenient/essential to consider equations of the form $F(x_1, \cdots, x_N; t_1, \cdots, t_k) = 0$ with $t = (t_1, \cdots, t_k)$ regarded as k-parameters. In such cases we generally make reasonable assumptions about the rank of the matrix of partial derivatives relative to t_j's so that the implicit function theorem becomes applicable and proceed in the natural manner. Such necessary assumptions are often implicit and not necessarily explicitly stated. These points and their significance will become more clear when we study Riemannian geometry in the next chapter, and the example that follows describes an elementary situation. ♡

Example 0.3.4.1 Assume $k = 1$ and $N = 3$ in the preceding remark so that we have a one parameter family of surfaces M_t defined by the equation $F(x_1, x_2, x_3; t) = 0$. It is understood that the surfaces in question are in the (x_1, x_2, x_3)-space and the hypotheses of the implicit function theorem are satisfied to avoid singularities and unnecessary pathologies. For $\frac{\partial F}{\partial t}(x_1, x_2, x_3; t) \neq 0$, we can locally solve $F = 0$ for $t = \Phi(x_1, x_2, x_3)$ and thus the surfaces M_t are also represented as integral manifolds for a 1-form $\theta = 0$. Differentiating F with respect to t we obtain the family of surfaces defined by $G(x_1, x_2, x_3; t) = 0$ where $G = \frac{\partial F}{\partial t}$. Solving this equation for t we obtain $t = \Psi(x_1, x_2, x_3)$. For each value of t, the system of equations

$$\Phi(x_1, x_2, x_3) = t, \quad \Psi(x_1, x_2, x_3) = t,$$

defines a curve γ_t in $\mathbb{R}^3$ called the *characteristic* of the surface M_t. Now as t varies, the system of curves γ_t gives a surface in $\mathbb{R}^3$ which we denote by Γ. Γ is called the *enveloping surface* of the family of surfaces $F(x_1, x_2, x_3; t) = 0$. Notice that Γ is an integral manifold for the

1-form $\omega = 0$ where $\omega = d(\Phi - \Psi)$. The enveloping surface Γ which is an integral manifold for $\omega = 0$ is obviously defined by the equation $H(x_1, x_2, x_3) = 0$ where H is obtained by eliminating t from the equations

$$F(x_1, x_2, x_3; t) = 0, \quad G(x_1, x_2, x_3; t) = 0.$$

Geometrically this locus is the (orthogonal) projection of the surface in $\mathbb{R}^4$ defined by $F = 0, G = 0$ on the three dimensional space with (x_1, x_2, x_3) coordinates. We shall return to this example in the subsection Flat Surfaces and Parallel Translation in the next chapter. ♠

Exercise 0.3.4.1 *Let $R > r > 0$ and consider the family of spheres of radius $r > 0$ (fixed) and centers on the circle $x_1^2 + x_2^2 = R^2, x_3 = 0$. Show the envelope of family of spheres is given by*

$$(x_1^2 + x_2^2 + x_3^2 + r^2 - R^2)^2 - 4r^2(x_1^2 + x_2^2) = 0.$$

Which familiar surface is this envelope?

Exercise 0.3.4.2 *Assume a one parameter family of surfaces is defined implicitly as*

$$F(x_1, x_2, x_3; s, t) = 0, \quad \psi(s, t) = 0.$$

Show that the equation of the enveloping surface is obtained by eliminating s and t from the equations

$$F = 0, \ \psi = 0, \ \frac{\partial F}{\partial s}\frac{\partial \psi}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial \psi}{\partial s} = 0.$$

0.3.5 Lie Algebras and Maurer-Cartan Equations

Consider $GL(m, \mathbb{R}) \subset \mathbb{R}^{m^2}$. A left invariant vector field has a simple description in this case. Let $A \in M_m(\mathbb{R})$ be an $m \times m$ matrix regarded as a tangent vector at I to $GL(m, \mathbb{R})$. Left translation by $g \in GL(m, \mathbb{R})$ is the linear map $h \rightarrow gh$, and its derivative is linear map $A \rightarrow gA$. Therefore the left (resp. right) invariant vector field ξ_A determined by A assigns the tangent vector gA (resp. Ag) at the point $g \in GL(m, \mathbb{R})$. The ordinary differential equation determined by the left invariant vector field ξ_A is

$$\frac{dg}{dt} = gA, \qquad (0.3.5.1)$$

on $GL(m, \mathbb{R})$. Then the solution curve to this equation passing through I is $g(t) = \exp(tA)$ (see also (0.3.3.3)) where

$$\exp(A) = \sum_k \frac{A^k}{k!}.$$

(Similar remark applies to the right invariant case by replacing gA with Ag in (0.3.5.1).) The map $A \to \exp(A)$ is called the *exponential map*. The considerations are valid if we replace $\mathbb{R}$ by $\mathbb{C}$, and we can even consider the complex analogue of the ordinary differential equation (0.3.5.1), however, we are generally only interested in the real structure. Clearly, exp satifies the equation

$$\exp((t+s)A) = \exp(tA)\exp(sA), \quad \text{for } s, t \in K, \tag{0.3.5.2}$$

so that, for every fixed matrix A, $t \to \exp(tA)$ is a group homomorphism $K \to GL(m, K)$ where $K = \mathbb{R}$ or $\mathbb{C}$. However, if A and B do not commute, then $\exp(A)\exp(B) \neq \exp(A+B)$ (see remark 0.3.5.1 below).

Lemma 0.3.5.1 *The tangent vector field to the curve* $\exp(tA)$*, where* A *is an* $m \times m$ *matrix is the restriction to* $\exp(tA)$ *of the left (or right) invariant vector field* ξ *on* $GL(m, K)$ *represented by the matrix* A*.*

Proof - Follows from the identification of A with a left (or right) invariant vector field on $GL(m, \mathbb{R})$. ♣

Exercise 0.3.5.1 *Show that the derivative of* exp *at* $\mathbf{0} \in M_m(K)$ *is the identity map and* exp *is an analytic diffeomorphism of a neighborhood of the origin onto a neighborhood of* $I \in GL(m, K)$*.*

Exercise 0.3.5.2 *Show that* $\exp : \mathcal{GL}(m, \mathbb{C}) \to GL(m, \mathbb{C})$ *is surjective, and the same conclusion is valid for* $SL(m, \mathbb{C})$*.*

Now if $G \subset GL(m, \mathbb{R})$ is an analytic group, and A is tangent to G at $e = I$, then for $g \in G$, both vectors gA and Ag will be tangent to G at g since multiplication on left or right by g maps a neighborhood of $e \in G$ onto a neighborhood of $g \in G$. Consequently the solution curves $\exp(tA)$ remain in G for all $t \in \mathbb{R}$. Let $\rho : G \to GL(N, \mathbb{R})$ be a representation Of G, and define the linear map $\rho' : \mathcal{G} \to \mathcal{GL}(N, \mathbb{R})$ as

$$\rho'(A) = \lim_{t \to 0} \frac{\rho(t\exp(A)) - I}{t} = \frac{d\rho(t\exp(A))}{dt}_{|t=0}. \tag{0.3.5.3}$$

Then $\exp(t\rho'(A))$ is the solution to the differential equation (0.3.5.1) on $GL(N,\mathbb{R})$ passing through I at $t=0$. It follows that we have the commutative diagram

$$\begin{array}{rcl} \mathcal{G} & \xrightarrow{\rho'} & \mathcal{GL}(N,\mathbb{R}) \\ \exp\downarrow & & \downarrow\exp \\ G & \xrightarrow{\rho} & GL(N,\mathbb{R}) \end{array} \tag{0.3.5.4}$$

Since there is no likelihood of confusion, we shall also use ρ rather than ρ' for the linear map of Lie algebras defined by (0.3.5.3).

The linear space $\mathcal{G}$ of left invariant vector fields on the Lie group G has an important algebraic structure. A left invariant vector field ξ maybe regarded as a homogeneous differential operator of first order on G which is invariant under the left action of the group. Therefore the composition of two left invariant vector fields, ξ and η, is a left invariant differential operator of second order. On the other hand, the *bracket* or *commutator* defined by

$$[\xi,\eta] = \xi\eta - \eta\xi \tag{0.3.5.5}$$

is a left invariant homogeneous first order differential operator since the second order terms cancel out for obvious reasons. Therefore $\mathcal{G}$ is closed under the bracket operation. Clearly $[\xi,\eta]+[\eta,\xi]=0$ and by a simple substitution from (0.3.5.5) we see that the *Jacobi identity* is valid:

$$[\xi,[\eta,\zeta]] + [\zeta,[\xi,\eta]] + [\eta,[\zeta,\xi]] = 0. \tag{0.3.5.6}$$

A vector space $\mathcal{G}$ together with an antisymmetric pairing $[,]:\mathcal{G}\times\mathcal{G}\rightarrow\mathcal{G}$, and satisfying the Jacobi identity is called a *Lie algebra*. In classical literature, the elements of the Lie algebra are referred to as *infinitesimal generators*. Naturally we denote the Lie algebras of $GL(m,K)$, $SU(m)$ etc. by $\mathcal{GL}(m,K)$, $\mathcal{SU}(m)$ etc.

To understand the commutator of left invariant vector fields and for important geometric reasons, we now introduce the Maurer-Cartan equations. Denote the matrix of 1-forms $g^{-1}dg$ on $GL(m,K)$ by ω. Since $dg^{-1} = -g^{-1}(dg)g^{-1}$ we obtain the important relation (known as *Maurer-Cartan equations*)

$$d\omega + \omega\wedge\omega = 0. \tag{0.3.5.7}$$

Notice that here by $\omega\wedge\omega$ we mean matrix multiplication except that ordinary multiplication of scalars is replaced by wedge product of 1-forms. In long hand notation (0.3.5.7) means

$$d\omega_{ij} + \sum_k \omega_{ik}\wedge\omega_{kj} = 0. \tag{0.3.5.8}$$

If $\varphi : G \to GL(m, K)$ is a homomorphism, then $\varphi^\star(\omega)$ is matrix of left invariant 1-forms on G and satisfies

$$d\varphi^\star(\omega) + \varphi^\star(\omega) \wedge \varphi^\star(\omega) = 0, \quad \text{or} \quad d\varphi^\star(\omega_{ij}) + \sum_k \varphi^\star(\omega_{ik}) \wedge \varphi^\star(\omega_{kj}) = 0.$$

Note also that if φ is injective then the matrix $\varphi^\star(\omega)$ contains a basis of left invariant 1-forms for G. *If G is a closed subgroup of $GL(m, K)$ we simply write ω instead of $\varphi^\star(\omega)$.* The Maurer-Cartan relations have profound geometric implications which to some extent will be exploited in this text.

We had noted earlier that $\mathcal{G}$ can be identified with the tangent space at the identity, which for matrix groups such as $SL(m, \mathbb{R})$, $SO(m)$ etc. was a linear space of matrices. The Maurer-Cartan equations enable us to translate the bracket operation $[,]$ on the Lie algebra into an algebraic operation in the corresponding linear spaces of matrices. To do so, first consider $G = GL(m, \mathbb{R})$ so that $\mathcal{G} = \mathcal{GL}(m, \mathbb{R}) = M_m(\mathbb{R})$ is the linear space of $m \times m$ real matrices. (We use the notation $\mathcal{GL}(m, \mathbb{R})$ rather than $M_m(K)$ to emphasize the Lie algebra structure.) The entries ω_{ij} of the matrix $\omega = g^{-1}dg$ form a basis for left invariant 1-forms. Let the dual left invariant vector fields be denoted by ξ_{ij} so that $\omega_{ij}(\xi_{kl}) = \delta_{ik}\delta_{jl}$. The formula for exterior differentiation simplifies into

$$d\beta(\eta, \zeta) = -\frac{1}{2}\beta([\eta, \zeta]) \tag{0.3.5.9}$$

for a left invariant 1-form β and left invariant vector fields η and ζ. Setting $\eta = \sum A_{kl}\xi_{kl}$, $\zeta = \sum B_{kl}\xi_{kl}$ and $\beta = \omega_{ij}$ and using the Maurer-Cartan equations $d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = 0$ and (0.3.5.9), we obtain after a simple calculation

$$\omega_{ij}([A, B]) = \sum_k (A_{ik}B_{kj} - B_{ik}A_{kj}). \tag{0.3.5.10}$$

This equation means that under the identification of left invariant vector fields on $G = GL(m, \mathbb{R})$ with $m \times m$ matrices, the bracket of left invariant vector fields translates into the commutator $[A, B] = AB - BA$ of matrices. In this manner, the complex operation of computing the commutator of differential operators becomes the much simpler problem of algebraically calculating brackets of matrices. A Lie algebra $\mathcal{G}$ is *abelian* if for all $\xi, \eta \in \mathcal{G}$ we have $[\xi, \eta] = 0$. A homomorphism $\rho : \mathcal{G} \to \mathcal{L}$ of Lie algebras is a linear mapping such that

$$\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)].$$

A representation of a Lie algebra $\mathcal{G}$ is a Lie algebra homomorphism $\rho : \mathcal{G} \to \mathcal{GL}(N, K)$ where $K = \mathbb{R}$ or $\mathbb{C}$. Sometimes it may be necessary to specify whether a representation is real or complex. The *trivial homomorphism* of a Lie algebra is a mapping taking everything to 0.

Remark 0.3.5.1 According to the *Baker-Campbell-Hausdorff formula* (abbreviated as BCH) we have

$$\exp(A)\exp(B) = \exp(\gamma(A,B)),$$

where

$$\gamma(A,B) = (A+B) + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] + [B,[B,A]]) + \cdots$$

(The n^{th} term of $\gamma(A,B)$ is the sum of certain n-fold brackets of A and B.) This formula has the limitation that the series for $\gamma(A,B)$ is generally divergent and generically converges only when both A and B are in a small neighborhood of the origin. For example, consider $g = \begin{pmatrix} -e^s & 0 \\ 0 & -e^{-s} \end{pmatrix} \in SL(2,\mathbb{R})$, and note that $g = \exp(A)\exp(B)$ where

$$A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}, \quad B = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}.$$

We will show that $\gamma(A,B)$ is necessarily divergent for $s \neq 0$. First we prove that if $s \neq 0$ then $g \neq \exp(C)$ for any real matrix C. In fact, the eigenvalues λ_i of C are either both real or they are complex conjugates of each other. In either case it is impossible to satisfy both equations

$$e^{\lambda_1} = -e^s, \quad \text{and} \quad e^{\lambda_2} = -e^{-s}.$$

Since $A^{-1}\exp(C)A = \exp(A^{-1}CA)$ we have shown $g \neq \exp(C)$ for $s \neq 0$. (See also exercise 0.3.5.2 above.) Since $g = \exp(A)\exp(B)$, this implies that the series $\gamma(A,B)$ is necessarily divergent for all $s \neq 0$. The proof of BCH can be found in many texts on Lie groups and/or Lie algebras (e.g. [Ho]).

Let $G \subset GL(m,\mathbb{R})$ be a Lie subgroup. In view of (0.3.5.4) and the convergence of the series $\gamma(\sqrt{t}A, \sqrt{t}B)$ in BCH for $t > 0$ sufficiently small, and (0.3.5.4) we have

$$\frac{\rho(\exp(t[A,B]))(v) - v}{t} = \frac{\rho(\exp(\sqrt{t}A)\exp(\sqrt{t}B))(v) - \rho(\exp(\sqrt{t}B)\exp(\sqrt{t}A))(v) - v}{t} + O(t^{1/2}).$$

It then follows easily that the linear map $\rho : \mathcal{G} \to \mathcal{GL}(N,\mathbb{R})$ is a representation of Lie algebras, i.e., $\rho([A,B]) = [\rho(A), \rho(B)]$. ♡

Example 0.3.5.1 The Lie algebra operation $[.,.]$ on $\mathcal{SO}(3)$ is related to the vector product $\times$ on $\mathbb{R}^3$ familiar from advanced calculus. Consider the basis

$$f_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

for $\mathcal{SO}(3)$ which is orthonormal relative to the inner product $\frac{1}{2}\mathrm{Tr}(AB)$. Furthermore,

$$[f_1, f_2] = f_3, \quad [f_2, f_3] = f_1, \quad [f_3, f_1] = f_2.$$

It follows that the linear mapping $f_j \rightarrow e_j$, where e_1, e_2, e_3 is the standard basis for $\mathbb{R}^3$, preserves inner products and maps Lie algebra operation $[.,.]$ into vector product. Jacobi identity for $\mathcal{SO}(3)$ translates into the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \times (\mathbf{b} \times \mathbf{a}) - \mathbf{b} \times (\mathbf{c} \times \mathbf{a}).$$

The invariance condition $\mathrm{Tr}(\mathrm{ad}(\xi)\eta, \zeta) + \mathrm{Tr}(\eta, \mathrm{ad}(\xi)\zeta) = 0$ translates into

$$(\mathbf{a} \times \mathbf{b}).\mathbf{c} - (\mathbf{c} \times \mathbf{a}).\mathbf{b} = 0,$$

where . denotes the standard inner product on $\mathbb{R}^3$. The standard identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$$

is proven by using tri-linearity and checking its validity on basis vectors e_j or f_j. ♠

Example 0.3.5.2 In this example we look at the group $SO(4)$ which plays an important role in geometry and has features different from other orthogonal groups. It is most easily understood by introducing Hamilton's quaternions. Let $\mathbf{H}$ be the division algebra of Hamilton quaternions, i.e., the real vector space of dimension 4 with basis $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and law of multiplication given by ($\mathbf{1}$ is the identity)

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

There is the operator of conjugation on $\mathbf{H}$ given by

$$q = a_\circ \mathbf{1} + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \longrightarrow q^\star = a_\circ \mathbf{1} - a_1 \mathbf{i} - a_2 \mathbf{j} - a_3 \mathbf{k}.$$

We define inner product on $\mathbf{H}$ via the positive definite quadratic form $||q||^2 = qq^\star$. Since $||qq'|| = ||q||\ ||q'||$, the unit sphere in $\mathbf{H}$ is a group with the inverse of q being $q^\star$. It is called the group of *unit quaternions*. Quaternions can be represented as 2×2 complex matrices. In fact, the mapping

$$\mathbf{1}\to\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \mathbf{i}\to\begin{pmatrix}i&0\\0&-i\end{pmatrix},\ \mathbf{j}\to\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ \mathbf{k}\to\begin{pmatrix}0&i\\i&0\end{pmatrix},$$

realizes $\mathbf{H}$ as a division algebra of 2×2 matrices over $\mathbb{R}$. Since $SU(2)$ is the set of complex matrices $\begin{pmatrix}a+ib&c+id\\c-id&a-ib\end{pmatrix}$ such that $a^2+b^2+c^2+d^2=1$, it can be identified be the unit quaternions. Note that multiplication of unit quaternions is identical with the group operation in $SU(2)$. This enables us to define a homomorphism $\delta: SU(2)\times SU(2)\to SO(4)$ by

$$\delta(q',q)(x)=q'xq^\star,\quad x\in\mathbf{H},\ q,q'\in SU(2).$$

Now $\ker\delta=\pm(I,I)$ and therefore $SO(4)\simeq SU(2)\times SU(2)/(\pm(I,I))$ for dimension reasons and connectivity. ♠

Example 0.3.5.3 It is useful to look at the Lie algebra version of the isomorphism of example 0.3.5.2. The Lie algebra of $K=SO(4)$ is the space of 4×4 skew symmetric matrices. To compute the decomposition of $\mathcal{K}$ corresponding to the product structure of $SO(4)$, we consider the basis for $\mathcal{SU}(2)$ consisting of matrices

$$\kappa_1=\begin{pmatrix}i&0\\0&-i\end{pmatrix},\quad \kappa_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \kappa_3=\begin{pmatrix}0&i\\i&0\end{pmatrix}.$$

Regarding $\exp(t\kappa_l)$ as a quaternion we compute the linear transformations

$$q\to\frac{d}{dt}_{|t=0}\exp(t\kappa_l)q,\quad q\to\frac{d}{dt}_{|t=0}q\exp(t\kappa_l),\qquad l=1,2,3,$$

and regard them as 4×4 matrices. After a simple calculation we see that the desired isomorphism at the level of Lie algebras translates into the decomposition $\mathcal{K}=\mathcal{K}_1\oplus\mathcal{K}_2$ where $\mathcal{K}_i$ is the set of real matrices of the form

$$\mathcal{K}_1:\ \begin{pmatrix}0&-a&-b&-c\\a&0&-c&b\\b&c&0&-a\\c&-b&a&0\end{pmatrix},\qquad \mathcal{K}_2:\ \begin{pmatrix}0&-a&-b&-c\\a&0&c&-b\\b&-c&0&a\\c&b&-a&0\end{pmatrix}$$

Note $\mathcal{K}_i\simeq\mathcal{SO}(3)$ and $SO(3)\simeq SU(2)/\pm I$. ♠

Quaternions can be used to give an alternative interpretation to the compact symplectic group. For an $n\times n$ matrix $A=(A_{ij})$ with entries from $\mathbf{H}$, define its conjugate as $A^\star=(A^\star_{ij})$ where

$$(A^\star)_{ij}=(A_{ji})^\star.$$

Then it is easily verified that $(AB)^\star = B^\star A^\star$. Consequently the subset of those A's with the property $A^\star A = I$ is a group which we temporarily denote by $U(n,\mathbf{H})$. This is the analogue of the unitary group where complex numbers are replaced by quaternions. It is readily verified that $U(n,\mathbf{H})$ is a compact group. To better understand $U(n,\mathbf{H})$ we make use of the isomorphism between the space $\mathbf{H}^n$ of n-tuples of quaternions and $\mathbb{C}^{2n}$ by writing each component of $\mathbf{q}=(q_1,\cdots,q_n)$ as

$$q_j = \mathbf{1}z_j + \mathbf{j}z_{n+j},$$

where z_k's are complex numbers. This gives the embedding $\jmath : U(n,\mathbf{H}) \rightarrow U(2n)$. The following exercise shows that $U(n,\mathbf{H}) \simeq USp(n)$:

Exercise 0.3.5.3 *With the above notation show that* $\mathrm{Im}\jmath$ *is the subgroup of* $2n\times 2n$ *unitary matrices leaving the bilinear forms*

$$\sum_{j=1}^{2n}\bar{z}_j w_j \quad \text{and} \quad \sum_{j=1}^{n}(z_j w_{n+j} - z_{n+j}w_j)$$

invariant. Deduce the isomorphism $U(n,\mathbf{H})\simeq USp(n)$, *and in particular* $USp(1)\simeq SU(2)$.

Example 0.3.5.4 In this example we use the algebra of quaternions to give a generalization of the upper half plane with the action of fractional linear transformation to dimension three. The *upper half space* is defined as

$$\mathcal{H}_3 = \{z + t\mathbf{j} \mid z\in\mathbb{C},\ t>0\}.$$

For $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{C})$, $\zeta\in\mathcal{H}_3$ define

$$g:\zeta \longrightarrow g\cdot\zeta = (a\zeta+b)(c\zeta+d)^{-1}.$$

Here $\zeta = z+t\mathbf{j}$ is regarded as a quaternion and all algebraic operations are carried in the algebra of quaternions. Naturally $\mathbf{i}$ is identified with $i=\sqrt{-1}$. It is straightforward to verify that $g\cdot\zeta$ has no component along the quaternion $\mathbf{k}$ and its $\mathbf{j}$-component is positive. The isotropy subgroup of the quaternion $\mathbf{j}$ is the special unitary group $SU(2) \subset SL(2,\mathbb{C})$. The upper half space can be identified with the space $\mathcal{P}_2$ of 2×2 positive definite hermitian matrices of determinant 1. In fact, consider the mapping

$$p : SL(2,\mathbb{C}) \longrightarrow \mathcal{P}_2, \quad A \longrightarrow AA^\star,$$

where $A^\star$ denotes the complex conjugate transpose of A. Then p is surjective and endows $p : SL(2,\mathbb{C}) \to \mathcal{P}_2$ with the structure of a principal fibre bundle with structure group $SU(2)$. Every $P \in \mathcal{P}_2$ has the unique decomposition of the form

$$P = AA^\star, \quad \text{where } A = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}.$$

To relate left action of $SL(2,\mathbb{C})$ on $\mathcal{P}_2 \simeq SL(2,\mathbb{C})/SU(2)$ to that on $\mathcal{H}_3$ we make use of a simple trick. Write $\zeta = z + t\mathbf{j}$ and note

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix} \begin{pmatrix} \mathbf{j} \\ 1 \end{pmatrix} = \begin{pmatrix} \zeta \\ 1 \end{pmatrix} \frac{1}{\sqrt{t}}$$

A matrix $\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in SU(2)$ is identified with the quaternion $u + \mathbf{j}v$ and

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} \mathbf{j} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{j} \\ 1 \end{pmatrix} (\bar{u} - \bar{v}\mathbf{j}).$$

To calculate the effect of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ we therefore write

$$gA \begin{pmatrix} \mathbf{j} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{j}' \\ 1 \end{pmatrix} \frac{1}{\sqrt{t'}} (\bar{u} - \bar{v}\mathbf{j})$$

for some positive real number t' and quaternion $u + \mathbf{j}v$. Setting $\zeta = z + t\mathbf{j}$ this equation yields

$$\begin{pmatrix} a\zeta + b \\ c\zeta + d \end{pmatrix} \frac{1}{\sqrt{t}} = \begin{pmatrix} \zeta' \\ 1 \end{pmatrix} \frac{1}{\sqrt{t'}} (\bar{u} - \bar{v}\mathbf{j}), \tag{0.3.5.11}$$

where $\zeta' = g \cdot \zeta \in \mathcal{H}_3$, t' is the component of ζ' along $\mathbf{j}$. Comparing the components of the vectors on both sides of (0.3.5.11) we obtain

$$\zeta' = (a\zeta + b)(c\zeta + d)^{-1}, \quad t' = \frac{t}{|c\zeta + d|^2}. \tag{0.3.5.12}$$

This simple trick is also applicable to $SL(2,\mathbb{R})$ acting on $\mathbf{H}_2$. ♠

Let $G \subset GL(m,\mathbb{R})$ be an analytic group. From the validity of BCH for A, B in a small neighborhood of $\mathbf{0} \in \mathcal{G}$, it follows that if $g \in G$ is sufficiently close to $e = I$, then $\mathcal{G}$ is invariant under the linear transformation $A \rightarrow gAg^{-1}$. Since G is generated by any neighborhood of identity, $\mathcal{G}$ is invariant under the conjugation action of G. We denote the linear transformation $A \rightarrow gAg^{-1}$ by $\mathrm{Ad}(g)$ and call it the *adjoint representation* of G. Recall that for $G = GL(m,K)$, and $m \times m$ matrix A representing a tangent vector at the identity, then the left (resp. right) invariant vector fields extending A is the tangent vector field gA (resp. Ag) at $g \in G$. Therefore, identifying $\mathcal{GL}(m,K)$ with left invariant vector fields, the adjoint representation describes the effect of right translation on a left invariant vector field. The representation of $\mathcal{G}$ corresponding to Ad is denoted by ad and it is easily verified that it is given by

$$\mathrm{ad}(A)(B) = AB - BA = [A,B].$$

Note that the Jacobi identity implies that ad is a representation of Lie algebras.

It should be pointed out that there are circumstances where the meaning of left invariant vector fields as differential operators and not just as matrices plays a pivotal role. Furthermore, the product of two left invariant vector fields does not correspond to their matrix product. In fact, the associative algebra generated by products of left invariant vector fields regarded as differential operators, is infinite dimensional and plays an important role in representation theory. It is called the *universal enveloping algebra* of $\mathcal{G}$.

Let $G \subset GL(m,\mathbb{R})$ be a Lie group and assume G acts on the manifold M smoothly. An important special case is when $M = K^N$ where $K = \mathbb{R}$, or $\mathbb{C}$ and we have a representation $\rho : G \rightarrow GL(N,K)$ so that G acts as group of linear transformations. Now we show that any smooth action of G on M allows one to assign to every left invariant vector field on G, a homogeneous first order differential operator on M, and in fact, induces a homomorphism of the universal enveloping algebra of G to the algebra of differential operators on M. This assignment of differential operators to left invariant vector fields is defined by

$$\xi(f)(x) = \lim_{t\rightarrow 0} \frac{f((\exp t\xi)(x)) - f(x)}{t} = \frac{d}{dt}_{|t=0} f((\exp(t\xi))(x)). \qquad (0.3.5.13)$$

for a smooth function f on M, $\xi \in \mathcal{G}$ and $x \in M$. Note that for $M = G$ and action of G on M by left translation the induced first order differential operator is the left invariant vector field itself.

Example 0.3.5.5 Let us compute left invariant differential operators as first order homogeneous differential operators on the simplest examples of Lie groups. On $\mathbb{R}^m$ left or right invariant vector fields are simply first order constant coefficient differential operators. On the torus $S^1 \times \cdots \times S^1$, they are linear combinations with constant coefficients of $\frac{\partial}{\partial \theta_j}$'s where θ_j's are the variables on S^1's. On the multiplicative group of positive real numbers, the operator $r\frac{\partial}{\partial r}$ is invariant under multiplication. On the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers, the operators

$$z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = r\frac{\partial}{\partial r}, \quad i(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) = \frac{\partial}{\partial \theta}$$

are left (or right) invariant, where (r, θ) is the polar coordinates representation. All these facts are simple consequences of the chain rule. Exhibiting left invariant vector fields on noncommutative matrix groups is more subtle than left invariant 1-forms. ♠

Example 0.3.5.6 Next consider $SO(2)$ acting on $\mathbb{R}^2$ as the group of rotations. Let $R(\theta) = \exp(\theta\kappa)$ be rotation through angle θ where $\kappa = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a basis for $\mathcal{SO}(2)$. Then $R(\theta)$ maps the point $\mathbf{x} = (x_1, x_2)$ to the point

$$(x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta).$$

Therefore differentiating $f(R(\theta)(\mathbf{x}))$ relative to θ at $\theta = 0$ we obtain the representation of κ as the differential operator

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}.$$

Similarly in $\mathbb{R}^3$, the differential operators

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}, \quad x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}, \quad x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}.$$

correspond to rotations in the (x_1, x_2), (x_2, x_3) and (x_3, x_1) planes respectively. ♠

Example 0.3.5.7 Let $G = SL(2, \mathbb{R})$ and $M = \mathcal{H} = \{z = x + iy | y > 0\}$ be the upper half plane with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ acting on $\mathcal{H}$ by fractional linear transformations:

$$z \longrightarrow \frac{az + b}{cz + d}.$$

We want to exhibit elements of $\mathcal{SL}(2, \mathbb{R})$ as linear differential operators on $\mathcal{H}$. It is convenient to introduce some notation. Let $g = k_\phi a_t u_x$ following the Iwasawa decomposition $G = KAU$, so that

$$k_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Bases for the Lie algebras $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{U}$ are given by

$$\kappa = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \upsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We use the right hand side of (0.3.5.13) and the chain rule to compute the desired differential operators. For example, to compute the operator representing κ we have to calculate

$$\frac{d}{d\theta}_{|\theta=0} f((\exp(\theta\kappa))(z), (\exp(\theta\kappa))(\bar{z})) = (\frac{dz}{d\theta}\frac{\partial}{\partial z} + \frac{d\bar{z}}{d\theta}\frac{\partial}{\partial \bar{z}})_{|\theta=0} f((\exp(\theta\kappa))(z), (\exp(\theta\kappa))(\bar{z})),$$

where f is a C^∞ function of (x, y) or equivalently of $(z, \bar{z})$. Consequently

$$\kappa \longleftrightarrow -(1+z^2)\frac{\partial}{\partial z} - (1+\bar{z}^2)\frac{\partial}{\partial \bar{z}}.$$

Similarly, we obtain the operators representing α and υ:

$$\alpha \longleftrightarrow 2(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}), \quad \upsilon \longleftrightarrow \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}.$$

As operators on $G = SL(2, \mathbb{R})$ the operators corresponding κ, α and υ are given in exercise 0.3.5.4. ♠

Exercise 0.3.5.4 *We continue with $G = SL(2, \mathbb{R})$ and the notation of example 0.3.5.7. Show that relative to the parametrization (ϕ, t, x) of the Iwasawa decomposition, the differential operators κ, α and υ are given by*

$$\kappa \longleftrightarrow e^{-2t}\frac{\partial}{\partial \phi} + x\frac{\partial}{\partial t} + (e^{-4t} - 1 - x^2)\frac{\partial}{\partial x}.$$

and

$$\alpha \longleftrightarrow \frac{\partial}{\partial t} - 2x\frac{\partial}{\partial x}, \quad \upsilon \longleftrightarrow \frac{\partial}{\partial x}.$$

0.3.6 Subgroups and Subalgebras

For the same reason that in the definition of a submanifold it was necessary to distinguish between immersed and embedded submanifolds, it is judicious to exercise some care in the definition of a Lie subgroup. Recall from example 0.1.1.1 that if $1, x_1, \cdots, x_n$ are linearly independent over the rationals, then the image of

$$\rho \; : \; t \longrightarrow (e^{2\pi i x_1 t}, \cdots, e^{2\pi i x_n t})$$

is dense in $T^n - S^1 \times \cdots \times S^1$. In particular, Im$\rho$, which is isomorphic to $\mathbb{R}$, is an immersed and not an embedded subgroup of T^n. By a *Lie subgroup* $H \subset G$ we mean a Lie group H together with an injective homomorphism $\jmath : H \rightarrow G$. The manifold structure on H need not be induced from that G. This kind of phenomenon occurs often when constructs a "submanifold" by invoking the theorem of Frobenius on integrability of a Pfaffian system. All Lie subgroups can be obtained via the theorem of Frobenius, and as long as one keeps in mind that the manifold structure of the subgroup may be different (and more natural) than that induced from the ambient group, no problem should arise. *In circumstances when it necessary to avoid the density phonomenon described in 0.1.1.1 we consider closed subgroups of a Lie group* G.

For a Lie subgroup $G \subset GL(m, \mathbb{R})$, taking commutators of two left invariant vector fields on G and $GL(m, \mathbb{R})$ coincide. Therefore the reduction of $[\eta, \zeta]$ to the algebraic operation of matrix bracket remains valid for Lie subgroups of $G \subset GL(m, \mathbb{R})$ as well. We summarize this important fact as

Proposition 0.3.6.1 *Let $G \subset GL(m, \mathbb{R})$ be a Lie group. Then under the natural identification of the Lie algebra $\mathcal{G}$ of G with the tangent space to G at $e = I \in G$, the commutator of two left invariant vector fields translates into the commutator of the corresponding matrices.*

Let $\eta_1, \cdots, \eta_r$ be left invariant 1-forms on the Lie group $GL(m, \mathbb{R})$. Then the equations $\eta_j = 0$ at $e = I \in GL(m, \mathbb{R})$ is a system of homogeneous linear equations and define a subspace of $\mathcal{G} \subset \mathcal{GL}(m, \mathbb{R})$. Then in view of the above proposition and the integrability theorem of Frobenius, the Pfaffian system $\eta_j = 0$, $j = 1, \cdots, r$ is integrable if and only if the corresponding subspace $\mathcal{G}$ is closed under $[,]$ of matrices. We have shown

Corollary 0.3.6.1 *A left invariant Pfaffian system on a Lie group $GL(m, \mathbb{R})$ is integrable if and only if the corresponding subspace of matrices is a Lie algebra.*

Clearly this corollary implies the more general

Corollary 0.3.6.2 *A left invariant Pfaffian system on a Lie group $G \subset GL(m, \mathbb{R})$ is integrable if and only if the corresponding subspace of matrices is a Lie algebra.*

Having clarified the notion of subgroup and subalgebra, we can now discuss invariant volume elements and Riemannian metrics on certain homogeneous spaces. Let $H \subset G$ be a Lie subgroup of the Lie group G. Then $\mathcal{H}$ is invariant under Ad(h) and therefore we have representation $\mathrm{Ad}_{\mathcal{G}/\mathcal{H}}(h)$ of H on $\mathcal{G}/\mathcal{H}$. This concept in often useful in understanding homogeneous spaces. For example, let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup and $\omega_1, \cdots, \omega_n$ be a basis of left invariant 1-forms on G. Let $\mathcal{H} \subset \mathcal{G}$ be a subalgebra defined by the equations

$$\omega_1 = 0, \cdots, \omega_k = 0,$$

and assume $H \subset G$ a closed subgroup (not necessarily connected) with Lie algebra $\mathcal{H}$. We want to see whether we can define a G-invariant volume element on G/H. A natural candidate is $dv_{G/H} = \omega_1 \wedge \cdots \wedge \omega_k$, however, *a priori*, $dv_{G/H}$ is defined on G and not on G/H. It is evident that right invariance of $dv_{G/H}$ under H is a necessary and sufficient for $dv_{G/H}$ to be defined on G/H. Invariance under right translation is identical with the the algebraic condition

$$\det(\mathrm{Ad}_{\mathcal{G}/\mathcal{H}}(h)) = 1 \qquad \text{for } h \in H. \tag{0.3.6.1}$$

Summarizing, we have

Proposition 0.3.6.2 *With the above notation and hypothesis, there is at most one (up to multiplication by a constant) G-invariant k-form on G/H and relation (0.3.6.1) is a necessary and sufficient condition for the G-invariant k-form $dv_{G/H}$ to be defined on G/H.*

Example 0.3.6.1 As a special case consider the *oriented inhomogeneous Grassmann manifold* $\tilde{\mathbf{G}}_{k,n}(\mathbb{R})$ which is defined as the oriented k-dimensional affine subspaces of $\mathbb{R}^{k+n}$. Let $SE(n)$ be the group of proper Euclidean motions of $\mathbb{R}^n$ (see example 0.2.3.3), and $SE(k) \subset SE(k+n)$ be closed subgroup of the group of rigid motions of $\mathbb{R}^{k+n}$ acting on the first k coordinates. Then $\tilde{\mathbf{G}}_{k,n}(\mathbb{R}) = SE(k+n)/SE(k) \times SO(n)$, where $SO(n)$ acts on the last n coordinates of $\mathbb{R}^{k+n}$. It is trivial that condition (0.3.6.1) is satisfied and therefore we have a G-invariant volume element on $\tilde{\mathbf{G}}_{k,n}(\mathbb{R})$. Similarly, (oriented or complex) flag manifolds carry invariant volume elements. ♠

Exercise 0.3.6.1 *Let $SO(n)$ be embedded in $SO(n+1)$ as the subgroup of matrices $\{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}\}$, $A \in SO(n)$. Show that $SO(n+1)/SO(n) \simeq S^n$ and the $SO(n)$-invariant volume element obtained by the above procedure is identical (up to multiplication by a nonzero constant) with the usual volume element on S^n given in example 0.3.1.2.*

Proposition 0.3.6.3 *Let $\mathcal{I} = \{\eta_1, \cdots, \eta_r\}$ be a integrable Pfaffian system consisting of left invariant 1-forms η_j on $GL(m, \mathbb{R})$. Set $\mathcal{G} = \{\xi \in \mathcal{GL}(m, \mathbb{R}) | \eta_j(\xi) = 0, \ j = 1, \cdots, r\}$, and let G be the maximal connected integral manifold for $\mathcal{I}$ passing through $e = I \in GL(m, \mathbb{R})$. Then G is an analytic group with Lie algebra $\mathcal{G}$.*

Proof - It suffices to show G is closed under multiplication and inversion. Let $g, h \in G$. By left invariance of the of the Pfaffian system $\mathcal{I}$, the 1-forms η_j vanish on $h^{-1}G$ as well as on G. Clearly $h^{-1}G \cap G \neq \emptyset$ since both sets contain $e = I$. Therefore $h^{-1}G = G$ and $h^{-1}g \in G$ proving the proposition. ♠

By an *analytic subgroup* we mean a connected Lie subgroup (with the manifold structure of the Lie subgroup not necessarily induced from the ambient group). An immediate consequence of proposition 0.3.6.3 is

Corollary 0.3.6.3 *Let $G \subset GL(m, \mathbb{R})$ be a Lie subgroup. Then there is a one to one correspondence between subalgebras of $\mathcal{G}$ and analytic subgroups of G.*

Together with lemma 0.3.5.1, this corollary implies

Corollary 0.3.6.4 *Let $\mathcal{G}$ be a subalgebra of $\mathcal{GL}(m, K)$. Then* exp *maps $\mathcal{G}$ into the corresponding Lie subgroup G.*

We can also construct G-invariant Riemannian metrics on certain homogeneous spaces of compact Lie groups G. It is a consequence of the following lemma, sometimes called *Weyl's unitary trick*, that every compact Lie group G carries Riemannian metric which is invariant under both left and right translations:

Lemma 0.3.6.1 *Let $\rho : G \rightarrow GL(m, \mathbb{C})$ be a representation of a compact group G. Then there is an hermitian inner product $<,>$ on $\mathbb{C}^m$ such that the linear transformations $\rho(g)$ are unitary with respect to an orthonormal basis relative to $<,>$. Similarly, if $\rho : G \rightarrow GL(m, \mathbb{R})$, then there is an inner product $<,>$ on $\mathbb{R}^m$ such that the linear transformations $\rho(g)$ are orthogonal with respect to an orthonormal basis relative to $<,>$.*

Proof - Let $(.,.)$ be any hermitian inner product, then we define

$$\prec v, w \succ = \int_G (\rho(g)(v), \rho(g)(w)) dv_G,$$

where dv_G is an invariant volume element on G. Clearly $\prec ., \succ$ has the required properties. ♣

Corollary 0.3.6.5 *Every compact Lie group G carries a Riemannian metric which is invariant under both left and right translations.*

Proof - Applying lemma 0.3.6.1 to the adjoint representation of G we obtain an inner product $\prec .,. \succ$ on $\mathcal{G}$ which is invariant under the adjoint action. It follows that left translation of $\prec .,. \succ$ is required Riemannian metric. ♣

Assume the algebra $\mathcal{G}$ admits of the decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M} \tag{0.3.6.2}$$

where $\mathcal{M}$ is the orthogonal complement of $\mathcal{K}$ in $\mathcal{G}$ relative to an inner product $\prec .,. \succ$ which is invariant under the adjoint action. We shall show in §5.1 that $\mathcal{M}$ is necessarily invariant under the adjoint action of K. Since K acts by orthogonal transformations on $\mathcal{M}$, the restriction of $\prec .,. \succ$ to $\mathcal{M}$ extends of a G-invariant Riemannian metric on G/K. If K is defined by the Pfaffian system

$$\omega_1 = 0, \cdots, \omega_k = 0,$$

and $\omega_1, \cdots, \omega_n$ is an orthonormal basis for $\mathcal{G}^\star$ relative to $\prec .,. \succ$ (or more precisely the dual inner product), then then the G-invariant inner product on G/K is $\omega_{k+1}^2 + \cdots + \omega_n^2$.

Example 0.3.6.2 Let $G = U(m+1)$, then for the inner product $\prec .,. \succ$ of the proof of corollary 0.3.6.5 we can take

$$\prec A, B \succ = -\mathrm{Tr}(AB)$$

where $A, B \in \mathcal{U}(m+1)$ which is the space of skew hermitian matrices. Now let $K = U(1) \times U(m) \subset U(m+1)$ where $U(1)$ (resp. $U(m)$) acts on the first coordinate (resp. last m coordinates). Then $\mathcal{M}$ is the set of matrices of the forms

$$\begin{pmatrix} 0 & z_1 & z_2 & \cdots & z_m \\ -\bar{z}_1 & 0 & 0 & \cdots & 0 \\ -\bar{z}_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\bar{z}_m & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The restriction of the inner product $\prec .,. \succ$ to $\mathcal{M}$ gives the standard hermitian inner $z_1\bar{z}_1 + \cdots + z_m\bar{z}_m$ on $\mathbb{C}^m$. Since $\mathbb{C}P(m) \simeq U(m+1)/U(1) \times U(m)$, we obtain a $U(m+1)$-invariant Riemannian metric on $\mathbb{C}P(m)$ which is generally called the *Fubini-Study* metric. ♠

Exercise 0.3.6.2 *Constrict $U(k+m)$-invariant Riemannian metric on the complex Grassmann manifold $\mathbf{G}_{k,m}$ by repeating the argument of example 0.3.6.2.*

Exercise 0.3.6.3 *Repeat exercise 0.3.6.2 for real Grassmann manifolds.*

0.3.7 Contact and Symplectic Forms

So far we have encountered two important tensors namely the Riemannian metric which is a contravariant symmetric 2-tensor and the volume element. In this subsection we introduce two other forms which appear in geometry and physics.

A symplectic form ω on an open set $U \subset \mathbb{R}^n$ is a nondegenerate closed 2-form. *Non-degeneracy* means that the restriction of ω to each tangent space is a non-degenerate (antisymmetric) bilinear form. Thus from linear algebra $n = 2m$ is an even integer, and non-degeneracy of ω is equivalent to $\omega \wedge \omega \wedge \cdots \wedge \omega$ (m-fold product) being a volume element. A *symplectic manifold* is a pair (N, ω) where N is a manifold and ω is a symplectic form on N. In other words, the restriction of ω to each coordinate neighborhood U is a symplectic form. The simplest example of a symplectic manifold is $\mathbb{R}^{2m}$ together with the symplectic form

$$\omega_\circ = dx_1 \wedge dx_{m+1} + dx_2 \wedge dx_{m+2} \wedge \cdots \wedge dx_m \wedge dx_{2m}, \tag{0.3.7.1}$$

where $x_1, \cdots, x_{2m}$ are standard coordinates in $\mathbb{R}^{2m}$. It follows from the definition of the symplectic group that $\omega_\circ$ is invariant under linear changes of coordinates by symplectic transformations (i.e., elements of $Sp(m, \mathbb{R})$). Another simple example of a symplectic manifold is an orientable surface M together with a volume element $\omega = dv_M$.

Example 0.3.7.1 The complex projective space $\mathbb{C}P(n)$ has the structure of a symplectic manifold. In fact, we consider the realization $\mathbb{C}P(n) \simeq G/K$ where $G = U(n+1)$ and $K = U(1) \times U(n)$ and the decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$ (see example 0.3.6.2). Let $\prec, \succ$ be the skew symmetric bilinear form on $\mathcal{M}$ defined by

$$\prec \xi, \eta \succ = -\frac{1}{2}\Im \mathrm{Tr}(\xi\eta) = \frac{1}{2}\sum_j (\xi_{j1}\eta_{j2} - \xi_{j2}\eta_{j1}),$$

where $\xi, \eta \in \mathcal{M}$ as in example 0.3.6.2 and $\xi_j = \xi_{j1} + i\xi_{j2}$. Just as in the case of a Riemannian metric, $\prec, \succ$ extends to G-invariant nondegenerate 2-form ω on $\mathbb{C}P(n)$. From $[\mathcal{M}, \mathcal{M}] \subseteq \mathcal{K}$ we see that $d\omega(\xi, \eta) = \frac{1}{2}\omega([\xi, \eta]) = 0$ for $\xi, \eta \in \mathcal{M}$ and consequently ω is closed which proves that $(\mathbb{C}P(n), \omega)$ is a symplectic manifold. It is clear from our construction that the symplectic form ω is invariant under $U(n+1)$. Later we show that up to multiplication by a non-zero scalar, it is the only $U(m+1)$-invariant symplectic form on $\mathbb{C}P(n)$. This symplectic form is an example of a Kähler form which plays an important role in complex geometry. ♠

Exercise 0.3.7.1 *By mimicking example 0.3.7.1 exhibit a $U(k+n)$-invariant symplectic form on the complex Grassmann manifold $\mathbf{G}_{k,n}$.*

Exercise 0.3.7.2 *Let* $\mathbf{s} : 0 < s_1 < \cdots < s_r < n$ *and* $\mathbf{F_s}$ *be the corresponding complex flag manifold. Exhibit a* $U(n)$*-invariant symplectic form on* $\mathbf{F_s}$*. If* $\mathbf{F_s}$ *is not a Grassmann manifold, show that the space of* $U(n)$*-invariant symplectic structures on* $\mathbf{F_s}$ *has dimension* > 1.

Probably the most important example of a symplectic manifold is the cotangent bundle $N = \mathcal{T}^\star M$ of any manifold M together with the canonically defined exact 2-form ω which we now describe. Let $\pi : \mathcal{T}^\star M \to M$ and for a tangent vector v to $\mathcal{T}^\star M$ at a point $\theta \in \mathcal{T}^\star M$ set

$$\varepsilon(\theta) = \theta(\pi_\star(v)),$$

and define $\omega = -d\varepsilon$. To understand ε and in particular show that $(\mathcal{T}^\star M, \omega)$ is a symplectic manifold we let $(x_1, \cdots, x_m)$ be a coordinate system on $U \subset M$. Then the differentials $dx_1, \cdots, dx_m$ form bases for the cotangent spaces $\mathcal{T}^\star_x M$ for $x \in U$. Therefore there are linear functions $\theta_{x,1}, \cdots, \theta_{x,m}$ (or simply θ_i) on each $\mathcal{T}^\star_x M$, $x \in U$, such that $v = \sum_i \theta_{x,i}(v) dx_i$, i.e., $\theta_{x,i}(v)$'s are the coefficients of the expression of v in terms of the basis $dx_1, \cdots, dx_m$ for $\mathcal{T}^\star_x M$ for every $x \in U$. Therefore

$$\varepsilon = \sum_i \theta_i dx_i, \quad \text{and} \quad \omega = -d\varepsilon = \sum_i dx_i \wedge d\theta_i. \tag{0.3.7.2}$$

It is clear from (0.3.7.2) that $(\mathcal{T}^\star M, \omega)$ is a symplectic manifold. If a Riemannian metric ds^2 is fixed on M, then the symplectic structure on $\mathcal{T}^\star M$ can be transported to the $\mathcal{T}M$ by invoking the isomorphism $\mathcal{T}^\star_x M \xrightarrow{\sim} \mathcal{T}_x M$ induced by ds^2. More precisely the linear functions θ_i are transported to the tangent space $\mathcal{T}_x M$ to yield $\phi_i = \sum_j g_{ij}\theta_j$. Now set $\tilde{\varepsilon} = \sum_i \phi_i dx_i$, then the symplectic form on the tangent bundle is

$$\tilde{\omega} = -d\tilde{\varepsilon} = \sum_i dx_i \wedge d\phi_i, \tag{0.3.7.3}$$

which has the same form as (0.3.7.1). This is in fact the general local normal form for any symplectic form, i.e., by a diffeomorphism we can locally write a symplectic form as (0.3.7.1). The method (due to J. Moser) for the proof of this fact, which we now describe has other implications as well.

Let $\omega_\circ$ be as given by (0.3.7.1) and ω be given. Let ξ^t, $t \in \mathbb{R}$, be a time dependent vector field on M. On $M \times \mathbb{R}$ we regard ξ^t as time independent vector field which for each fixed t is tangent to the manifold $M \times \mathbb{R}$. Thus for fixed t, ξ^t defines a parameter group φ^t_s (relative to s) of diffeomorphisms of $M \times \mathbb{R}$. Let θ be a differential form $M \times \mathbb{R}$ and assume θ does not involve the differential dt. We want to compute the derivative of θ relative to the *diagonal* in the sense

$$\frac{d}{dt}(\varphi_t^t)^\star(\theta).$$

This derivative involves two kinds of terms, viz., differentiation relative to the horizontal component M and the vertical component $\mathbb{R}$. The derivative along horizontal component gives

$$\lim_{h\to 0}\frac{(\varphi_{t+h}^t)^\star(\theta)-(\varphi_t^t)^\star(\theta)}{h}=(\varphi_t^t)^\star\mathbf{L}_{\xi^t}(\theta)$$

where in the application of the Lie derivative $\mathbf{L}_{\xi^t}$, t is a fixed number. The vertical term gives

$$(\varphi_t^t)^\star(\frac{d\theta}{dt}),$$

where the derivative $\frac{d\theta}{dt}$ means differentiating the coefficients of the differential form θ relative to t. Thus, in view of H. Cartan's formula, we obtain

$$\frac{d}{dt}(\varphi_t^t)^\star(\theta)=(\varphi_t^t)^\star\left[\frac{d\theta}{dt}+i_{\xi^t}d\theta+di_{\xi^t}\theta\right]. \tag{0.3.7.4}$$

We emphasize that in the application of i_{ξ^t} and d in second and third terms on the right hand side, t is regarded as a fixed number. We apply (0.3.7.4) to the differential form $\theta=(1-t)\omega_\circ+t\omega$. Then $d\theta=0$ (exterior derivative does not involve differentiation relative to t) and we obtain

$$\frac{d}{dt}(\varphi_t^t)^\star(\theta)=(\varphi_t^t)^\star\left[\frac{d\theta}{dt}+d\sigma_t\right], \tag{0.3.7.5}$$

where $\sigma_t=i_{\xi^t}\theta$. We can now prove

Proposition 0.3.7.1 (Darboux) - *Let ω be a symplectic form on $U\subset\mathbb{R}^{2n}$, and $\mathbf{0}\in U$. Then there is a neighborhood $B\subset U$ of $\mathbf{0}$ and a diffeomorphism $\phi:B\to B$ such that $\phi^\star(\omega)=\omega_\circ$*

Proof - By a linear transformation we may assume that ω and $\omega_\circ$ are identical at $\mathbf{0}$. Let B be a neighborhood of $\mathbf{0}$ such that $\theta=(1-t)\omega_\circ+t\omega$ is non-degenerate in B for all $t\in I$. Let β be a 1-form such that $d\beta=\omega_\circ-\omega_1$, and determine the time dependent vector field ξ^t by

$$i_{\xi^t}\theta=\beta.$$

Here d is exterior differentiation on $\mathbb{R}^{2n}$, and the existence of ξ^t follows from non-degeneracy of θ. For this choice of ξ^t, right hand side of (0.3.7.5) vanishes and integrating the left hand side we obtain

$$(\varphi_1^1)^\star(\omega) - (\varphi_\circ^\circ)^\star(\omega_\circ) = 0.$$

Since $\varphi_\circ^\circ = \text{id.}$, the required result follows. ♣

The method of proof of proposition 0.3.7.1 can be applied to directly improve example 0.3.1.4 to a global result:

Exercise 0.3.7.3 *Let M be a compact orientable manifold with volume elements ω_1 and ω_2. Assume*

$$\int_M \omega_1 = \int_M \omega_2.$$

Show that there is a diffeomorphism ϕ of M such that $\phi^\star(\omega_2) = \omega_1$.

A notion related to symplectic structure is that of contact form. Let $U \subset \mathbb{R}^{2m+1}$ be an open subset and α be a 1-form such that

$$\alpha \wedge \underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{m \text{ times}}$$

is a volume element (i.e., nowhere vanishing) is called a *contact form.* A *contact form* on an odd dimensional manifold is a 1-form whose restriction to any coordinate neighborhood is a contact form, equivalently $\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha$ is a volume element on M. Note that the condition of contactness is more or less opposite to that of being integrable (see theorem of Frobenius). With coordinates $(x_1, \cdots, x_{2m}, t)$ on $\mathbb{R}^{2m+1}$, a basic example of a contact form is

$$\alpha_\circ = \sum_{j=1}^m x_j dx_{m+j} - cdt \tag{0.3.7.6}$$

where $c \neq 0$ is any constant. The following example gives a method for constructing contact forms from symplectic forms:

Example 0.3.7.2 Let (N, ω) be a symplectic manifold and ξ be a vector field such that $\mathbf{L}_\xi \omega = \omega$. Let M be a hypersurface in N which is everywhere transverse to ξ. Set $\alpha = i_\xi \omega$. Then

$$\omega = \mathbf{L}_\xi \omega = d i_\xi \omega = d\alpha,$$

which implies

$$\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha = \alpha \wedge \omega \wedge \cdots \wedge \omega.$$

From transversality of ξ to M and the fact that ω is a symplectic form, it easily follows that α is a contact form on the manifold M. A instance of this situation is when $N = \mathbb{C}^{m+1}\backslash 0$ with the usual symplectic structure

$$\omega_\circ = dx_1 \wedge dy_1 + \cdots + dx_{m+1} \wedge dy_{m+1},$$

where $z_j = x_j + iy_j$'s are standard coordinates in $\mathbb{C}^{m+1}$. Let M be the hypersurface S^{2m+1} : $\sum |z_j|^2 = 1$ and let

$$\xi = \frac{1}{2}\Big[\sum_{j=1}^{m+1} x_j dx_j + \sum_{j=1}^{m+1} y_j dy_j\Big].$$

It follows that the restriction of

$$\alpha = i_\xi \omega_\circ = \frac{1}{2}\sum_{j=1}^{m+1}(x_j dy_j - y_j dx_j).$$

is a contact form on S^{2m+1}. It is known that every compact orientable manifold of dimension three admits of a contact structure, however, the proof involves ideas which we have not discussed. ♠

Remark 0.3.7.1 The notion of contact manifold is more general than the description in terms of 1-forms given here. An odd dimensional manifold N is a *contact manifold* if it admits of a covering $\{U_j\}$ and 1-forms α_j defined on U_j such that α_j is a contact form on U_j and

$$\mathrm{Ker}(\alpha_{j,x}) = \mathrm{Ker}(\alpha_{k,x}) \quad \text{for all} \;\; x \in U_j \cap U_k.$$

Here $\alpha_{j,x}$ is the linear function $\mathcal{T}_x N \to \mathbb{R}$ determined by α_j. There is no requirement that the α_j's can be patched together to yield a globally defined 1-form with the required properties. When we have a globally defined contact 1-form on M, then there is well-defined global transversal direction to the subspaces $\mathrm{Ker}(\alpha_{j,x})$ which is not the case in general. This issue will not be of concern to us. ♡

A motivation for studying symplectic and contact forms is physics. When a problem of classical physics is formulated as the solution to a variational problem, the integrand is often a contact form. The system of differential equations characterizing critical points of the variational problem, is transformed into the more convenient form of a first order system by realizing it on a symplectic or contact manifold. This formulation has far-reaching consequences for both classical and quantum physics. The aspect of this subject related to analytical mechanics can be found in classics such as [Whit] or in more introductory expositions such as [tH] where some applications to quantum physics are also discussed. Here we introduce the related mathematical notions and briefly specialize to some situations in physics to demonstrate the theory.

The underlying symplectic manifold (often called *phase space*) for most (but not all) systems of time independent first order ordinary differential equations arising in classical physics, is the cotangent bundle of a manifold (M is generally called the *configuration space*). Given a function f on a symplectic manifold (N, ω), the *Hamiltonian vector field* Υ_f is defined by the requirement

$$df = \omega(\Upsilon_f, .).$$

An important consequence of this definition is that the symplectic form ω is invariant under the Hamilton flow Υ_f:

$$\mathbf{L}_{\Upsilon_f}\omega = d\omega(\Upsilon_f, .) + i_{\Upsilon_f} d\omega = 0.$$

Let us consider the cotangent bundle $N = \mathcal{T}^\star M$ with the symplectic form $\omega = \sum dx_i \wedge dx_{m+i}$ with $x_{m+i} = \theta_i$ in the notation of (0.3.7.2). Then the vector field Υ_f is

$$\Upsilon_f = \sum_{i=1}^{m} \frac{\partial f}{\partial x_{m+i}} \frac{\partial}{\partial x_i} - \sum_i \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_{m+i}},$$

which is equivalent to the system of ordinary differential equations

$$\frac{dx_i}{dt} = \frac{\partial f}{\partial x_{m+i}}, \qquad \frac{dx_{m+i}}{dt} = -\frac{\partial f}{\partial x_i}. \tag{0.3.7.7}$$

Many equations of physics are of the form (0.3.7.7) where f is replaced by the total energy or Hamiltonian H and x_{m+i}'s and x_i's are replaced by p_i's and q_i's in the traditional notation of physics literature. A system of the form (0.3.7.7) is called a *Hamiltonian system*.

Let f_1 and f_2 be functions on the symplectic manifold (M, ω). The *Poisson bracket* of f_1 and f_2 is denoted by $\{f_1, f_2\}$ and is defined as the function $\Upsilon_{f_1}(f_2)$. From the definition of Hamiltonian vector field Υ_f we have

$$\Upsilon_{f_1}(f_2) = df_2(\Upsilon_{f_1}) = \omega(\Upsilon_{f_2}, \Upsilon_{f_1}),$$

which implies that $\{f_1, f_2\} = -\{f_2, f_1\}$. From $\{f, g\} = \Upsilon_f(g)$ it follows that the condition $\{f, g\} = 0$ means that the function g is constant along integral curves of Υ_f (or flow of (0.3.7.7)). It is clear that for $\omega = \sum dx_i \wedge d\theta_i$ we have

$$\{f_1, f_2\} = \sum_i \left(\frac{\partial f_1}{\partial \theta_i}\frac{\partial f_2}{\partial x_i} - \frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial \theta_i}\right).$$

Note also that $\{f_1, f_2\} = 0$ means that the vector fields Υ_{f_1} and Υ_{f_2} commute since $\Upsilon_{\{f_1,f_2\}} = [\Upsilon_{f_1}, \Upsilon_{f_2}]$ which is easily verified.

A function g on N is called an *integral* of the Hamiltonian system (0.3.7.7) if $\Upsilon_f(g) = \{\Upsilon_f, \Upsilon_g\} = 0$. A set of functions $g_1, \cdots, g_k$ on N are said to be *in involution* if $\{\Upsilon_{g_i}, \Upsilon_{g_j}\} = 0$ for all i, j. Clearly the system (0.3.7.7) has one integral namely the function f itself. Let $\dim N = n = 2m$. The system (0.3.7.7) is *completely integrable* if there are m functions $g_1, \cdots, g_m$ such that

1. The functions g_i's are involution,
2. The functions g_i's are invariant under the flow of Υ_f, i.e., $\{g_i, f\} = 0$.
3. The differentials dg_i's are linearly independent on a dense (necessarily open) subset.

Often one takes $g_1 = f$ so that condition (2) becomes a consequence of (1). For $c = (c_1, \cdots, c_m) \in \mathbb{R}^m$, let N_c be the subset of N defined by the equations $g_1 = c_1, \cdots, g_m = c_m$. Condition (c) of complete integrability implies that N_c is a submanifold of dimension m.

Lemma 0.3.7.1 *Let f and g be two functions in involution on the symplectic manifold (N, ω) and denote by N'_α the subset of N defined by $f(y) = \alpha$ which we assume is a submanifold. Then N'_α is invariant under the flow φ of Υ_g.*

Proof - $\Upsilon_g(df) = 0$ implies that the vector field Υ_g is tangent to the submanifold N'_α which proves the required invariance. ♣

In particular, for a completely integrable Hamiltonian system the submanifolds N_c are invariant under the above described action of $\mathbb{R}^m$. Let φ^i_t denote the one parameter group corresponding to the vector field Υ_{g_i}. Since $[\Upsilon_{g_i}, \Upsilon_{g_j}] = 0$, the actions of the one parameter groups $\varphi^i_{t_i}$ and $\varphi^j_{t_j}$ commute. Therefore if the system (0.3.7.7) is completely integrable then we have an action of $\mathbb{R}^m$ on N by

$$t = (t_1, \cdots, t_m) : y \longrightarrow \varphi^1_{t_1}(\varphi^2_{t_2} \cdots (\varphi^m_{t_m}(y)) \cdots). \tag{0.3.7.8}$$

Notice that the order of applying the one parameter groups φ^j is immaterial since they commute. The inverse of the mapping in (0.3.7.8) gives a parametrization of N relative to which the action of $\mathbb{R}^m$ is by translation (linear). To understand the significance of this we need one further observation about symplectic manifolds and Hamiltonian systems. We can now state the following important and almost obvious proposition:

Proposition 0.3.7.2 *Assume the Hamiltonian system (0.3.7.7) is completely integrable with the functions $g_1 = f, g_2, \cdots, g_n$ effecting complete integrability. Assume N_c is a compact submanifold, and the orbits of the above described action of $\mathbb{R}^m$ on N_c are of dimension m. Then each connected component of N_c is an m-torus $T^m \simeq \mathbb{R}^m/\mathbf{Z}^m$. The action of $\mathbb{R}^m$ on each connected component of N_c is induced by the translation action of $\mathbb{R}^m$. In particular, if φ^i has one closed orbit on an N_c of dimension > 1, then φ^i has a continuum of closed orbits.*

Proof - Since the orbits are of dimension m, they are open submanifolds of N_c and by compactness of N_c, $\mathbb{R}^m$ acts transitively on each connected component. Therefore each connected component is of the form $\mathbb{R}^m/L$ where L is the isotropy subgroup of a point which is necessarily a *lattice* (i.e., $g(\mathbf{Z}^m)$ for some $g \in GL(m, \mathbb{R})$) for dimension reasons. This proves the proposition. ♣

Exercise 0.3.7.4 *Let f be a (locally defined) function on the symplectic manifold (N, ω), and P be a submanifold of N which is transverse to the Hamiltonian vector field Υ_f (i.e., $\mathcal{T}_xP$ and Υ_f span $\mathcal{T}_xN$). Then for sufficiently small open sets $U \subset N$ and for every $x \in U$ there are unique $y \in P$ and $t \in (-\delta, \delta)$ with $\varphi_t^f(y) = x$, where φ_t^f is the one parameter group corresponding to the Hamiltonian vector field Υ_f and we require $t = 0$ if $x \in P$. Define $h(x) = t$, and for $c_1, c_2 \in (-\delta, \delta)$, a sufficiently small interval, set $N_{c_1,c_2} = f^{-1}(c_1) \cap h^{-1}(c_2)$.*

1. *Show that $dh(\Upsilon_f) = 1$ or equivalently $\{f, h\} = 1$.*
2. *Prove that $\{\Upsilon_f, \Upsilon_h\} = 0$ and deduce that the one parameter groups φ_t^f and φ_s^h commute.*
3. *Show that N_{c_1,c_2} is a submanifold of codimension 2.*
4. *Prove that $(N_{c_1c_2}, \omega_{|N_{c_1c_2}})$ is a symplectic manifold.*
5. *Show how the preceding can be used to give an alternate proof of proposition 0.3.7.1.*

The existence of n functions $g_1 = f, \cdots, g_n$ effecting complete integrabiilty is related to the existence of a maximal number of conservation laws. One generally takes g_j's to be conserved quantities in the physical problem under consideration. Relative to the coordinate system adopted to the description of the system according proposition 0.3.7.2, the differential equations (0.3.7.7 take the simple form

$$\frac{dy_j}{dt} = 0, \quad \frac{dy_{m+j}}{dt} = \gamma_j, \qquad j = 1, \cdots, m. \tag{0.3.7.9}$$

The first m equations express the invariance of the tori N_c of proposition 0.3.7.2 under the flow. The second set of m equations exhibit the flow along the tori N_c linearly as required by proposition 0.3.7.2. Naturally, one can take the coordinates $y_1, \cdots, y_m$ as the conserved quantities (assuming they exist) of the problem under study. It is customary to call the variables $y_{m+1}, \cdots, y_{2m}$ as the *angle* and $y_1, \cdots, y_m$ as the *action* variables. For the symplectic form in the standard representation (0.3.7.1), the quantities x_j and x_{m+j} are called *conjugate variables.*

Remark 0.3.7.2 The quantities γ_j depend on $c = (c_1, \cdots, c_m)$ and vary continuously with c. Therefore under some genericity assumption, for a completely integrable system, there are tori N_c for which the orbits of the flow of Υ_f are closed (periodic). For systems which are not completely integrable, the existence of periodic orbits for the system of differential equations is a subtle issue. Contact forms are useful in detecting periodic solutions to (0.3.7.7). Specifically, one can show that certain compact hypersurfaces with a contact structure necessarily contain periodic orbits of the Hamiltonian system. A digression into the issue of the existence of periodic orbits is not appropriate in this context and so will not pursued any further (see for example [HZ]). ♡

A diffeomorphism preserving a symplectic form is called a *symplectic* diffeomorphism or a *canonical transformation.* A mapping

$$F \ : \ x_j = x_j(y_1, \cdots, y_{2m}), \qquad \text{for} \ \ j = 1, 2, \cdots, 2m \tag{0.3.7.10}$$

is a canonical trasnsformation relative to the standard symplectic form (0.3.7.1) if and only if the derivative $DF(y) \in Sp(m, \mathbb{R})$ for every $y = (y_1, \cdots, y_{2m})$. It is readily verified that the form of the system of equations (0.3.7.7) is preserved by symplectic diffeomorphisms, i.e., it remains Hamiltonian with the Hamiltonian given by the function f expressed relative to the new variables. Implementing proposition 0.3.7.2 and reducing the Hamiltonian system to the form (0.3.7.9) requires constructing symplectic diffeomorphisms which transform the differential equations to the desired form. Obtaining the required transformation in a specific situation often adds to our insight about the physical problem under consideration.

In example 0.3.1.6 we noted that the group of volume preserving diffeomorphisms of a surface (or equivalently canonical transformations of the symplectic form) is infinite dimensional. In general, the group of canonical transformations of a symplectic manifold is infinite dimensional. Although symplectic transformations exist in abundance, they should be exhibited in a manner that the computation of the transformed equations and the verification of the condition $DF \in Sp(m, \mathbb{R})$ become practical. We now show how this can be done (by a method due to Jacobi) and demonstrate the principle by applying to the one body problem and deriving Kepler's equation of classical physics which has a quantum analogue. The following observation simplifies the verification of the condition $DF(y) \in Sp(m, \mathbb{R})$:

Lemma 0.3.7.2 *The diffeomorphism of the form (0.3.7.10) is symplectic if and only if one and therefore all the expressions*

1. $\alpha_1 = \sum_{k=1}^m x_{m+k}dx_k - \sum_{k=1}^m y_{m+k}dy_k$;
2. $\alpha_2 = \sum_{k=1}^m x_k dx_{m+k} + \sum_{k=1}^m y_{m+k}dy_k$;
3. $\alpha_3 = \sum_{k=1}^m x_{m+k}dx_k + \sum_{k=1}^m y_k dy_{m+k}$;
4. $\alpha_4 = \sum_{k=1}^m x_k dx_{m+k} - \sum_{k=1}^m y_k dy_{m+k}$.

become exact differentials upon substitution in terms of y_j's and dy_j's for x_k's and dx_k's.

Proof - The diffeomorphism F (0.3.7.10) is symplectic if and only if $d\alpha_j = 0$ (for one and therfore all j), from which the required result follows. ♣

Now we construct four types of canonical transformations corresponding to the 1-forms α_j, $j = 1, 2, 3, 4$. For example for α_1, let $S = S(x_1, \cdots, x_m, y_1, \cdots, y_m)$ be a function with the property

$$\det\left(\frac{\partial^2 S}{\partial x_i \partial y_j}\right) \neq 0,$$

and define

$$\frac{\partial S}{\partial x_k} = x_{m+k}, \qquad \frac{\partial S}{\partial y_k} = -y_{m+k}.$$

By the implicit function theorem we can solve the equations $\frac{\partial S}{\partial y_k} = -y_{m+k}$, $k = 1, \cdots, m$, for x_k's in terms of y_j, y_{m+j}'s and so we obtain a mapping F expressing x_j, x_{m+j}'s in terms y_k, y_{m+k}'s. Substituting in α_1 we get

$$\alpha_1 \rightarrow \sum_{j=1}^m \frac{\partial S}{\partial x_j}dx_j + \sum_{j=1}^m \frac{\partial S}{\partial y_j}dy_j$$

which is an exact differential as required by lemma 0.3.7.2. The fact that we can solve for x_j, x_{m+j}'s in terms of y_k, y_{m+k}'s and vice versa shows that the mapping F is in fact a (local) diffeomorphism. Similar construction applies to the other cases:

2. (α_2): $S = S(x_{m+1}, \cdots, x_{2m}, y_1, \cdots, y_m)$, $\det(\frac{\partial^2 S}{\partial x_{m+j}\partial y_k}) \neq 0$; $\frac{\partial S}{\partial x_{m+j}} = x_j$, $\frac{\partial S}{\partial y_j} = y_{m+j}$.
3. (α_3): $S = S(x_1, \cdots, x_m, y_{m+1}, \cdots, y_{2m})$, $\det(\frac{\partial^2 S}{\partial x_j \partial y_{m+k}}) \neq 0$; $\frac{\partial S}{\partial x_j} = x_{m+j}$, $\frac{\partial S}{\partial y_{m+j}} = y_j$.

4. (α_4): $S = S(x_{m+1}, \cdots, x_{2m}, y_{m+1}, \cdots, y_{2m})$, $\det(\frac{\partial^2 S}{\partial x_{m+j} y_{m+k}}) \neq 0$; $\frac{\partial S}{\partial x_{m+j}} = x_j$, $\frac{\partial S}{\partial y_{m+j}} = -y_j$.

In this manner we have at our disposal, in an explicit manner, an infinite dimensional space of symplectic diffeomorphisms which we can use to advantage to simplify a given Hamiltonian system. A function S of the above form is often called a *generating function*.

Recall that we like to obtain a symplectic diffeomorphism which implements the conclusion of proposition 0.3.7.2 (in the completely integrable case), or equivalently exhibits the action-angle variables explicitly. This requires making use of conservation laws expressed by the functions $g_1 = f, g_2, \cdots, g_m$ which are involution. This effectively helps determine the function S and the corresponding symplectic diffeomorphism. For instance by setting the Hamiltonian $H = f$ equal to y_1 (regarded as a constant) and substituting $\frac{\partial S}{\partial x_j}$ for x_{m+j} in the expression for f should lead to a valid identity. In other words, we must require that S satisfy the first order (generally non-linear) partial differential equation

$$f(\frac{\partial S}{\partial x_1}, \cdots, \frac{\partial S}{\partial x_1}, x_1, \cdots, x_m) = y_1. \tag{0.3.7.11}$$

This equation, known as the *Hamilton-Jacobi* equation, is often quite helpful in understanding the behavior of the system of ordinary differential (0.3.7.7). In (0.3.7.11) y_j is regarded as constant defining the total energy or Hamiltonian of the system.

Exercise 0.3.7.5 below demonstrates the above considerations in the simplest realistic physical system, namely a single *harmonic oscillator*. This simple system can be integrated easily without the above theory and is included here only to demonstrate the principles involved. Such a system is described by the Hamiltonian $H = \frac{1}{2m}p^2 + \frac{k}{2}q^2$ (Hooke's Law) which yields the differential equations

$$\frac{dq}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = -kq. \tag{0.3.7.12}$$

(Here m is the mass of the particle and k is Hooke's constant, but the physical significance of these quantities is not material for our calculation. (q, p) are the coordinates (x_1, x_2) and (Q, P) below correspond to (y_1, y_2) in preceding notation.)

Exercise 0.3.7.5 *Show that a generating fyunction $S = S(q, Q)$ preserving the energy surface $H(q, p) = Q$ of the harmonic oscillator satisfies the differential equation*

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{k}{2}q^2 = Q.$$

Solve this equation to obtain the generating function

$$S(q,Q) = \sqrt{\frac{m}{k}}\sin^{-1}\left(\sqrt{\frac{k}{2Q}}q\right).$$

Show that the corresponding symplectic diffeomorphism is

$$q = -\sqrt{\frac{2Q}{k}}\sin\lambda P, \quad p = \sqrt{\frac{m}{2Q\cos^2\lambda P}}.$$

Prove also that this diffeomorphism transforms (0.3.7.12) into the standard form

$$\frac{dQ}{dt} = 0, \quad \frac{dP}{dt} = -1,$$

The following example is an application of the above theory to a classical problem in physics and serves to demonstrate some aspects of the theory:

Example 0.3.7.3 The Lagrangian for the one body problem in the gravitaional field generated by a massive body has polar coordinate representation

$$L = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\varphi}^2 + r^2\sin^2\varphi\dot{\theta}^2\right] + \frac{Km}{r},$$

where the constant $K = GM$ is the product of the gravitational cosntant and the mass M of the massive body. Since the variable θ does not explicitly appear in L, $\dot{\theta}$ is a conserved quantity (angular momentum). This means motion takes place in a plane with constant θ. Without loss of generality we may choose coordinates so that $\theta \equiv 0$ and the Lagrangian reduces to

$$L = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\varphi}^2\right] + \frac{Km}{r}.$$

The corresponding Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\varphi^2\right) - \frac{Km}{r}.$$

To find the appropriate symplectic diffeomorphism, we seek a generating function of the special form

$$S(r,\varphi,q_1,q_2) = R(r,q_1) + \Phi(\varphi,q_2).$$

Substituting in the Hamiltonian or total energy which is a conserved quantity, we obtain the corresponding Hamilton-Jacobi equation:

$$\left(\frac{dR}{dr}\right)^2 + \frac{1}{2m}\left[\frac{1}{r^2}\left(\frac{d\Phi}{d\varphi}\right)^2\right] - \frac{Km}{r} = q_1. \tag{0.3.7.13}$$

The law of conservation of angular momentum suggests that we should set $\frac{\partial S}{\partial \varphi} = q_2$ (a constant). Substituting in (0.3.7.13) we obtain an ordinary differential equation which we can solve to obtain the expression

$$S = c + q_2\varphi + \int_{r_\circ}^{r} \sqrt{2m\left(q_1 + \frac{Km}{u}\right) - \frac{q_2^2}{u^2}}du. \tag{0.3.7.14}$$

Let p_j denote the variable conjugate to q_j. Since $p_2 = -\frac{\partial S}{\partial q_2}$ we obtain

$$p_2 = -\varphi + q_2 \int_{r_\circ}^{r} \frac{1}{\sqrt{2m\left(q_1 + \frac{Km}{u}\right) - \frac{q_2^2}{u^2}}}\frac{du}{u^2}.$$

Making a change of variable $u = \frac{1}{s}$ we obtain

$$p_2 + \varphi = -q_2 \int_{s_\circ}^{s} \frac{1}{\sqrt{2m\left(q_1 + Kms\right) - q_2^2 s^2}}ds.$$

Let $\alpha < \beta$ be the roots of the equation $\lambda^2 - \frac{2Km^2}{q_2^2}\lambda - \frac{2mq_1}{q_2^2} = 0$, and set

$$s = \frac{\beta + \alpha}{2} + \frac{\beta - \alpha}{2}y$$

to transform the integral to

$$p_2 + \varphi = \int_{v_\circ}^{v} \frac{dy}{\sqrt{1 - y^2}}.$$

Integrating this equation we obtain after a starightforward calculation

$$\frac{1}{r} = \frac{1 + \epsilon\cos(\varphi + p_2)}{a(1 - \epsilon^2)}, \tag{0.3.7.15}$$

where $\alpha = \frac{1}{a(1+\epsilon)}$ and $\beta = \frac{1}{a(1-\epsilon)}$. (0.3.7.15) is the equation of an ellipse in polar coordinates with eccentricity ϵ and major axis $2a$. Since time is the variable conjugate to energy[7], we can relate time and the position of the body of mass m by

$$t = \frac{\partial S}{\partial q_1} = -\frac{m}{q_2}\int_{s_\circ}^{s} \frac{dx}{x^2\sqrt{(x-\alpha)(\beta-x)}}.$$

This integral can evaluated if we make a change of variable to *eccentric anomaly* u of conic (see figure (XXXX)) to obtain

$$\gamma t = u - \epsilon \sin u, \quad \text{where} \quad \gamma = \sqrt{\frac{K}{a^3}}. \tag{0.3.7.16}$$

This is Kepler's equation. ♠.

[7] In the formulation of a time dependent system as a variational problem, the contact form which appears as the integrand contains the terms Edt which exhibits energy and time as conjugate variables.

0.4 Mappings of Manifolds

0.4.1 Constructing Manifolds

In this subsection we study how new manifolds can be constructed from old ones. Let M and N be manifolds of dimension m, and $C_1 \subset M$ and $C_2 \subset N$ be "small" balls (e.g. each contained in one coordinate neighborhood), $M' = M \setminus C_1$ and $N' = N \setminus C_2$. M' and N' are manifolds with boundary and $\partial M' \simeq S^{m-1} \simeq \partial N'$. The idea is to identify the boundaries of M' and N' to obtain a new manifold $M \sharp N$. In general, $M \sharp N$ depends on the identification map of the boundaries $f : \partial M' \to \partial N'$. We carry out this construction a little differently[8]. A convenient of describing M' and N' is by smooth real valued functions f and h on M and N such that

$$M' = \{x \in M \mid f(x) \leq 0\}, \quad N' = \{x \in N \mid h(x) \leq 0\},$$

with df and dh nonzero on (and therefore near) the zero sets $f = 0$ and $h = 0$ (i.e., boundaries of M' and N'). We are assuming that the sets $\{x \in M \mid f(x) \geq 0\}$ and $\{x \in N \mid h(x) \geq 0\}$ are small closed discs. For $\epsilon > 0$ and sufficiently small the sets

$$M_\epsilon = \{x \in M \mid -\epsilon < f(x) < \epsilon\}, \quad N_\epsilon = \{x \in N \mid -\epsilon < h(x) < \epsilon\}$$

are (small) tubular neighborhoods of $\partial M'$ and $\partial N'$ in M and N. We denote the boundary components of M_ϵ by M_ϵ^+ and M_ϵ^- corresponding to $f(x) = \epsilon$ and $f(x) = -\epsilon$ respectively. Let $r > 1$, $B_r \subset \mathbb{R}^m$ the ball of radius r centered at the origin, and

$$A_r = \{x \in B_r \mid \frac{1}{r} < ||x|| < r\}, \quad \partial A_r = S_i \cup S_o,$$

with $S_i = \{x \mid ||x|| = \frac{1}{r}\}$ and $S_o = \{x \mid ||x|| = r\}$. Let $\phi_1 : A_r \to M_\epsilon$ and $\phi_2 : A_r \to N_\epsilon$ be diffeomorphisms. We assume that ϕ_j's are obvious extensions of diffeomorphisms of S^{m-1} onto $\partial M'$ and $\partial N'$ so that we have diffeomorphisms

$$\phi_1(S_o) \xrightarrow{\sim} M_\epsilon^-, \quad \phi_1(S_i) \xrightarrow{\sim} M_\epsilon^+, \quad \phi_2(S_o) \xrightarrow{\sim} N_\epsilon^-, \quad \phi_2(S_i) \xrightarrow{\sim} N_\epsilon^+.$$

Define the involution $\jmath : A_\epsilon \to A_\epsilon$ by

$$\jmath(x) = \frac{x}{||x||^2}.$$

[8] It will be clear that the construction is valid for manifolds with boundary provided $M_\epsilon \cap \partial M = \emptyset$ and $N_\epsilon \cap \partial N = \emptyset$ where M_ϵ and N_ϵ are defined below. For simplicity of notation we assume M and N are without boundary.

which is an analytic diffeomorphism. Let $\psi : M_\epsilon \to N_\epsilon$ be defined by

$$\psi(x) = \phi_2 \jmath \phi_1^{-1}(x). \quad (0.4.1.1)$$

Set $M'' = M' \cup M_\epsilon$ and $N'' = N' \cup N_\epsilon$, and

$$M\sharp_\psi N = M'' \cup N''/\{x \sim \psi(x)\}, \quad (0.4.1.2)$$

i.e., $M\sharp_\psi N$ is the union of M'' and N'' with the points x and $\psi(x)$ identified. It is trivial that $M\sharp_\psi N$ so constructed has the structure of a manifold. It is also clear that if we replace ϵ by ϵ' with $0 < \epsilon' < \epsilon$, then the resulting manifolds $M\sharp_\psi N$ will be diffeomorphic. The functions f and h played only an auxilliary role and $M\sharp_\psi N$ is independent of their choice as long as the necessary hypotheses are fulfilled. Furthermore, if M and N are compact then so is $M\sharp_\psi N$. Now assume the manifolds M and N are oriented, ϕ_1 is orientation preserving and ϕ_2 is orientation reversing relative to the standard orientation for $B_r \subset \mathbb{R}^m$. Since $\jmath$ is orientation reversing, $\psi = \phi_2 \jmath \phi_1^{-1}$ is orientation preserving and $M\sharp_\psi N$ is also orientable with orientation compatible with those of M and N. The manifold $M\sharp_\psi N$ depends on the the diffeomorphism ψ and the degree of this dependence is clarified by

Lemma 0.4.1.1 *Let ψ be as above, and $\psi' = \phi_2' \jmath \phi_1'^{-1}(x)$ be another such diffeomorphism. If ψ and ψ' are isotopic, then $M\sharp_\psi N$ and $M\sharp_{\psi'} N$ are diffeomorphic.*

Proof - Let $\tilde{M}_\epsilon$ denote the image of M_ϵ in $M\sharp_\psi N$, and $F' : M_\epsilon \times I \to N_\epsilon$ be an isotopy with $F'(.,0) = \psi$ and $F'(.,1) = \psi'$. Then F' yields an isotopy $F : \tilde{M}_\epsilon \times I \to M\sharp_\psi N$ with $(.,0) = \mathrm{id}$ and $F(.,1) = \psi'\psi^{-1}$. By lemma 0.4.7.1, after possibly replacing ϵ be a smaller positive number ϵ', F extends to an isotopy $F : M\sharp_\psi N \times I \to M\sharp_\psi N$. Since $F(.,1) = \psi'\psi^{-1}$, $F(.,1)$ gives the desired diffeomorphism. ♣

A consequence of lemma 0.4.1.1 and corollary 0.4.7.2 is

Corollary 0.4.1.1 *Assume the manifolds M and N are oriented, ϕ_1 is orientation preserving and ϕ_2 is orientation reversing, and both ϕ_j's extend to diffeomorphisms of the disc. Then, up to diffeomorphisms, $M\sharp N$ is independent of the particular choice of ϕ_j's satisfying the hypotheses.*

Proof - The required result follows from lemma 0.4.1.1 and the fact that any embedding of a disc is isotopic to a linear embedding (see example 0.4.7.5). ♣

Exercise 0.4.1.1 *For an orientable manifold M of dimension m show that $M\sharp S^m \simeq M$ and $M\sharp\mathbb{R}^m$ is diffeomorphic to M with one point removed.*

$M\sharp_\psi N$ is called the *connected sum of M and N relative to ψ*. When the hypotheses of corollary 0.4.1.1 are fulfilled, we simply write $M\sharp N$ and refer to it as the *connected sum of M and N*. The $\sharp$-construction can be vastly generalized and is an important tool in topology. For instance, let $B^2_r \subset \mathbb{R}^2$ denote the open disc of radius $r > 0$. Assume $r > 1 > \frac{1}{r}$ and $N_r \simeq S^1 \times B^2_r \subset \mathbb{R}^3 \subset S^3$ be a solid torus. Then $N_{r,\frac{1}{r}} = N_r \backslash \overline{N_{\frac{1}{r}}}$ is a tubular neighborhood of ∂N_1. Let $M = S^3 \backslash N_{\frac{1}{r}}$, M_1 and M_2 be two copies of M, and denote the corresponding copies of $N_{r,\frac{1}{r}}$ in M_i by $N^i_{r,\frac{1}{r}}$. The boundary of $N_{r,\frac{1}{r}}$ has two components which we denote by $\partial_r N_{r,\frac{1}{r}}$ and $\partial_{\frac{1}{r}} N_{r,\frac{1}{r}}$ respectively, and their images in M_i by the addition of superscript ^{i}. Let $\phi_i : N_{r,\frac{1}{r}} \rightarrow N^i_{r,\frac{1}{r}}$ be diffeomorphisms extending to diffeomorphisms of $\partial_r N_{r,\frac{1}{r}}$ (resp. $\partial_{\frac{1}{r}} N_{r,\frac{1}{r}}$) onto $\partial_r N^i_{r,\frac{1}{r}}$ (resp. $\partial_{\frac{1}{r}} N^i_{r,\frac{1}{r}}$). The map $\jmath$ is defined as an orientation reversing involution of $N_{r,\frac{1}{r}}$ leaving ∂N_1 pointwise fixed. Near every $x \in \partial N_1$, $\jmath$ is like a reflection relative to the tangent plane $\mathcal{T}_x \partial N$. In particular, $\jmath$ interchanges the boundary components $\partial_r N_{r,\frac{1}{r}}$ and $\partial_{\frac{1}{r}} N_{r,\frac{1}{r}}$. Then by the same formulae (0.4.1.1) and (0.4.1.2) we define the $M_1\sharp_\psi M_2$. The fact that $M_1\sharp_\psi M_2$ has the structure of a manifold is clear. $M_1\sharp_\psi M_2$ depends on the maps ϕ_i. To emphasize this dependence we introduce $\psi = \phi_2 \jmath \phi_1^{-1}$ and use the notation $M_1\sharp_\psi M_2$. The fact that we deleted solid tori from S^3 is not critical and similar construction can be carried out if, for example, we delete solid surfaces of genus g from S^3. The construction is not limited to S^3 and can be extended to arbitrary manifolds, however since the construction proceeds in the obvious manner we will not dwell on a more formal description.

Example 0.4.1.1 There is an important concept in complex geometry known as blow-up, σ-process or quadratic transform. In this example we consider a special case of this notion for complex manifolds and relate it to the $\sharp$ construction. Let $\mathbb{C}^m \subset \mathbb{C}P(m)$ be the open subset defined by the inhomogeneous coordinates $z_m = 1$ and $z_\circ, \cdots, z_{m-1} \in \mathbb{C}$ arbitrary. Denote homogeneous coordinates for $\mathbb{C}P(m-1)$ by $w_\circ, \cdots, w_{m-1}$. Let $X \subset \mathbb{C}^m \times \mathbb{C}P(m-1)$ (resp. $Y \subset \mathbb{C}P(m) \times \mathbb{C}P(m-1)$) be defined by the equations

$$w_j z_k - w_k z_j = 0, \quad j, k = 0, 1, \cdots, m-1, \tag{0.4.1.3}$$

and $\pi_1 : \mathbb{C}^m \times \mathbb{C}P(m-1) \longrightarrow \mathbb{C}^m$ be the projection on the first factor. If $(z_\circ, z_1, \cdots, z_{m-1}) \neq \mathbf{0}$, then the solution to (0.4.1.3) is $w_j = z_j$. Thus if we denote the restriction of π_1 to X by π_1 again, then

$$\pi_1 : X \setminus \pi_1^{-1}(\mathbf{0}) \longrightarrow \mathbb{C}^m \setminus \mathbf{0}$$

is a complex analytic diffeomorphism. On the other hand, $\pi_1^{-1}(\mathbf{0})$ is complex analytically diffeomorphic to $\mathbb{C}P(m-1)$. The assignment of X to $\mathbb{C}^m$ or Y to $\mathbb{C}P(m)$ is called the *blow-up* of $\mathbb{C}^m$ or $\mathbb{C}P(m)$ at a point (here $\mathbf{0} \in \mathbb{C}^m$.) It is not difficult to see that this construction is independent of the choice of complex analytic coordinates, i.e., different choices of complex analytic coordinates lead to holomorphically diffeomorphic complex manifolds. Observe that given $(z_\circ, z_1, \cdots, z_{m-1}) \neq \mathbf{0}$, we have in Y,

$$\lim_{\lambda\to 0}(\lambda z_\circ, \cdots, \lambda z_{m-1}) = [w_\circ, \cdots, w_{m-1}], \quad \text{where } w_j = z_j; \tag{0.4.1.4}$$

and also

$$\lim_{\lambda\to\infty}[\lambda z_\circ, \cdots, \lambda z_{m-1}, 1] = [z_\circ, \cdots, z_{m-1}, 0] \tag{0.4.1.5}$$

which lies in the hyperplane at infinity ($z_m = 0$). The blow-up of $\mathbb{C}P(m)$ at a point has a simple and familiar topological description as well. We regard $\mathbb{C}P(m)$ as a real manifold of dimension $2m$, then $\mathbb{C}P(m)\sharp\mathbb{C}P(m)$ is an orientable manifold of dimension $2m$. We now show that $\mathbb{C}P(m)\sharp\mathbb{C}P(m)$ is (real analytically) diffeomorphic to the blow-up of $\mathbb{C}P(m)$ at one point[9]. Consider the mapping $\Delta : \mathbb{C}^m \setminus \mathbf{0} \to \mathbb{C}^m \times \mathbb{C}P(m-1)$ defined by

$$\Delta : (z_\circ, \cdots, z_{m-1}) \longrightarrow (z_\circ, \cdots, z_{m-1}, [w_\circ, \cdots, w_{m-1}])$$

where w_j's are related to z_k's by the equations (0.4.1.3). Now $\mathrm{Im}(\Delta)$ has a real analytic involution defined by

$$\tau((z_\circ, \cdots, z_{m-1}, [w_\circ, \cdots, w_{m-1}])) = (\frac{z_\circ}{||z||^2}, \cdots, \frac{z_{m-1}}{||z||^2}, [w_\circ, \cdots, w_{m-1}]),$$

where $||z||^2 = |z_\circ|^2 + \cdots + |z_{m-1}|^2$. τ leaves the unit sphere $S^{2m-1} \subset \mathbb{C}^m$ pointwise fixed. In view of (0.4.1.4) and (0.4.1.5), τ extends to a real analytic involution of the blow-up of $\mathbb{C}P(m)$ at one point and maps the hyperplane at infinity ($z_m = 0$) diffeomorphically onto $\pi_1^{-1}(\mathbf{0})$. Therefore $\mathbb{C}P(m) \setminus$ (one point) is real analytically diffeomorphic to the blow-up of $\mathbb{C}^m$ at the origin. From this and the $\sharp$ construction, it immediately follows that the blow-up of $\mathbb{C}P(m)$ at one point is real analytically diffeomorphic to $\mathbb{C}P(m)\sharp\mathbb{C}P(m)$. ♠

[9]It is customary to write $\mathbb{C}P(2)\sharp\overline{\mathbb{C}P}(2)$ for the blow-up of $\mathbb{C}P(2)$ at one point where $\overline{\mathbb{C}P}(2)$ denotes the complex projective plane with the reverse orientation. It seems to the author that this notation is rather confusing and may create misconceptions. For example it suggests that the orientation on one of the copies of $\mathbb{C}P(2)$ is the reverse of that imposed by its complex structure. The situation is even more confusing if we blow-up $\mathbb{C}P(2)$ at two points to obtain $\mathbb{C}P(2)\sharp\overline{\mathbb{C}P}(2)\sharp\overline{\mathbb{C}P}(2)$ or maybe $\mathbb{C}P(2)\sharp\overline{\mathbb{C}P}(2)\sharp\mathbb{C}P(2)$! This confusing notation is a consequence of the fact that the involution $\jmath : A_\epsilon \to A_\epsilon$ is not incorporated in the prevalent definition of the $\sharp$ construction. The definition given above using the involution $\jmath$ is more compatible with our intuitive notion of joining surfaces than the prevalent one.

0.4.2 Critical Points and Critical Values

Understanding the structure and geometric implications of critical points of maps of manifolds is an important problem. The general case is far from understood. Even understanding the significance and implications of critical points of a smooth mapping of a manifold into the circle presents challenging problems. In this section we gather two basic facts about critical points which properly speaking belong to foundations of differential topology. A general reference for the material of this section is [Hi].

Let $U \subset \mathbb{R}^m$ be an open subset, and $F : U \to \mathbb{R}^n$ a C^1 mapping. A point $x \in U$ where $DF(x)$ has rank $\min(m, n)$ is called a *regular* point of F. Otherwise it is called a *critical point*. The set of critical points of F will be denoted by C_F. A point $y \in \mathbb{R}^n$ is a regular value for F if every $x \in F^{-1}(y)$ is a regular point. In particular, $y \notin \mathrm{Im}F$ is a regular value. It is clear that the notions of critical point and regular point extend to mappings of manifolds since the defining conditions are invariant under diffeomorphisms. Recall that a subset of a (metrizable) topological space is called *residual* if it is a countable intersection of dense open sets. By the Baire category theorem a residual set is dense. Various versions of the following fundamental theorem, often known as *Sard's theorem*, are due to A. Brown, A. P. Morse and A. Sard:

Theorem 0.4.2.1 *Let $F : M \to N$ be a C^r map where $r > \max(0, m - n)$. Let μ be any measure on N absolutely continuous relative to the Lebesgue measure on each coordinate neighborhood. Then $F(C_F)$ has μ-measure zero in N, and the set of regular values of F is residual in N. If $n > m$ then $N \setminus F(M)$ is dense in N.*

Remark 0.4.2.1 The differentiability assumption in theorem 0.4.2.1 is essential and subtle problems emerge when sufficient differentiability is not assumed. ♡

We shall not prove this basic theorem, but will discuss some important consequences of it in this and the following subsections.

Example 0.4.2.1 Let $M \subset \mathbb{R}^m$ be an open subset, $x_1, \cdots, x_m$ the standard coordinates on $\mathbb{R}^m$, and $f : M \to \mathbb{R}$ a C^2 function. Let $< ., . >$ denote the standard inner product on $\mathbb{R}^m$. We use theorem 0.4.2.1 to show that for almost all $\lambda \in \mathbb{R}^m$, the function $f_\lambda(x) =< \lambda, x > + f(x)$ is a Morse function, i.e., all its critical points are nondegenerate. To this end define the submanifold $N\imath M \times \mathbb{R}^m$ by the relation

$$\xi + \mathrm{grad} f(x) = 0, \quad \text{where} \quad x \in M, \quad \text{and} \quad \xi \in \mathbb{R}^m,$$

where $\mathrm{grad} f = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_m})$ is computed relative to the standard metric on $\mathbb{R}^m$. Let $\psi : N \to \mathbb{R}^m$ be the restriction of the projection $M \times \mathbb{R}^m \to \mathbb{R}^m$ to N, and λ be a regular value for ψ. Taking $(x_1, \cdots, x_m)$ as coordinates on N and differentiating ψ as a function on N (relative to coordinates $x_1, \cdots, x_n$), we see that its derivative is negative of the Hessian of f. Therefore by the regularity of λ, $\mathsf{H}(f)$ is everywhere nonsingular on N. Since critical points of f_λ lie on N and the Hessian of f_λ is $\mathsf{H}(f)$, we have proven that all critical points of f_λ are nondegenerate. ♠

With a little care, example 0.4.2.1 can be generalized significantly. Let $M\imath\mathbb{R}^N$ be an embedded submanifold, $U \supset M$ an open set containing M and $f : U \to \mathbb{R}$ a C^k function where $k \geq N-m+2$. Let $U_1, U_2, \cdots$ be open subsets of $\mathbb{R}^N$ such that $\cup U_j \supset M$. We assume U_i's sufficiently small so that after possibly an orthogonal change of cartesian coordinates in $\mathbb{R}^N$, $M \cap U_i$ is represented as a graph

$$\mathbf{x} = (x_1, \cdots, x_m) \to (\mathbf{x}, h_{m+1}(\mathbf{x}), \cdots, h_N(\mathbf{x})).$$

Let W_i denote the linear subspace $\{(x_1, \cdots, x_m, 0, \cdots, 0)\}$ relative to linear coordinates noted above, and $\pi_i : \mathbb{R}^N \to W_i$ be orthogonal projection. Consider the submanifold $N\imath M \times \mathbb{R}^N$ defined by the relation

$$\xi + \pi_i \mathrm{grad} f(x) = 0,$$

where grad is relative to the standard metric on $\mathbb{R}^N$. The restriction of the projection $M \times \mathbb{R}^N \to \mathbb{R}^N$ to the submanifold N composed with orthogonal projection π_i gives a mapping $\psi : N \to W_i$ which satisfies the hypothesis of theorem 0.4.2.1 and therefore the complement of the set of its regular value has Lebesgue measure zero. Denote this set of regular values by R_i and its complement by R_i'. Differentiating the function ψ_i on N we see that $\lambda \in R_i$ implies that the $m \times m$ matrix $(\frac{\partial^2 f}{\partial x_j \partial x_k})$, $j, k = 1, \cdots, m$, is nonsingular everywhere on $M \cap U_i$. Let $\lambda \in \cap_i R_i$ whose complement $\cup R_i'$ has measure zero. The critical points of the function $f_\lambda(x) = <\lambda, x> + f(x)$ on $M \cap U_i$ lie on the manifold N and therefore are nondegenerate. Notice that since $\cup R_i'$ has measure zero, we can require $\lambda \neq 0$ to have arbitrarily small norm greater than zero. Summarizing,

Proposition 0.4.2.1 *Let $M\imath\mathbb{R}^N$ be an embedded submanifold, and f a C^k function defined on a neighborhood of M where $k \geq N - m + 2$. For $\epsilon > 0$ there is $\lambda \in \mathbb{R}^N$ such that $<\lambda, \lambda> \leq \epsilon$ and the restriction of f_λ to M is a Morse function on M.*

Remark 0.4.2.2 The assumption that M is an embedded submanifold of $\mathbb{R}^N$ is no loss of generality by the Whitney Embedding theorem (proven later in this chapter) which also implies that we can assume $N \leq 2m + 1$. Furthermore by the implicit function theorem we can locally realize M as an affine subspace in $\mathbb{R}^N$ so that we can locally extend a function

f to a neighborhood. The local extensions of f can be patched together by a partition of unity to give an extension of f to a neighborhood of M in $\mathbb{R}^N$. Therefore proposition 0.4.2.1 implies that for $k \geq m+3$ a C^k function f on M can be approximated by a Morse function arbitrarily closely in an appropriate topology. For M compact, the appropriate topology is C^k topology which will be introduced in the subsection Smoothing and Transversality. ♡

Example 0.4.2.2 Let $M \subset \mathbb{R}^{m+1}$ be a compact hypersurface, and consider the function

$$f(p) =< p - x, p - x >,$$

where $p \in M$ and $x \in \mathbb{R}^{m+1}$. The function f, defined on M, depends on the choice of the origin x in the affine space space $\mathbb{R}^{m+1}$. f may not be a Morse function on M. Perturbing f by a small vector ϵ yileds the function

$$f_\epsilon(p) =< p - x - \epsilon, p - x - \epsilon >= f(p)+ < \epsilon, \epsilon > +2 < \epsilon, x > -2 < \epsilon, p > .$$

The argument of example 0.4.2.1 is applicable to show that for almost all (small) vectors ϵ, the function f_ϵ is a Morse function. The details are left to the reader. ♠

Using the above arguments one easily proves the following:

Exercise 0.4.2.1 *Let $M\imath\mathbb{R}^N$ and $f : M \to \mathbb{R}$ be a Morse function. Let $x, y \in M$ be two critical points of f with $f(x) = f(y)$. Show that there is a $\lambda \in \mathbb{R}^N$ with $< \lambda, \lambda >$ arbitrarily small such that f_λ is a Morse function and $f_\lambda(x) \neq f_\lambda(y)$.*

The basic result about the structure of critical points of a Morse function is given in proposition 0.4.2.2 below. Recall from our earlier discussion that at a critical point, the Hessian of a function transforms like linear change of basis for a quadratic form. This observation is used in the proof of proposition 0.4.2.2 which gives the canonical form for a Morse function near a critical point. Applications of this result will be discussed in later chapters.

Proposition 0.4.2.2 (Morse Lemma) *Let $f : M \to \mathbb{R}$ be a Morse function. Then for every critical point $x \in M$ of f, there is a coordinate chart containing x relative to which f has the representation (0.3.3.7):*

$$f(x) = -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_m^2.$$

Proof - We may assume $M = \mathbb{R}^m$, the critical point of f is the origin and $f(\mathbf{0}) = 0$. From the Taylor expansion of f and the fact that $\mathbf{0}$ is a critical point we obtain the expression

$$f(x) = \sum_{i,j} x_i x_j h_{ij}(x), \tag{0.4.2.1}$$

for some smooth functions h_{ij} with $h_{ij} = h_{ji}$. After possibly a linear change of coordinates we may assume that $h(0)$ is a diagonal matrix with ± 1 along the diagonal. Clearly the symmetric matrix $h(x) = (h_{ij}(x))$ is $\frac{1}{2}\mathsf{H}(f)(x)$. We claim that there is a matrix valued function R such that

$$R(x)' h(x) R(x) = h(\mathbf{0}), \tag{0.4.2.2}$$

where superscript $'$ signifies the transposition of the matrix. This is a consequence of the implicit function theorem. In fact, consider the mapping

$$\Phi : \mathbb{R}^m \times \mathbb{R}^{m^2} \longrightarrow \mathbb{R}^{\frac{1}{2}m(m+1)}, \quad \Phi(x, Y) = Y' h(x) Y.$$

Now $\Phi(\mathbf{0}, I) = h(\mathbf{0})$ and

$$D_2\Phi(\mathbf{0}, I)(\Psi) = \Psi' h(\mathbf{0}) + h(\mathbf{0})\Psi,$$

where D_2 denotes differentiation relative to the Y variable. Since $h(\mathbf{0})$ is nonsingular, the map $D_2\Phi(\mathbf{0}, I)$ is onto the space of symmetric matrices. Let $N \subset \mathbb{R}^{m^2}$ be its kernel, then the implicit function theorem is applicable and gives a mapping S from a neighborhood of the origin in $\mathbb{R}^m \times N$ to a complement C of N in $\mathbb{R}^{m^2}$ such that

$$\Phi(x, S(x, X)) = h(\mathbf{0}).$$

Now set $R(x) = S(x, \mathbf{0})$ and note that $R(\mathbf{0}) = I$ and therefore the matrix $R = (R_{ij}(\mathbf{x}))$ is invertible in a neighborhood of the origin. Let $\rho = (\rho_{ij}(\mathbf{x})) = R^{-1}$ and make the substitution (change of variable) $y_j = \sum_k \rho_{jk}(\mathbf{x}) x_k$ in (0.4.2.1) and use (0.4.2.2) to obtain the desired result. ♣

Corollary 0.4.2.1 *Let M be a compact manifold admitting of a Morse function f with only two critical points. Then M is homeomorphic to a sphere.*

Proof - The two critical points p and q are necessarily a minimum and a maximum. We may assume $f(p) = 0$ and $f(q) = 1$. It follows from the Morse lemma and example **??** that $f^{-1}[0, a]$ (and $f^{-1}[b, 1]$) are diffeomorphic to the disc. Therefore for every $0 < a < 1$ we have a diffeomorphism ψ of $f^{-1}[0, a]$ onto S^m with a disc (around north pole) removed. Now let $a \to 1$, then $f^{-1}([1 - a, 1])$ shrinks to q and therefore ψ extends to a homeomorphism of M

onto the one point compactification of the disc which is S^m. (Note that there is no guarantee that the extension of ψ to the point q remains a diffeomorphism. See example 0.4.2.3) ♣

Corollary 0.4.2.1 appears rather innocuous at first sight, however a deeper examination of it reveals some remarkable facts. The following example is the first step:

Example 0.4.2.3 We construct a manifold which is not at all obvious to be homeomorphic to S^7, however, by exhibiting a Morse function with only two critical points we deduce from corollary 0.4.2.1 that it is so. Identify S^4 with the one point compactification of $\mathbb{R}^4$ and the regard the latter space as the space of quaternions $\mathbf{H}$. Denote by q_∞ the point at infinity for $\mathbf{H} \subset S^4$. Let

$$\mathbf{H}_\circ = \{q \in \mathbf{H} \mid ||q|| < 2\}, \quad \mathbf{H}_\infty = \{q \in \mathbf{H} \mid ||q|| > \frac{1}{2}\}, \quad \mathbf{H}_1 = \mathbf{H}_\circ \cap \mathbf{H}_\infty.$$

Identify S^3 with the group of unit quaternions. Let $\bar{\mathbf{H}}_\infty = \mathbf{H}_\infty \cup \{q_\infty\}$. We construct a smooth manifold by exhibiting a transition function ψ

$$\begin{array}{ccc} \bar{\mathbf{H}}_\infty \times S^3 & & \mathbf{H}_\circ \times S^3 \\ \cup & & \cup \\ \mathbf{H}_1 \times S^3 & \xrightarrow{\psi} & \mathbf{H}_1 \times S^3 \end{array}$$

defined by

$$\psi(u,v) = (u, \frac{u^j v u^k}{||u||}), \quad \text{where} \quad j + k = 1.$$

Here $u^j v u^k$ is calculated according to multiplication of quaternions. Clearly we obtain a smooth manifold which we denote by $M_{j,k}$. Now consider the function $f : M_{j,k} \to \mathbb{R}$ defined as

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By setting $f(q_\infty, v) = 0$ we obtain an extension of f to a smooth function on $M_{j,k}$. Furthermore, it is elementary that the only critical points of f are the points $(0, \pm 1)$. It follows from corollary 0.4.2.1. that $M_{j,k}$ is homeomorphic to the sphere S^7. What is more remarkable is that if $j - k \not\equiv \pm 1 \bmod 7$, then $M_{j,k}$ is not diffeomorphic to S^7, but the proof of this fact requires techniques we have not introduced. This is Milnor's celebrated example of a manifold homeomorphic but not diffeomorphic to S^7. In particular, this shows that "homeomorphic" in the assertion of corollary 0.4.2.1 cannot be improved to "diffeomorphic". ♠

0.4.3 Smoothing and Transversality

Let M and N be manifolds and let $\mathcal{M}^r(M,N)$ be the space of maps of class r from M to N. There are standard ways of endowing $\mathcal{M}^r(M,N)$ with a topology. Let $\{U_i,\varphi_i\}$ be fixed coverings of M by coordinate charts and we assume that every U_i is relatively compact. It is convenient to fix an embedding of N in some Euclidean space $\mathbb{R}^q$ with the standard inner product, although the end result is independent of the embedding. (This is no loss of generality since it will be shown in the subsection on Whitney Embedding Theorem that every manifold can be embedded in some Euclidean space.) Let $F^i = F\varphi_i^{-1} : U_i \to \mathbb{R}^q$, and consider the semi-norms on the space of mappings of U_i into $\mathbb{R}^q$:

$$||F||_{i,s} = \sup_{x\in U_i}[||F^i(x)|| + ||DF^i(x)|| + \cdots + ||D^sF^i(x)||].$$

The semi-norms define a topology on $\mathcal{M}^r(M,\mathbb{R}^q)$ and by standard real variable arguments endow it with the structure of a metric space (in fact, a Fréchet space). $\mathcal{M}^r(M,N)$ is the subset of this metric space defined by the requirement $F(x)\in N$. The induced topology on $\mathcal{M}^r(M,N)$ is known as the C^r *topology*.

It is often useful to approximate a map with one with greater degree of smoothness. The main tool for accomplishing this is by using a standard convolution argument from elementary analysis. Let $\psi_\epsilon : \mathbb{R}^m \to \mathbb{R}_+$ be a non-negative C^∞ function depending only on $||x||$. Assume furthermore

1. ψ_ϵ vanishes outside the ball of radius $\epsilon > 0$ centered at the origin;
2. ψ_ϵ has mass 1, i.e.

$$\int_{\mathbb{R}^m} \psi_\epsilon(x)dx_1\cdots dx_m = 1.$$

Then for any continuous function $h:\mathbb{R}^m \to \mathbb{R}$

$$h_\epsilon = \psi_\epsilon \star h(x) = \int_{\mathbb{R}^m} \psi_\epsilon(x-y)h(y)dy_1\cdots dy_m$$

is C^∞. Furthermore, if h is C^r, $r\geq 0$, then $h_\epsilon \to h$ as $\epsilon\to 0$ in C^r topology. (Here $N=\mathbb{R}$.) Note also that supph_ϵ is contained in an ϵ neighborhood of support of h. While this allows one to smooth out any real valued function, we still have to smooth out a mapping from M to N. To do so let $\{U_i,\varphi\}$ be a covering of M by coordinate charts, and ϕ_i be a partition of unity subordinate to this covering. Then $\phi_i h$ is a real values function on U_i with the same degree of smoothness as h. Now it is an exercise to show that if h is C^r, then

$$\sum_i (\phi_i h)_\epsilon$$

yields the desired approximation to h in the topology of $\mathcal{M}^r(M, \mathbb{R})$. The case of general $N \subset \mathbb{R}^q$ requires an additional observation since once we modify the function h, its values may no longer lie in N. The following lemma allows us to circumvent this problem:

Lemma 0.4.3.1 *Let $U' \subset U \subset \mathbb{R}^m$ open subsets, $h : U \to V' \subset \bar{V}' \subset V \subset \mathbb{R}^n$, with U, V and V' open and relatively compact. Assume h is C^r and its restriction to U' is C^∞. Let $U'' \subset U'$ be such that $\bar{U}'' \cap U \subset U'$. Then there is a sequence of C^∞ mappings $h_n : U \to V$ converging to h such that $h_n = h$ on U''.*

Proof - Let β_1 and β_2 be real valued non-negative C^∞ functions on U such that

1. β_1 is identically 1 on a neighborhood of U'' and vanishes outside of U';
2. β_2 is identically 1 on $U \setminus U'$;
3. $\beta_1 + \beta_2 \equiv 1$.

Then for $\epsilon > 0$ sufficiently small, $\beta_1 h + (\beta_2 h)_\epsilon$ yields the required sequence. ♣

Now we can find smooth approximations to mappings $h \in \mathcal{M}^s(M, N)$. In fact consider relatively compact open subsets

$$U_i'' \subset \bar{U}_i'' \subset U_i' \subset \bar{U}_i' \subset U_i \subset M, \quad \text{and } V_p' \subset \bar{V}_p' \subset V_p \subset N,$$

such that $\cup U_i'' = M$, $\cup V_p' = N$ with $(U_i, \varphi_{i,M})$'s and $(V_p, \varphi_{p,N})$'s coordinate charts. We may also assume that for every i there is $p = p(i)$ such that $h(U_i) \subset V_p'$. It is clear that $\varphi_{p(1),N} h \varphi_{1,M}^{-1}$ can be approximated by a C^∞ map. We then proceed inductively, using lemma 0.4.3.1 to modify h only outside the union of certain U_i'''s to obtain the required C^∞ approximation to h. The details are straightforward.

As an application of the concept and existence of smoothing and theorem 0.4.2.1 we discuss the notion of transverse mappings and the transversality theorem. Let M and N be manifolds, $N' \subset N$ an embedded submanifold and $K \subset M$ a subset. A mapping F (which is always assumed to be at least C^1) is *transverse* to N' along K if for every $x \in K$ either $F(x) \notin N'$ or $F(x) \in N'$ and

$$DF(\mathcal{T}_x M) + \mathcal{T}_{F(x)} N' = \mathcal{T}_x N.$$

Note that the sum is not required to be direct. In particular, if $N' = y$ is a point, then transverse to N' means y is a regular value, Often no mention of K is made in transvesality statements which means $K = M$. Since N' is an embedded submanifold we can represent it locally as the zero set Z_h of a (smooth) function h on N, and by the implicit function theorem, N' is locally a linear space of dimension n' in $\mathbb{R}^n$. Assume $K = M$, and let $p : \mathbb{R}^n \rightarrow \mathbb{R}^{n-n'}$ a linear projection mapping $\mathbb{R}^{n'}$ to the origin. If F is transverse to N', locally pF is a submersion from an open set $U \subset M$ to a subset of $\mathbb{R}^{n-n'}$. It follows that $F^{-1}(N') \cap U = (pF)^{-1}(\mathbf{0})$ and

Lemma 0.4.3.2 *Let $F : M \rightarrow N$ be tranverse to the embedded submanifold N'. Then $F^{-1}(N')$ is a submanifold of M.*

The following key corollary depends on theorem 0.4.2.1:

Corollary 0.4.3.1 *Let K be a compact subset of the manifold M, $N = \mathbb{R}^n$ and $N' = \mathbb{R}^{n'}$ a linear subspace of N. Then the set of mappings of M to N which are transverse to N' along K is open and dense in $\mathcal{M}^r(M, N)$.*

Proof - Let $p : \mathbb{R}^n \rightarrow \mathbb{R}^{n-n'}$ be the canonical projection with kernel N'. Then a map $h \in \mathcal{M}^r(M, N)$ is transverse to N' along K if and only if for every $x \in (ph)^{-1}(\mathbf{0}) \cap K$, the linear map $Dh(x) : \mathcal{T}_x M \rightarrow \mathbb{R}^{n-n'}$ is surjective. The open-ness statement of the corollary follows. To prove density we note that we can assume that h is C^∞. Then theorem 0.4.2.1 is applicable to ph and so the set of its regular values are dense. Let z_j be a sequence of points in N tending to the origin as $j \rightarrow \infty$ and such that $p(z_j)$ is a regular value for ph. Set $h_j(x) = h(x) - z_j$. Then $h_j \rightarrow h$ in $\mathcal{M}^r(M, N)$ and h_j is tranverse to N'. ♣

We can now deduce the global version of transversality theorem for mappings of a compact manifold into another manifold. Let $\mathcal{M}^r(M, N; N') \subset \mathcal{M}^r(M, N)$ be the subset of C^r maps of M to N which are transverse to N'. Similarly, let $\mathcal{M}^r(M, N; K, N') \subset \mathcal{M}^r(M, N)$ be the subset of C^r maps of M to N which are transverse to N' along K. Then we have

Proposition 0.4.3.1 (Transversality Theorem) *Let $K\imath M$ be a compact subset of the manifold M, and N' an embedded submanifold of N. Then $\mathcal{M}^r(M, N; K, N')$ is open and dense in $\mathcal{M}^r(M, N)$.*

Proof - The proposition follows easily from its local version corollary 0.4.3.1 and patching together procedure described in lemma 0.4.3.1. The details are left to the reader. ♣

Remark 0.4.3.1 The assumption of compactness in proposition 0.4.3.1 is not essential, and it is not difficult to see how the above can be adopted to the case where M is non-compact. There is a point that sometimes requires attention. When M is non-compact, it is convenient to endow $C^\infty(M, N)$ with a stronger topology which is not even metrizable. Situations of this kind occur often in analysis; for example, the standard topology of $C^\infty_\circ(\mathbb{R}^n)$ is not metrizable. We shall not dwell on these matters here. For a discussion of the strong topology on $C^r(M, N)$, when M is non-compact see [Hi]. ♡

Remark 0.4.3.2 Let $F : M \rightarrow \mathbb{R}^N$ be an immersion so that $F(M)$ is allowed to have self-intersections. Let $F(x) = F(y)$ where $x, y \in M$ are distinct points, U_x and U_y be small neighborhoods of x and y in M. Then applying the Transversality Theorem with $N' = F(U_y)$ we may assume that $F(U_x)$ and $F(U_y)$ intersect transversally. By using an argument similar to one used in the proof of lemma 0.4.3.1 we may assume that all self intersections of the immersion F are transverse. In particular this implies that for $N \geq 2m + 1$ we may assume that the immersion $F : M \rightarrow \mathbb{R}^N$ is actually an embedding. For $N = 2m$ the transversality property implies that self intersections are isolated points, and if $F(x) = F(y)$, $x \neq y \in M$, then

$$F_\star(\mathcal{T}_x M) + F_\star(\mathcal{T}_y M) = \mathbb{R}^{2m}.$$

This implies that near $F(x) = F(y)$ we have two *branches* of $F(M)$, viz., $F(U_x)$ and $F(U_y)$. ♡

Remark 0.4.3.3 The smoothing procedure described above can be used to show that a C^1 manifold has a unique structure of a C^∞ manifold and this is why we did not distinguish between different degrees of smoothness of the manifolds under consideration. Notice however, a topological manifold may have none or many structures of a smooth manifold. This is a subtle matter. ♡

0.4.4 Kronecker Index

Let M be an oriented manifold of dimension m with P and Q embedded oriented submanifolds of dimensions p and q respectively. We assume P and Q are in *general position* which means that if $x \in P \cap Q$, then $\dim(\mathcal{T}_x P \cap \mathcal{T}_x Q) = p + q - m$ if $p + q \geq m$. In other words, if $p+q < m$, then P and Q do not intersect, and $\mathcal{T}_x P + \mathcal{T}_x Q = \mathcal{T}_x M$ if $p+q \geq m$. This condition is equivalent to the inclusion of P in M to be transverse to Q, and by proposition 0.4.3.1 is valid after arbitrarily small perturbation of the inclusion. Recall that the normal bundle of the submanifold P of M is the quotient of the tangent bundle of M restricted to P by the tangent bundle of P. Orientations for any two of M, P and its normal bundle, determine an orientation for the third. For example, let $\{e_1, \cdots, e_p\}$ and $\{e_1, \cdots, e_p, e_{p+1}, \cdots, e_m\}$ be positively oriented basis for $\mathcal{T}_x P$ and $\mathcal{T}_x M$ respectively. Then we may regard $\{e_{p+1}, \cdots, e_m\}$ as a basis for the normal bundle of P at x and declare it to be positively oriented.

By the *intersection* of P and Q we mean their set-theoretic intersection together with the orientation of the normal bundle to $P \cap Q$ given by the following rule: Let $\{e_1, \cdots, e_r\}$ be a basis for $\mathcal{T}_x P \cap \mathcal{T}_x Q$ where $r = p + q - m$. Extend this basis to positively oriented bases $\{e_1, \cdots, e_p\}$ and $\{e_1, \cdots, e_r, e_{p+1}, \cdots, e_m\}$ for $\mathcal{T}_x P$ and $\mathcal{T}_x Q$ respectively. Now $\{e_1, \cdots, e_m\}$ is a basis for $\mathcal{T}_x M$. If this basis is positively oriented, we declare $\{e_{r+1}, \cdots, e_m\}$ to be a positively oriented basis for the normal bundle to $P \cap Q$ at $x \in P \cap Q$. Otherwise we declare it to be negatively oriented. The intersection of P and Q together with this choice of orientation for the normal bundle is denoted by $I(P, Q)$. By a simple argument

$$I(P, Q) = (-1)^{(m-p)(m-q)} I(Q, P), \tag{0.4.4.1}$$

where the factor $(-1)^{(m-p)(m-q)}$ refers to the orientation. Notice that the sign of the orientation depends on the codimensions of the submanifolds.

The case $p + q = m$ is of special interest since, in this case, the set-theoretic intersection of P and Q consists of a discrete set of points, and the normal bundle is the restriction of the tangent bundle of M. In this case the intersection is a discrete set of points together with a number ± 1 at each intersection point. It is customary to write $\mathrm{KI}(P, Q; x) = \pm 1$ according as the basis $\{e_1, \cdots, e_p, e_{p+1}, \cdots, e_m\}$ is a positively or negatively oriented basis for $\mathcal{T}_x M$ for $x \in P \cap Q$. $\mathrm{KI}(P, Q; x)$ is called the *Kronecker index of P and Q at $x \in P \cap Q$.* If the number of intersections in finite, then we define the *Kronecker index of P and Q* as

$$\mathrm{KI}(P, Q) = \sum_{x \in P \cap Q} \mathrm{KI}(P, Q; x). \tag{0.4.4.2}$$

It is a simple consequence of the argument in example 0.2.1.5 that $\mathrm{KI}(P, Q; x) = 1$ if P and Q are complex submanifolds of the complex manifold M.

Example 0.4.4.1 In this example we derive a formula for the Kronecker index of the intersection of the graph $\Gamma(f)$ of a C^1 function $f : M \to M$ and the diagonal Δ (i.e., graph of the identity map) regarded as submanifolds of $M \times M$. Of course we are assuming that $\Gamma(f)$ and Δ are in general position. Since the tangent space to $\Gamma(f)$ at (x, x) is the span of the column vectors $\{(\xi, Df(x)(\xi)) | \xi \in \mathbb{R}^m\}$, $\mathrm{KI}(\Gamma(f), \Delta; (x, x)) = \pm 1$ according as

$$\det \begin{pmatrix} I & I \\ Df(x) & I \end{pmatrix} = \det(I - Df(x))$$

is positive or negative, i.e.,

$$\mathrm{KI}(\Gamma(f), \Delta; (x, x)) = \frac{\det(I - Df(x))}{|\det(I - Df(x))|}.$$

This formula has applications to Fixed Point theorems. ♣

Proposition 0.4.4.1 *Let M be an oriented manifold of dimension m. Let P and Q be compact oriented submanifold of M of dimensions p and q respectively, and assume that $\partial Q \neq \emptyset = \partial P$. Assume also that $p + q = m + 1$ and P and Q are in general position in M. Then* $\mathrm{KI}(P, \partial Q) = 0$.

Proof - From transversality it follows that $P \cap Q$ consists of a finite number of circles and smooth line segments Λ_i. Denote the end-points of Λ_i by a_i and b_i. Then $a_i, b_i \in \partial Q$. Consider coordinate neighborhoods $U_i \subset Q$ and $V_i \subset P$ such that $U_i \cap V_i$ contains a neighborhood Λ_i' of a_i in Λ_i. Furthermore, let $\{y_1, \cdots, y_q\}$ be positively oriented coordinate functions in U_i such that $\Lambda_i \cap U_i$ is defined by $0 \leq y_q < 1$ and $y_j = 0$ for $j < q$. We may assume $\partial Q \cap U_i$ is defined by $y_q = 0$, and set R_i be the submanifold of U_i defined by the equation $y_q = 1 - \epsilon$ for some $\epsilon > 0$ small. The intersection of R_i and Λ_i consists of a single point c. Since the orientations induced from Q on $\partial Q \cap U_i$ and R_i are given by $dy_1 \wedge \cdots \wedge dy_{q-1}$ and $-dy_1 \wedge \cdots \wedge dy_{q-1}$ respectively, we see that

$$\mathrm{KI}(P, \partial Q; a_i) + \mathrm{KI}(P, R_i; c) = 0.$$

It is clear that there is a sequence of submanifolds $R_{i1}, \cdots, R_{il}$ of dimension $q-1$ of Q such that each R_{ij} intersects Λ_i transversally at a single point c_{ij}, $c_{i1} = a_i$, $c_{il} = b_i$ and

$$\mathrm{KI}(P, R_{ij}; c_{ij}) + \mathrm{KI}(P, R_{ij+1}; c_{ij+1}) = 0.$$

Adding these relations we obtain $\mathrm{KI}(P, \partial Q; a_i) + \mathrm{KI}(P, \partial Q; b_i) = 0$. It now follows easily that $\mathrm{KI}(P, \partial Q) = 0$ as desired. ♣

Example 0.4.4.2 The mapping of $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (-\frac{1}{1+t^2}, t - \frac{2t}{1+t^2})$$

is an immersion and has precisely one self intersection which is at $f(\pm 1)$. The image of f, near $(-\frac{1}{2}, 0)$ looks approximately like $\propto$. We generalize this and construct an immersion $F : \mathbb{R}^m \to \mathbb{R}^{2m}$ with precisely one self intersection. Let $t_1, \cdots, t_m$ denote standard cartesian coordinates in $\mathbb{R}^m$, and define

$$\xi = (1+t_1^2)(1+t_2^2)\cdots(1+t_m^2).$$

Define $F = (F_1, \cdots, F_{2m}) : \mathbb{R}^m \to \mathbb{R}^{2m}$ by

$$\begin{array}{cccc} F_1(t) = t_1 - \frac{2t_1}{\xi}, & F_2(t) = t_2, & \cdots, & F_m(t) = t_m \\ F_{m+1}(t) = \frac{1}{\xi}, & F_{m+2}(t) = \frac{t_1 t_2}{\xi}, & \cdots, & F_{2m}(t) = \frac{t_1 t_m}{\xi}. \end{array}$$

It is clear that for $\sum t_i^2$ large, the map F is approximately

$$F_i(t) \sim t_i, \quad F_{i+m}(t) \sim 0, \qquad \text{for } i = 1, 2, \cdots, m.$$

It is elementary that F is injective except for a *double point*, namely,

$$F(1,0,\cdots,0)=F(-1,0,\cdots,0).$$

Computing the matrix of partial derivatives one sees that F is an immersion. For example, for $m=4$ this matrix is given by

$$\begin{pmatrix} 1-\frac{2(1-t_1^2)}{\xi(1+t_1^2)} & 0 & 0 & 0 & \frac{-2t_1}{\xi(1+t_1^2)} & \frac{t_2(1-t_1^2)}{\xi(1+t_1^2)} & \frac{t_3(1-t_1^2)}{\xi(1+t_1^2)} & \frac{t_4(1-t_1^2)}{\xi(1+t_1^2)} \\ \frac{4t_1t_2}{\xi(1+t_2^2)} & 1 & 0 & 0 & \frac{-2t_2}{\xi(1+t_2^2)} & \frac{t_1(1-t_2^2)}{\xi(1+t_2^2)} & \frac{-2t_1t_2t_3}{\xi(1+t_2^2)} & \frac{-2t_1t_2t_4}{\xi(1+t_2^2)} \\ \frac{4t_1t_3}{\xi(1+t_3^2)} & 0 & 1 & 0 & \frac{-2t_3}{\xi(1+t_3^2)} & \frac{-2t_1t_2t_3}{\xi(1+t_3^2)} & \frac{t_1(1-t_3^2)}{\xi(1+t_3^2)} & \frac{-2t_1t_3t_4}{\xi(1+t_3^2)} \\ \frac{4t_1t_4}{\xi(1+t_4^2)} & 0 & 0 & 1 & \frac{-2t_4}{\xi(1+t_4^2)} & \frac{-2t_1t_2t_4}{\xi(1+t_4^2)} & \frac{-2t_1t_3t_4}{\xi(1+t_4^2)} & \frac{t_1(1-t_4^2)}{\xi(1+t_4^2)} \end{pmatrix}$$

from which the assertion regarding maximality of its rank follows easily. At the point of self intersection the matrix of partial derivatives is

$$A_\pm=\begin{pmatrix} 1 & 0 & 0 & 0 & \mp\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \pm\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \pm\frac{1}{2} \end{pmatrix} \tag{0.4.4.3}$$

corresponding to neighborhoods of $(\pm1,0,0,0)$. Similar formula is valid for all m. It is a simple consequence of the transversality theorem (and the arguments leading to its proof) that we can require an immersion $F: M\to\mathbb{R}^{2m}$, as usual $\dim M=m$, to have only transversal self intersections. This means that if $F(x)=F(y)$, then

$$F_\star(\mathcal{T}_xM)+F_\star(\mathcal{T}_yM)=\mathbb{R}^{2m}.$$

Thus at the self intersection point $p=F(x)=F(y)$ we have two “branches” of M passing through p which we denote by P and Q. If $m=2k$, then we define the *self intersection number* at p by

$$\mathrm{KI}(F;p)=\mathrm{KI}(P,Q;p).$$

Note that since m is even, $\mathrm{KI}(P,Q;p)=\mathrm{KI}(Q,P;p)$ and the definition is meaningful. Therefore for m even, we obtain

$$\frac{\det\begin{pmatrix} A_+ \\ A_-\end{pmatrix}}{|\det\begin{pmatrix} A_+ \\ A_-|\end{pmatrix}}. \tag{0.4.4.4}$$

for the self-intersection number of the mapping F at $p=F(\pm1,0,\cdots,0)$. ♠

The singularities of a vector field has an interpretation in terms of transversality of intersections. A vector field is a section of the tangent bundle, it maybe regarded as a submanifold of dimension m of the tangent bundle $\mathcal{T}M$ of M. Identifying M with the zero section of the tangent bundle, we may regard a vector field ξ and M as two m-dimensional submanifolds of $\mathcal{T}M$. Let us assume that ξ and M are in general position so that we can apply our notions of intersections to them. Essentially the same calculation that allowed us to define the notion of simplicity of a singular point makes it possible to assign an index ± 1 to simple singular point. In fact that calculation shows that the non-vanishing and the sign of $\det(D\xi)$ are independent of the choice of positively coordinate system. Consequently, the sign of the determinant $\det(D\xi) \neq 0$ is defined as the *index* of the vector field ξ at the singular point. To formulate this in terms of intersections we transform the problem by mapping a neighborhood of the zero section of $\mathcal{T}M$ onto a neighborhood of the diagonal in $M \times M$ diffeomorphically. To do this we can for example fix a Riemanian metric on M and consider the mapping $(y, \eta) \to (y, \mathrm{Exp}_y \eta)$, which does the job if $\|\eta\|$ is small. In particular, the manifold M becomes identified with the diagonal. Let $\phi_t(y)$ be the 1-parameter group associated to ξ. For small $t > 0$ we consider the submanifold $M(t) = \{(y, \phi_t(y)) | y \in M\}$, then our problem is that of computing the intersection of the diagonal and $M(t)$. First we make sure that $M(t)$ and M (i.e., the diagonal) intersect transversally. To do so notice that the Taylor expansion of $\phi_t(x)$ is

$$\phi_t(x) = x + t\xi_x + t^2 r(t, x).$$

Therefore

$$(D\phi_t)(x) - I = t(D\xi)(x) + t^2 Dr(t, x).$$

Since $M(t)$ is the graph of the function $y \to \phi_t(y)$, non-vanishing of $\det(D\xi)(x)$ implies that $M(t)$ and M intersect transversally at (x, x) for $t > 0$ small. We have, by example 3.1,

$$\mathrm{KI}(M(t), M) = \sum \frac{\det(D\phi_t(x) - I)}{|\det(D\phi_t(x) - I)|},$$

where the summation is over all the singularities x of the vector field ξ. The transversality assumption implies that the singularities of ξ form a discrete set which we assume to be finite. Now

$$\frac{\det(D\phi_t(x) - I)}{|\det(D\phi_t(x) - I)|} = \frac{\det((D\xi)(x) + tDr(t, x))}{|\det((D\xi)(x) + tDr(t, x))|} = \frac{\det(D\xi(x))}{|\det(D\xi(x))|},$$

since $t > 0$ is small. This quantity is of course ± 1 according as the mapping $y \to \xi_y$ is orientation preserving or reversing near x.

Let us look a different way at the vector field ξ at an isolated singular point x. We identify a neighborhood of x with an open subset $U \subset \mathbb{R}^m$, via an orientation preserving diffeomorphism, and regard ξ as a tangent vector field on U. Assuming the only singularity of ξ in U is x, we define

$$g : U \setminus \{x\} \longrightarrow S^{m-1} \qquad \text{by} \quad g(y) = \frac{\xi_y}{\|\xi_y\|}.$$

Let $dv_{S^{m-1}}$ denote the volume element on S^{m-1}, $\epsilon > 0$ be sufficiently small so that the closure of the ball D_ϵ of radius ϵ in contained in U. Define the *index* of ξ at x by

$$\mathrm{Ind}(\xi, x) = \frac{1}{c_{m-1}} \int_{\partial D_\epsilon} g^\star(dv_{S^{m-1}}),$$

where $c_{m-1} = \int_{S^{m-1}} dv_{S^{m-1}}$. We shall see below that $\mathrm{Ind}(\xi, x)$ is independent of the choice of the orientation preserving diffeomorphism and $\epsilon > 0$. The transversality assumption implies that $\det(D_x\xi) \neq 0$ and hence the mapping $y \to \xi_y$ is a diffeomorphism onto a neighborhood of 0. From differential calculus we know that $y \to \xi_y$ maps small spheres centered at x onto compact convex hypersurfaces containing the origin in their interiors. It follows that $g_{|\partial D}$ is a diffeomorphism onto S^{m-1}. Consequently, under the assumption of transversality, $\mathrm{Ind}(\xi, x) = \pm 1$ depending on whether $g_{|\partial D}$ is orientation preserving or reversing. Hence the index of a vector field is simply the intersection number discussed above. Note also that the above argument shows that $\mathrm{Ind}(\xi, x)$ is independent of the choice of the orientation preserving diffeomorphism, and hence it is meaningfully defined on a manifold.

One should note that the definition of the index of a vector field did not require the assumption of transversality. If the vector field of ξ does not satisfy the transversality assumption and vanish to some finite order, then by an arbitrarily small perturbation we can make it transverse, and we may assume that that the perturbation is the identity on $S_\epsilon(x) = \partial D$. Let ξ' be the perturbed vector field. Then the singular point x of ξ bifurcates into several, say n, singular points $x_1, \cdots, x_n$ of ξ' in the interior of D. We may apply the above consideration to ξ'. Let D^i be a small closed disc centered at x_i with $D^i \cap D^j = \emptyset$ for $i \neq j$. Let $V = D \setminus (\cup D^i)$ and modify the definition of g by setting $g(y) = \frac{\xi_y}{\|\xi_y\|}$ for $y \in V$. Then by Stokes' theorem

$$\int_{\partial D} g^\star(dv_{S^{m-1}}) - \sum \int_{\partial D^i} g^\star(dv_{S^{m-1}}) = \int_V dg^\star(dv_{S^{m-1}}) = 0.$$

Therefore the index is equal to the sum of the indices of the n bifurcated points. Notice that the argument involving Stokes'theorem also proves independence on the index from the choice of $\epsilon > 0$. We shall further discuss these concepts later in connection with the linking number.

For a vector field ξ with a finite number of singular points we we define the *index* of ξ as

$$\mathrm{Ind}(\xi) = \sum \mathrm{Ind}(\xi, x),$$

where the summation is over all singular points of ξ. In view of the above considerations, the index of ξ is simply the intersection number of the diagonal and the mapping $y \to \phi_t(y)$, where $t > 0$ is sufficiently small.

Exercise 0.4.4.1 *Let f be a real-valued function defined in a neighborhood of $\mathbf{0}$ in $\mathbb{R}^m$. Assume that the vector field* $\mathrm{grad}(f)$ *has an isolated singularity at $\mathbf{0}$. Then* $\mathrm{grad}(f)$ *satisfies the transvesality condition discussed above if and only if the Hessian $H(f) = (\partial^2 f/\partial x_i \partial x_j)$ is non-singular at $\mathbf{0}$. Prove that if* $\det(H(f)) \neq 0$, *then*

$$\mathrm{Ind}(\mathrm{grad} f, \mathbf{0}) = (-1)^{\nu(f)},$$

where $\nu(f)$ is the number of negative eigenvalues of $H(f)$.

Example 0.4.4.3 Let G be a compact analytic group and K a closed connected subgroup. Denote the Lie algebras of right invariant vector fields on G and K by $\mathcal{G}$ and $\mathcal{K}$ respectively. We have $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{M}$ is the orthogonal complement of $\mathcal{K}$ in $\mathcal{G}$ relative to a fixed bi-invariant metric. Let $\xi \in \mathcal{K}$ and ξ' denote the vector field induced on $M = G/K$ by ξ. This means

$$\xi'_{gK} = \frac{d}{dt}_{|t=0} \exp(t\xi) gK.$$

Since a one parameter subgroup tangent to K at one point lies entirely in K, the singularities of ξ' are precisely the point set $\{gK | \mathrm{Ad}(g)\xi \in \mathcal{K}\}$. We want to understand the nature of the singular points of ξ'_{gK} when $K = T$ is a maximal torus and ξ is a generic element, i.e., $\exp \xi$ generates T. Clearly in this case the singular set becomes $\{gT | g \in N(T)\}$ where $N(T)$ is the normalizer of T in G. Since the isotropy subgroup of the left action of G on M at gT is gTg^{-1}, the tangent space to M at gT may be identified $\mathrm{Ad}(g^{-1})(\mathcal{M})$. Therefore a neighborhood of the point gT, $g \in N(T)$, in M has coordinatization

$$\eta \longrightarrow (\exp \eta) gT, \quad \eta \in \mathcal{M} \text{ and } \|\eta\| \text{ small.}$$

By Baker-Campbell-Hausdorf formula

$$\exp(\xi)\exp(\eta) = \exp(\xi + \eta + \frac{1}{2}[\xi, \eta] + \cdots).$$

Therefore if D denotes differentiation with respect to η, then

$$D_{\eta=0}\xi'_{gT} = \frac{1}{2}\mathrm{ad}\xi : \eta \longrightarrow \frac{1}{2}[\xi, \eta].$$

By compactness, $\mathcal{M}$ is also invariant under the action of $\mathcal{T}$ (the Lie algebra of T). Therefore to understand the singularity of ξ at gT, we have to look at $\det(\mathrm{ad}\xi)$, where $\mathrm{ad}\xi$ is regarded as a linear operator of $\mathcal{M}$. It is immediate that this determinant is independent of $g \in N(T)$ so that *all the singularities have the same sign*. Furthermore, gT, $g \in N(T)$, is a simple zero since by maximality of T and genericity of ξ, $\mathrm{ad}\xi$, as an operator on $\mathcal{M}$, has only non-zero eigenvalues. Note also that non-vanishing of $\det(\mathrm{ad}\xi)$ also implies that ξ has only isolated zeros. ♠

0.4.5 Whitney Embedding Theorem

In this subsection we mainly consider the question of immersing and embedding of manifolds in Euclidean spaces and some of their ramifications. Let M be a compact C^k manifold and (U_i, φ_i), $i = 1, \cdots, N$, a finite covering of M by coordinate charts. It is clear that there are open sets $V_i \subset W_i$, $i = 1, \cdots, N$, such that

$$\bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i,$$

and V_i's cover M as well. Let ψ_i be a C^k function on U_i such that ψ_i is identically 1 on V_i and vanishes outside of W_i. The mapping ψ_i and therefore $\psi_i\varphi_i$ extend by zero to M. Consider the mapping

$$\Psi : M \longrightarrow \mathbb{R}^{mN}, \quad \Psi(x) = (\psi_1(x)\varphi_1(x), \cdots, \psi_N(x)\varphi_N(x)).$$

It is immediate that Ψ embeds M in $\mathbb{R}^{mN}$. Now we can use theorem 0.4.2.1 to improve this obsevation by embedding M in $\mathbb{R}^{2m+1}$ and immersing it in $\mathbb{R}^{2m}$. This is done inductively by showing that if $\Psi_r : M \to \mathbb{R}^r$ is an embedding, then we can embed M in $\mathbb{R}^{r-1}$ if $r > 2m+1$; and if $r = 2m+1$ we can immerse it in $\mathbb{R}^{2m}$. The trick in doing this is to show that there is $y \in \mathbb{R}^M$ such that if $\pi_y : \mathbb{R}^r \to \mathbb{R}^{r-1}$ is orthogonal projection on the orthogonal complement $\mathbb{R}y^\perp \simeq \mathbb{R}^{r-1}$, then $\pi_y\Psi_r$ is also an embedding or an immersion if $r = 2m+1$. This requires establishing

1. The derivative $D(\pi_y\Psi_r)$ is injective everywhere if $r \geq 2m+1$;

2. $\pi_y\Psi$ is injective if $r > 2m+1$.

To prove the first assertion we fix a Riemannian metric g on M and let $\mathcal{T}_1M$ be the *unit tangent bundle* of M, i.e., $\{(x,\xi)\in\mathcal{T}_xM|g(\xi,\xi)=1, x\in M\}$. Then we have a map

$$\Phi_r:\mathcal{T}_1M\rightarrow S^{r-1}\subset\mathbb{R}^r,\quad \Phi_r(x,\xi)=\frac{\Psi_{r\star}(\xi)}{||\Psi_{r\star}(\xi)||}.$$

(Note that $\Psi_{r\star}$, at $x\in M$, is a linear map of $\mathcal{T}_xM$ to $\mathbb{R}^r$ and $||\Psi_{r\star}(\xi)||\neq 0$ since Ψ_r is an embedding.) This is well-defined since $\Psi_{r\star}(\xi)\neq 0$. Note that if $\pm y\in(S^{r-1}\setminus\Phi_r(\mathcal{T}_1M))$, then the map $\pi_y\Psi_r$ is an immersion. Let $\tilde{\Phi}_r:\mathcal{T}_1M\rightarrow\mathbb{R}P(r-1)$ be the composition of Φ_r and the projection $S^{r-1}\rightarrow\mathbb{R}P(r-1)$ where antipodal points are identified. For $r\geq 2m+1$, $2m-1\leq r-2<r-1$ and $\mathbb{R}P(r-1)\setminus\tilde{\Phi}_r(\mathcal{T}_1M)$ is open and dense in $\mathbb{R}P(r-1)$ proving the existence of y such that $\pi_y\Psi_r$ is an immersion. This proves that for $r\geq 2m$, M can be immersed in $\mathbb{R}^r$.

Next let $\Delta_M=\{(x,x)|x\in M\}\subset M\times M$ and consider the mapping

$$P:M\times M\setminus\Delta_M\longrightarrow S^{r-1},\quad P(x,x')=\frac{\Psi_r(x)-\Psi_r(x')}{||\Psi_r(x)-\Psi_r(x')||}.$$

To ensure that there is $y\in\mathbb{R}^r$ such that the immersion $\pi_y\Psi_r$ is an embedding we have to prove the existence of

$$y\in S^{r-1}\setminus\mathrm{Im}(P).$$

Since $r>2m+1$, $2m<r-1$ and by theorem 0.4.2.1 $S^{r-1}\setminus\mathrm{Im}(P)$ is dense. Therefore for $\pm y\in S^{r-1}\setminus(\mathrm{Im}(P)\cup\mathrm{Im}(\Phi_r))$ both conditions (1) and (2) are fulfilled and we have proven

Theorem 0.4.5.1 (Whitney Embedding Theorem) *Let M be a C^k compact manifold of dimension m and $k\geq 2$. Then for $r\geq 2m$ there is an immersion of M in $\mathbb{R}^r$, and for $r\geq 2m+1$ there is an embedding of M in $\mathbb{R}^r$.*

Remark 0.4.5.1 The version of Whitney embedding theorem given above can be improved. One direction is that the compactness assumption can be removed. In fact an examination of the proof shows that if we represent M as an expanding union $U_1\subset U_2\subset\cdots$ of open subsets, then each U_i can be embedded in $\mathbb{R}^{2m+1}$ (resp. immersed in $\mathbb{R}^{2m}$) and we obtain the desired result by an application of Zorn's lemma. One can also prove that M can be embedded in $\mathbb{R}^{2m}$. The proof given above does not preclude $\mathrm{Im}(\pi_y\Psi_{2m+1})$ from having self-intersections and this is the issue that should be addressed. One may be tempted to think that by a perturbation of an immersion one can remove self-intersections and obtain an embedding. However the immersion of the circle as figure ∞ in $\mathbb{R}^2$ shows that the issue of removing self-intersections cannot be resolved by a perturbation argument. We shall return to this matter later. ♡

Remark 0.4.5.2 Whitney Embedding theorem is also valid for manifolds with boundary. In fact if M is a compact manifold of dimension m with boundary $\partial M \neq \emptyset$, and $H \subset \mathbb{R}^{2m+1}$ denotes the closed half space defined by $x_{2m+1} = 0$, then there is an embedding $\psi : M \rightarrow H$ such that $\psi(\partial M) \subset \partial H$. A similar statement is true about immersions. The proof is a technical extension of the same for manifolds without boundary and will not be presented here. ♡

Example 0.4.5.1 Let $\psi : M \rightarrow \mathbb{R}^{2m}$ be an immersion of the compact oriented manifold M of dimension $m = 2k$. We had noted earlier that we may assume that all self intersections are isolated points (see remark 0.4.3.2) and therefore finite in number. In example 0.4.4.2 we defined a self intersection at such a point. If the number of such points is finite then the *self intersection number* is naturally defined as

$$\mathrm{KI}(\psi) = \sum_{p \text{ self intersections}} \mathrm{KI}(\psi; p). \tag{0.4.5.1}$$

We now show that example 0.4.4.2 and Whitney's theorem imply that we can immerse M into $\mathbb{R}^{2m}$ with arbitrary pre-assigned self intersection number. To show this let $R > 0$ be fixed, and ϕ be a smooth function on $\mathbb{R}$ with values in $[0, 1]$ such that $\phi(u) = \phi(-u)$ and

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We furthermore require ϕ to have nonzero derivative everywhere on the interval $(R, R+1)$. Modifying $F(t)$ of example 0.4.4.2 by multiplying the components $F_{m+1}, \cdots, F_{2m}$ with $\phi(|t|)$ for $R > 0$ sufficiently large, "flattens out" the image of F for large values of $|t|$. We denote the flattened out modification of F by $\tilde{F}$. For the given immersion ψ we can assume that a neighborhood of a point of non-self intersection $\psi(x)$ lies in a linear space of dimension m. It is now clear that composing the given immersion ψ near $\psi(x)$ with the mapping $\tilde{F}$ (properly scaled) we add a new point of self intersection. The self intersection number at the new point of self intersection is given by equation (0.4.4.4). Let $T : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ be a reflection relative to a hyperplane. Then replacing the mapping $\tilde{F}(t)$ with $t \rightarrow T\tilde{F}(t)$ (for a proper choice of the hyperplane) we create a self intersection point with the opposite self intersection number. The claim that we can construct an immersion of M with arbitrary self intersection number follows immediately. It should be pointed out that the same argument works for m odd or for a nonorientable manifold if we work in $\mathbf{Z}/2$ so that at every intersection point $\mathrm{KI}(F; p) = 1 \equiv -1$ and define addition in (0.4.5.1) to be in $\mathbf{Z}/2$ rather than $\mathbf{Z}$. ♠

Example 0.4.5.2 In this example we explicitly embed $\mathbb{R}P(n)$ in S^{2n} and therefore in $\mathbb{R}^{2n}$ by removing one point from S^{2n}. The key idea in this example is to construct a symmetric bilinear map $P : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+k+1}$ which is *strongly nondegenerate*, i.e.,

$$P(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad \text{implies} \quad \mathbf{x} = \mathbf{0} \text{ or } \mathbf{y} = \mathbf{0}.$$

Given such a map we define an immersion $p : \mathbb{R}P(n) \rightarrow S^{n+k}$ by

$$p(\mathbf{x}) = \frac{P(\mathbf{x},\mathbf{x})}{||P(\mathbf{x},\mathbf{x})||},$$

where $||.||$ denotes the standard Euclidean norm and $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \mathbf{0}$. Notice that since P is quadratic, its restriction to S^n gives a mapping of $\mathbb{R}P(n)$ to $\mathbb{R}^{n+k+1}$. The derivative of p is given by

$$Dp(\mathbf{x})(\xi) = \frac{P(\mathbf{x},\xi)}{||P(\mathbf{x},\mathbf{x})||} + g(\mathbf{x},\xi)P(\mathbf{x},\mathbf{x}) = P(\mathbf{x}, \frac{\xi}{||P(\mathbf{x},\mathbf{x})||} + g(\mathbf{x},\xi)\mathbf{x}),$$

for some scalar valued function $g : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. If ξ is tangent to $S^n \subset \mathbb{R}^{n+1}$, then it is orthogonal to $\mathbf{x}$ and

$$\frac{\xi}{||P(\mathbf{x},\mathbf{x})||} + g(\mathbf{x},\xi)\mathbf{x} \neq \mathbf{0},$$

which implies $Dp(\mathbf{x})(\xi) \neq \mathbf{0}$ and Dp has maximal rank everywhere. A simple example of such a strongly nondegenerate symmetric bilinear map P with $k = n$ is

$$P(x_\circ, x_1, \cdots, x_n) = (y_\circ, y_1, \cdots, y_{2n}), \quad \text{where} \;\; y_k = \sum_{i+j=k} x_i x_j.$$

It is elementary that this choice of P yields an embedding of $\mathbb{R}P(n)$ in S^{2n}. In fact, if $p(\mathbf{x}) = p(\mathbf{x}')$, then by multiplying $\mathbf{x} = (x_\circ, \cdots, x_n)$ by a positive scalar we may assume $||P(\mathbf{x},\mathbf{x})|| = ||P(\mathbf{x}',\mathbf{x}')||$. Therefore $p(\mathbf{x}) = p(\mathbf{x}')$ implies $x'_\circ = \pm x_\circ$. Assuming $x_\circ \neq 0$, the relation $x_\circ x_1 = x'_\circ x'_1$ implies $x'_1 = \pm x_1$. Thus we see that $x_j = \pm x'_j$ with the same sign $\pm$ for all j. If $x_\circ = x_1 = \cdots = x_{j-1} = 0$ then we start by looking at the component y_{2j} and proceeding in the same manner. ♠

Exercise 0.4.5.1 *Identifying $\mathbb{R}^4$ with the quaternions $\mathbf{H}$, explain why the pairing $(q, q') \rightarrow qq'$ fails to satisfy the hypotheses of the above example and does not yield an immersion of $\mathbb{R}P(3)$ to S^3.*

The algebraic problem of constructing strongly nondegenerate symmetric bilinear forms and thereby obtaining embeddings of real projective spaces is not a trivial one. There are more powerful but less explicit methods for dealing with the immersion/embedding problem of manifolds which we will touch upon in later chapters. The requirement of symmetry of the strongly nondegenrate bilinear form can also be relaxed and still obtain immersions. The following example shows that the general construction of example 0.4.5.2 can be improved for n odd.

Example 0.4.5.3 Assume $n+1 = 2m$ and identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ so that a point $(x_1, \cdots, x_{2m})$ is represented as $\xi = (\xi_1, \cdots, \xi_m)$ where $\xi_j = x_{2j-1} + ix_{2j}$. Consider the symmetric bilinear pairing $P : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^{2m-1}$ defined by

$$P(\xi, \eta) = (P_2, \cdots, P_{2m}), \quad \text{where } P_l = \sum_{j+k=l} \xi_j \eta_k.$$

It is clear that P yields a strongly nondegenerate pairing $\mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}^{4m-2}$. Thus we obtain an immersion of $\mathbb{R}P(2m-1)$ into S^{4m-3}. In particular we can immerse $\mathbb{R}P(3)$ into S^5 or $\mathbb{R}^5$. However this result is not sharp either since one can immerse any compact orientable manifold of dimension three in $\mathbb{R}^4$. This issue will be discussed later in the text. The method of examples 0.4.5.2 and 0.4.5.3, as it stands, will not yield an immersion of $\mathbb{R}P(3)$ into S^4 since there is no strongly nondegenerate symmetric bilinear pairing $\mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^5$. The proof of this algebraic fact is omitted since it is not relevant to the methods used in this text. For a table of immersions and embeddings of real projective spaces the reader is referred to [DMD] which contains references to original papers. ♠

Exercise 0.4.5.2 *Show that by restricting the mapping*

$$\psi : \mathbb{R}^3 \to \mathbb{R}^4, \qquad \psi(x_1, x_2, x_3) = (x_1x_2, x_2x_3, x_3x_1, x_1^2 - x_2^2)$$

to $S^2 \subset \mathbb{R}^3$*, we obtain an embedding of* $\mathbb{R}P(2)$ *into* $\mathbb{R}^4$.

Exercise 0.4.5.3 *Let* $M \subset \mathbb{R}^{m+1}$ *be a compact orientable hypersuface. Show that* $M \times S^n$ *can be embedded in* $\mathbb{R}^{m+n+1}$.

Exercise 0.4.5.4 *Construct an immersion of* $T^2 \backslash \{\text{point}\}$ *onto an open subset of* $\mathbb{R}^2$.

Exercise 0.4.5.5 *Show that the mapping*

$$(z_\circ, z_1, z_2) \longrightarrow \frac{(z_\circ \bar{z}_1, z_1\bar{z}_2, z_2\bar{z}_\circ, |z_\circ|^2 - |z_1|^2)}{\zeta}$$

where $\zeta = |z_\circ|^2 + |z_1|^2 + |z_2|^2$ *induces an embedding of* $\mathbb{C}P(2)$ *into* $\mathbb{R}^7$.

Exercise 0.4.5.6 *Let* M *be a compact manifold of dimension* m *and* $\psi : M \to \mathbb{R}^N$, $N \geq 2m+1$, *a smooth map. Show that for every* $\epsilon > 0$ *there is an embedding* $\Psi : M \to \mathbb{R}^N$ *such that* Ψ *is* ϵ *close to* ψ *relative to the* sup*-norm in* $\mathbb{R}^N$. (Hint - Consider an embedding $\phi : M \to \mathbb{R}^l$ and the the embedding $(\psi, \phi) : M \to \mathbb{R}^{2m+1+l}$. Now use the projection argument of the proof of Whitney's theorem.)

Exercise 0.4.5.7 *Let M be a manifold of dimension m, and $f, g : M \rightarrow \mathbb{R}^{2m+2}$ immersions. Show that there is a smooth map $F : M \times I \rightarrow \mathbb{R}^{2m+2}$ such that $F(.,0) = f$, $F(.,1) = g$ and for every $u \in I$, $x \rightarrow F(x,u)$ is an immerion of of M into $\mathbb{R}^{2m+2}$.*

Exercise 0.4.5.8 *Let M be a compact manifold of dimension m with ∂M consisting of two connected components N_1 and N_2. Let $H \subset \mathbb{R}^{2m+1}$ be the closed subset defined by the iequalities $0 \leq x_{2m+1} \leq 1$. Denote the boundary components of H by $\partial_\circ H$ and $\partial_1 H$. Show that there is an embedding $\psi : M \rightarrow H$ such that $\psi(N_i) \subset \partial_i H$.*

The idea that allowed us to embed a compact manifold M in $\mathbb{R}^{mN}$ for some sufficiently large N is applicable to yield a mapping of a bundle $E \rightarrow X$ to the tautological bundle over a Grassmann manifold. The construction for complex, real or oriented real bundles is the same and the mapping is into the complex, real or the Grassmann manifold of oriented k-planes. To make this construction precise, let us consider real k-plane bundles for definiteness, and recall that $\mathbf{G}_{k,n}(\mathbb{R})$ denotes the Grassmann manifold of k dimensional linear subspaces of $\mathbb{R}^{k+n}$. Let $\pi : E \rightarrow X$ be a k-plane bundle over the compact[10]manifold M. Let $\{U_1, \cdots, U_N\}$ and $\{V_1, \cdots, V_N\}$ be coverings of M with open sets such that $V_j \subset \bar{V}_j \subset U_j$ and that E is a trivial k-plane bundle on each U_j. We fix trivializations

$$\theta_j = (\theta_{j1}, \theta_{j2}) : \pi^{-1}(U_j) \simeq U_j \times \mathbb{R}^k \subset \mathbb{R}^m \times \mathbb{R}^k$$

for every j. Let ψ_j be a C^∞ non-negative function which is identically 1 on $\bar{V}_j$ vanishes outside of U_j. Now every mapping $\phi_j \theta_j$ extends to a smooth mapping $E \rightarrow \mathbb{R}^m \times \mathbb{R}^k$. The mapping

$$\Theta' : E \rightarrow \mathbb{R}^{(m+k)N}, \quad \Theta'(e) = (\psi_1(\pi(e))\theta_1(e), \cdots, \psi_N(\pi(e))\theta_N(e)),$$

is an embedding which is an affine map on each fibre of the bundle $E \rightarrow M$. We can easily modify Θ' to make it linear on each fibre. In fact, consider $\Theta : E \rightarrow \mathbb{R}^{kN}$ defined by

$$\Theta(e) = (\psi_1(\pi(e))\theta_{12}(e), \cdots, \psi_N(\pi(e))\theta_{N2}(e)).$$

Notice that under Θ every fibre of $\pi : E \rightarrow M$ is mapped onto a k-dimensional subspace of $\mathbb{R}^{kN}$, and therefore it induces a map $\theta : M \rightarrow \mathbf{G}_{k,kN-k}(\mathbb{R})$ and we have the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Theta} & \mathcal{E} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\theta} & \mathbf{G}_{k,q}(\mathbb{R}) \end{array} \tag{0.4.5.2}$$

[10]The assumption that M is a compact manifold is inessential and is only a matter of convenience. Similar arguments work for more general topological spaces (see e.g. [Hi] for not necessarily compact C^r manifolds).

where $q = kN - k$ and Θ is injective on every fibre of $\pi : E \to M$. The above construction realizes the bundle $E \to M$ as the pull back $\theta^\star(\mathcal{E})$ of the tautological bundle over a Grassmann manifold. The map θ is generally not injective. The same argument works for complex or real oriented vector bundles. We can prove the following important proposition:

Proposition 0.4.5.1 *Every complex, real or real oriented k-plane bundle $E \to M$ over a compact manifold can be realized as the pull back of the tautological bundle over $\mathbf{G}_{k,q}$, $\mathbf{G}_{k,q}(\mathbb{R})$ or $\mathbf{G}^\circ_{k,q}(\mathbb{R})$ where $q \geq m$.*

Proof - We have already established the proposition for $q = kN - k$, and it remains to show any $q \geq m$ works. We follow the idea of the proof of the Whitney Embedding Theorem. For $q = kN - k$ we established a map from E to $\mathbb{R}^{q+k}$ which was linear on the fibres of the vector bundle $E \to M$. We want to reduce the dimension $q + k$ by composing the map with a projection onto a linear subspace of codimension 1 and proceed inductively. Now the projection $p : \mathbb{R}^{q+k} \to \mathbb{R}^{q+k-1}$ should be such that $p\theta$ remains a linear isomorphism on each fibre of the bundle. A straightforward application of theorem 0.4.2.1 just as in the proof of theorem 0.4.5.1 shows such a projection exists as long as $q > m$ completing the proof of the proposition. ♣

Remark 0.4.5.3 It can be shown that one can set $q = \dim M$, in proposition 0.4.5.1, however, this would not significantly affect the application of this proposition. ♡

0.4.6 Immersions of the Circle

In view of the transversality theorem every immersion of the circle in $\mathbb{R}^q$, $q \geq 3$, may be perturbed to an embedding by an arbitrarily small perturbation. However, the situation is different for $q = 2$ and in this subsection we analyze immersions $\gamma : S^1 \to \mathbb{R}^2$. We often regard such a mapping as a C^1 function $\gamma : [0, 2\pi] \to \mathbb{R}^2$ such that $\gamma(0) = \gamma(2\pi)$, $\gamma'(0) = \gamma'(2\pi)$ and γ' nowhere vanishing. We assume that all crossing of γ are *normal* which means that if $\gamma(t_1) = \gamma(t_2)$ then the vectors to $\gamma'(t_1)$ and $\gamma'(t_2)$ are linearly independent vectors in the plane. Two immersions $\gamma, \delta : S^1 \to \mathbb{R}^2$ can be *deformed* into or are *deformations* of each other if there is a C^1 map $F : S^1 \times I \to \mathbb{R}^2$ such that

1. $F(t, 0) = \gamma(t)$ and $F(t, 1) = \delta(t)$;

2. For every $u \in I$, $F(., u) : S^1 \to \mathbb{R}^2$ is an immersion.

This is clearly an equivalence relation. We want to classify immersions of S^1 into the plane up to this equivalence relation. The second requirement (namely, $F(., u)$ is an immersion for every u) is essential for otherwise all curves will be equivalent to the constant mapping taking S^1 to the origin via $F(t,u) = u\gamma(t)$. Since $\gamma : S^1 \rightarrow \mathbb{R}^2$ is an immersion, the mapping $\mathsf{G}_\gamma : S^1 \rightarrow S^1$ given by

$$\mathsf{G}_\gamma(t) = \frac{\gamma'(t)}{||\gamma'(t)||},$$

is well-defined. Let $d\theta = \frac{1}{i}\frac{dz}{z}$ denote the standard arc length[11] on S^1. We define the *winding number* of γ as

$$W(\gamma) = \frac{1}{2\pi}\int_{S^1} \mathsf{G}^\star_\gamma(d\theta). \tag{0.4.6.1}$$

Writing $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ the differential $\mathsf{G}^\star_\gamma(d\theta)$ is

$$\mathsf{G}^\star_\gamma(d\theta) = d\tan^{-1}(\frac{\gamma_2'(t)}{\gamma_1'(t)}) = \frac{\gamma_1'(t)\gamma_2''(t) - \gamma_1''(t)\gamma_2'(t)}{\gamma_1'^2(t) + \gamma_2'^2(t)}dt. \tag{0.4.6.2}$$

Now define

$$\Phi_\gamma(t) = \int_0^t \mathsf{G}^\star_\gamma(d\theta),$$

which, in view of (0.4.6.2), measures the number of times the tangent vector field to γ has wound around the circle. The orientation of γ determines the sign of $\Phi_\gamma(t)$. In particular, $\Phi_\gamma(2\pi)$ differs from $\Phi_\gamma(0)$ by an integral multiple of 2π since $\gamma'(2\pi) = \gamma'(0)$. By the fundamental theorem of calculus

$$W(\gamma) = \frac{1}{2\pi}(\Phi_\gamma(2\pi) - \Phi_\gamma(0)) = k \in \mathbf{Z},$$

which shows that the winding number of an immerison of S^1 into the plane is an integer. Note also that if t is the arc length along the curve γ then

$$\mathsf{G}^\star_\gamma(d\theta) = \kappa dt,$$

where κ is the curvature of the plane curve γ.

The integrality of the winding number implies that its value, which depends continuously on deformations of the curve γ, is constant on each equivalence class (of deformations). The

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[11] $d\theta$ is an unfortunate notation since it is not an exact differential on S^1. On the other hand, if we regard θ as a smooth many valued function on S^1, then $d\theta$ becomes an exact differential. There is no need to introduce further general theory to clarify this idea.

converse is also true and will be proven below. First we need some preliminaries. It is clear that the immersion γ can be deformed into one with $\gamma(0) = \mathbf{0}$ and $\gamma'(0)$ pointing along the positive x_1-axis. Replacing γ by $r\gamma$ for some fixed real number $r \neq 0$ we can furthermore assume that if $W(\gamma) \neq 0$ then γ has length $2\pi|W(\gamma)|$. For γ parametrized by arc length we use complex notation and set

$$\gamma'(t) = e^{i\varphi_\gamma(t)}, \quad t \in [0, 2\pi W(\gamma)]. \tag{0.4.6.3}$$

We can now prove

Proposition 0.4.6.1 *Two immersions $\gamma, \delta : S^1 \to \mathbb{R}^2$ are equivalent, with respect to the equivalence relation defined above, if and only if $W(\gamma) = W(\delta)$.*

Proof - The only if has already been established. Let γ, δ be curves with winding number $W(\gamma) = W(\delta)$. Since a reflection has the effect of multiplying the winding number by -1, we may assume $W(\gamma) = W(\delta) \geq 0$. Without loss of generality we may assume that $\delta(0) = \gamma(0) = \mathbf{0}$ and their tangents at 0 are identical. First consider the case where $W(\gamma) \neq 0$ so that we can assume γ and δ have length $2\pi W(\gamma)$ and are parametrized by arc length. For every $u \in I = [0, 1]$ define the function

$$F(t, u) = \int_0^t e^{i((1-u)\varphi_\gamma(s) + u\varphi_\delta(s))} ds - C(t, u),$$

where $C(t, u)$ is any complex valued function with the following properties:

1. $C(0, u) = 0$ for all $u \in [0, 1]$;
2. $C(2\pi W(\gamma), u) = \int_0^{2\pi W(\gamma)} e^{i((1-u)\varphi_\gamma(s) + u\varphi_\delta(s))} ds$;
3. $|\frac{dC}{dt}(t, u)| < 1$;
4. $\frac{dC}{dt}(0, u) = \frac{dC}{dt}(2\pi W(\gamma), u) = 0$.

Since we can assume that for every $u \in I$, the function $(1-u)\varphi_\gamma(t) + u\varphi_\delta(t)$ is not constant as a function of t (since otherwise one directly constructs a deformation of γ to δ), the Cauchy-Schwartz inequality implies

$$|\int_0^{2\pi W(\gamma)} e^{i((1-u)\varphi_\gamma(s) + u\varphi_\delta(s))} ds| < 2\pi W(\gamma).$$

From this, the existence of the function C follows easily. Condition (3) on C ensures that the map $t \to F(t,u)$ has non-vanishing derivative. It is now immediate that $F(t,u)$ defines the desired deformation of γ to δ. This proves the proposition for $W(\gamma) \neq 0$. The case of $W(\gamma) = 0$ is done similarly except that here we normalize the domain of γ and δ to be the unit interval since $W(\gamma) = 0$. The details are left to the reader. ♣

Looking at the immersion γ as a mapping of $I = [0,1]$ into the plane such that $\gamma(0) = \gamma(1)$ and $\gamma'(0) = \gamma'(1)$, we endow the image of γ with a definite orientation, namely that of increasing value of $t \in I$. At a self intersection, which we have assumed to be a normal crossing, there are two branches of γ. To be more precise, let $0 < t_1 < t_2 < 1$ be such that $\gamma(t_1) = \gamma(t_2)$ and by the hypothesis the tangent vectors $\gamma'(t_1)$ and $\gamma'(t_2)$ are linearly independent vectors in $\mathbb{R}^2 = \mathcal{T}_{\gamma(t_i)}\mathbb{R}^2$. Let P and Q be the images of neighborhoods of t_1 and t_2 respectively under γ. P and Q have natural orientations in the increasing direction of t. Fixing an orientation for $\mathbb{R}^2$, the self intersection number of γ at $\gamma(t_1) = \gamma(t_2)$ is

$$\mathrm{KI}(P, Q; \gamma(t_1)).$$

Notice that this self interesection number is $\mathbf{Z}$-valued in spite of the fact that immersed manifold is odd dimensional. This is because in dimension one we can choose order of the "branches" of the immersed manifold at self intersection points in a consistent manner which is not possible in higher dimensions. Let ν_+ and ν_- be the number of positive and negative self intersections. Our goal is to express $W(\gamma)$ in terms of $\nu_\pm$. To do so we first note the following:

Lemma 0.4.6.1 *Assume γ is an embedding so that it has no self-intersections. Then $W(\gamma) = \pm 1$ according as its orientation is counter-clockwise or clockwise.*

While one can give an elementary proof of this lemma, it is more instructive to present the simple and elegant proof of it based on theory of covering spaces. This is done in chapter 4, and for the time being we assume the validity of this lemma. An examination of special cases suggests that the statement of the lemma is indeed very plausible. One often refers to an embedding $\gamma : S^1 \to M$ as a *simple closed curve* in M.

For definiteness let us fix the standard orientation for $\mathbb{R}^2$ and assume

1. (*Normalization 1*) $\gamma(0) = \mathbf{0}$ and $\gamma'(0)$ points along the positive x_1-axis;

2. (*Normalization 2*) $\gamma(t)$ and $\gamma(1-t)$ for $t > 0$ small lie in the half plane $x_2 > 0$.

We can now state

Proposition 0.4.6.2 *Let γ be an immersion of S^1 into the plane. Then, with the above normalizations,*

$$W(\gamma) = 1 + \nu_-(\gamma) - \nu_+(\gamma) = 1 - \mathrm{KI}(\gamma).$$

The second equality in the proposition is the definition of self-intersection number. Lemma 0.4.6.1 establishes the validity of the proposition for simple closed curves in the plane. The effect of self intersections on the winding number $W(\gamma)$ is easy to understand by a decomposition argument. The decomposition of the image of an immersion near a self intersection shows that we have to analyze the effect of the winding number of figure ∞ on $W(\gamma)$. We can remove a self intersection as shown in figure (**??**), and this maybe regarded as a special case of $\sharp$ construction but the present situation is specially simple and does not require the general construction. Notice that removing a self intersection $\gamma(t_1) = \gamma(t_2)$, $t_1 < t_2$, as shown in figure (**??**) results in two curves γ_1 and γ_2 corresponding approximately to values of $t \in [0, t_1] \cup [t_2, 1]$ and $t \in [t_1, t_2]$. Near t_1 and t_2 it is necessary to modify the mapping γ to remove the self intersection $\gamma(t_1) = \gamma(t_2)$ and obtain two immersed closed curves. The curves γ_1 and γ_2 may intersect, and their intersection number is

$$\mathrm{KI}(\gamma_1, \gamma_2) = \sum_{p \text{ intersection point}} \mathrm{KI}(\gamma_1, \gamma_2; p). \tag{0.4.6.4}$$

We have

Lemma 0.4.6.2 *Let γ_1 and γ_2 be two immersions of S^1 into the plane, then* $\mathrm{KI}(\gamma_1, \gamma_2) = 0$.

Proof - Clearly intersection numbers are invariant under deformations of the curves γ_i. Let $\mathbf{x} \neq \mathbf{0}$ and consider the curves γ_1 and $\delta_\alpha(t) = \alpha\mathbf{x} + \gamma_2(t)$. Then for α sufficiently large, $\mathrm{KI}(\gamma_1, \delta_\alpha) = 0$, whence the required result. ♣

With these preliminaries out of the way we can now prove the proposition.

Proof of proposition 0.4.6.2 - The proof is by induction on the number of self intersection points. We have already established the result when γ is an embedding. In removing a self intersection $p \in \mathrm{Im}(\gamma)$ two cases occur. Either near p both curves lie on the same side of a virtual line L as shown in the figure or on opposite sides. In the former case the normalization conditions 1 and 2 are identical for both curves γ_1 and γ_2 and therefore by induction on the number of self intersections we may state

$$W(\gamma_1) = 1 + \nu_-(\gamma_1) - \nu_+(\gamma_1), \quad \text{and} \quad W(\gamma_2) = 1 + \nu_-(\gamma_2) - \nu_+(\gamma_2). \tag{0.4.6.5}$$

Joining γ_1 and γ_2 together to reconstruct γ, we regenerate the intersection point p and clearly the intersection number at p (in the former case) is -1. Thus adding $W(\gamma_1)$ and $W(\gamma_2)$ (in view of lemma 0.4.6.2) and adding the intersection number at p we obtain:

$$W(\gamma) = 1 + \nu_-(\gamma) - \nu_+(\gamma). \tag{0.4.6.6}$$

In the latter case the normalization condition is reversed for the curve γ_2 and the intersection number at p is $+1$. Taking account of the sign changes in the appropriate manner we similarly arrive at the same formula thereby completing the proof of the proposition. ♣

Remark 0.4.6.1 The proposition remains valid, with some minor modifications, for continuous but only piece-wise smooth curves. In this case the quantity $\mathsf{G}_\gamma^\star(d\theta)$ is defined in the complement of the set $\{s_1, \cdots, s_n\}$ of the points where γ is not differentiable. We assume that at the points of non-differentiability, one-sided derivatives exist but are not equal. The angle between the tangents at the point s_i will be denoted by θ_i. Then the winding number of γ is defined by

$$2\pi W(\gamma) = \sum_i \theta_i + \int_0^1 \mathsf{G}_\gamma^\star(d\theta).$$

The essential point is that we can approximate a piece-wise smooth closed curve with a smooth immersion of the circle (by smoothing the corners), and this process does not affect the winding number as defined above. This is can be made formal and rigorous by smoothing techniques introduced earlier, but such pedantry is superfluous. ♡

0.4.7 Homotopy and Isotopy

In the preceding subsection we considered the notion of deformation of curves in the plane with the additional condition that the derivative remained non-zero at all times. Here we give some useful definitions and examples of different kinds of deformations of spaces and/or maps which will be useful in the sequel. For reasons that will become clear later it is convenient to make use of pairs (X, A) where X is a Hausdorf second countable topological space, and $A \subseteq X$ a subspace. A map $f : (X, A) \to (Y, B)$ is a continuous map of X into Y mapping A into B. The pair $(X, \emptyset)$ is identified with X. Two maps $f_\circ, f_1 : (X, A) \to (Y, B)$ are *homotopic relative to the subspace* X' of X if there is a continuous map $F : X \times I \to Y$, $(I = [0, 1])$, such that

1. $F(x, 0) = f_\circ(x)$ for all $x \in X$,
2. $F(x, 1) = f_1(x)$ for all $x \in X$,
3. $F(x, t) = f_\circ(x)$ for all $x \in X'$ and all $t \in I$.

For $X' = \emptyset$ we simply say $f_\circ$ and f_1 are *homotopic*. Initially we will use the concept of homotopy in the case $X' = \emptyset$, and the importance of relative homotopy (i.e., $X' \neq \emptyset$) will become clear later. It is easy to see that the relation of being homotopic is an equivalence relation. A topological space X is *contractible* if the identity map of X is homotopic to a constant map $X \to X$, (a constant map $\underline{y} : X \to Y$ means $\underline{y}(x) = y$ for all $x \in X$). For example, $\mathbb{R}^n$ and the disc are contractible, and we can take the homotopies to be $F(x,t) = tx$ in both cases. A map $f : X \to Y$ is a *homotopy equivalence* if there is $h : Y \to X$ such that both hf and fh are homotopic to the identity map. In such a case we say that X and Y are *homotopically equivalent* or of the same *homotopy type*. For example, any constant map of a contractible space to itself is a homotopy equivalence.

Related to homotopy is the concept of retract. A subspace $A \subseteq X$ is a *retract* of X if the inclusion map $i : A \to X$ has a left inverse, i.e., if there exists a map $j : X \to A$ such that $ji = id_A$. If furthermore the map j can be chosen such that ij is homotopic to the identity map id_X, then we say that A is a *deformation retract* of X.

One can easily think of many homotopic maps and deformation retracts. For example, the sphere $S^{n-1} = \{x \in \mathbb{R}^n | \|x\| = 1\}$ is a deformation retract of $\mathbb{R}^n \setminus \mathbf{0}$ with the map $j(x) = x/\|x\|$. The following examples are a little less obvious:

Example 0.4.7.1 Let ξ be a vector field on the manifold M, and ϕ_t the corresponding one parameter group of transformations of M. The mapping $x \to \phi_1(x)$ is homotopic to the identity map and the homotopy is given by $F(x,t) = \phi_t(x)$. ♠

Example 0.4.7.2 Let $f : S^n \to S^n$ with no fixed points. We show that f is homotopic to the antipodal map. Since f has no fixed points, x and $-f(x)$ are not antipodal points. Let $t \to \phi_t(x)$ be the unique shortest great circle (segment) starting at x with $\phi_1(x) = -f(x)$. Then $\phi_t(x)$ gives the desired homotopy. ♠

Exercise 0.4.7.1 *Show that if f and g are maps of of S^m to itself such that $f(x)$ and $g(x)$ are distinct for all x, then f and $-g$ are homotopic.*

Example 0.4.7.3 Let f be a real-valued function on the manifold M and assume that f has no critical points on $[a,b]$. Let $M_a = f^{-1}((-\infty, a])$, and assume that $f^{-1}([a,b])$ is compact. Then M_a is a deformation retract of M_b. To prove this let ψ be a non-negative C^∞ function identically 1 on $f^{-1}([a,b])$ and vanishing outside a compact neighborhood of $f^{-1}([a,b])$. Fix a Riemannian metric g on M and let ξ be the vector field

$$\xi = \frac{-\psi \cdot \mathrm{grad} f}{g(\mathrm{grad} f, \mathrm{grad} f)}.$$

Since ξ vanishes outside a compact set, the one parameter ϕ_t corresponding to ξ is defined for all time t. Now

$$f(\phi_t(x)) - f(x) = \int_0^t \frac{d}{ds} f(\phi_s(x)) ds = \int_0^t g(\xi, \mathrm{grad} f) ds.$$

Hence $f(\phi_t(x)) \leq f(x)$ for all $t > 0$, and if $x \in M_b$ and $t \geq f(x) - a$ then $f(\phi_t(x)) \in M_a$. Define

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Then $j_1 i = id_A$, and ij_1 is homotopic to $\mathrm{id}._X$, and the homotopy is given by $F(x,t) = j_t(x)$. Notice that we have also proved that M_a and M_b are diffeomorphic. The assumption of compactness of $f^{-1}([a,b])$ cannot be removed. In fact, let M be the cylinder $I \times S^1$, $N = M\backslash\{p\}$ where $p = (\frac{1}{2}, e^{i\theta_\circ})$ is an interior point, and $f(t, e^{i\theta}) = t$. Then f has no critical point but $f^{-1}([0,\alpha])$ and $f^{-1}([0,\beta])$ are not homotopic for $\alpha < \frac{1}{2} < \beta$. This is perhaps plausible, yet a rigorous proof is given in chapter 3 exercise **??** using homology. ♠

A notion more restrictive than homotopy is isotopy. Two injective continuous maps $f_\circ, f_1 : X \to Y$ are called *isotopic* if there is a homotopy $F : X \times I \to Y$ such that for every $t \in I$ the map $F(.,t) : X \to Y$ is also injective. The homotopy $F(x,t)$ of example 0.4.7.1 is in fact an isotopy. If $X = M$ and $Y = N$ are manifolds, we assume that that the maps f_j are smooth (unless stated to the contrary) and we introduce the notion of *smoothly isotopic* by the additional requirement that for all $t \in I$, $F(.,t)$ is a diffeomorphism. Similarly, two immersions $f_\circ, f_1 : M \to N$ are called *isotopic immersions* if there is a smooth homotopy $F : M \times I \to N$ such that for every $t \in I$ the map $F(.,t) : M \to N$ is also an immersion. This is the generalization of the notion of deformation of immersions of the circle investigated in the preceding subsection. It is straightforward to show that homotopy, isotopy, smooth isotopy and isotopic immersion are equivalence relations.

Exercise 0.4.7.2 *Exhibit a smooth isotopy between antiopodal map $\jmath : S^m \to S^m$ and the identity map for m an odd integer. Show that for m even, $\jmath$ is not smoothly isotopic to to the identity map.*(Look at $\det(\jmath)$. It is shown is chapter 3 that $\jmath$ is not even homotopic to the identity map.)

Example 0.4.7.4 Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation with $\det(A) > 0$. Since $GL(m, \mathbb{R})$ has two connected components corresponding to the signs of determinant, there is a smooth curve $\gamma : I \to GL(m, \mathbb{R})$ with $\gamma(0) = I$ and $\gamma(1) = A$. Thus γ gives a smooth isotopy between the diffeomorphism A and I by $F(x,t) = \gamma(t)(x)$. Similarly every two linear transformations of $\mathbb{R}^m$ with negative determinant are smoothly isotopic. Since $GL(m, \mathbb{C})$ is connected, any nonsingular linear transformation of $\mathbb{C}^m$ is smoothly isotopic to the identity map. This suggests that if we look at more complex groups such as the group $\mathrm{Diff}(M)$ of diffeomorphisms of a compact manifold M (with the appropriate topology), then smooth isotopy classes of diffeomorphisms are the same as path components of this group. ♠

A (smooth) embedding $\gamma : S^1 \to \mathbb{R}^3$ (or S^3) is called a *knot*. We often use the word knot to denote both the embedding γ and its image. One way of distinguishing between trivial and non-trivial knots is to define a knot γ to be trivial (or the *unknot*) if γ is smoothly isotopic to a linear embedding of S^1 into $\mathbb{R}^3$, i.e., γ is smoothly isotopic to the restriction of an injective linear map of $\mathbb{R}^2$ into $\mathbb{R}^3$.

Example 0.4.7.5 Let $m \leq n$, D^m denote the unit disc in $\mathbb{R}^m$ centered at the origin $\mathbf{0}$ and $f : D^m \rightarrow \mathbb{R}^n$ be a smooth embedding. We show that f is smoothly isotopic to a linear embedding. Composing f with a translation we may assume $f(\mathbf{0}) = \mathbf{0}$. Define $F : D^m \times I \rightarrow \mathbb{R}^n$ as

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Then F defines the necessary smooth isotopy. The above construction also shows that any diffeomorphism of $\mathbb{R}^m$ is smoothly isotopic to an affine isomorphism.

In particular, if a knot $\gamma : S^1 \rightarrow \mathbb{R}^3$ extends to an embedding of the disc D^2 into $\mathbb{R}^3$ then the knot is trivial. Phrased differently, if a knot (regarded as the image of γ now) bounds a disc, then it is the unknot. This observation will be useful in the study of knots (see the proof of Fary-Milnor theorem in chapter 2, §1.4). For any knot there is always a surface whose boundary is the given knot, but the surface is generally not a disc except for the unknot. The relevance of the bounding surface to the study of knots is discussed in the next volume, and special cases are considered in chapter 4. ♠

Exercise 0.4.7.3 *Show that every diffeomorphism of the circle $S^1 = \partial D^2$ extends to a diffeomorphism of the disc D^2. Deduce that every diffeomorphism of the circle is smoothly isotopic to either the identity map or complex conjugation.*

We noted in example 0.4.7.1 that the one parameter group ϕ_t of a vector field gives a homotopy between $\phi_\circ$ and ϕ_1 assuming that the latter quantity is defined. It is clear that ϕ_t is in fact a smooth isotopy. In particular, let $\xi = -\text{grad}f$ where f is a Morse function on a Riemanian manifold M, $f^{-1}([a, b])$ is compact and f has no critical values in $[a, b]$, then a slight modification of the construction in example 0.4.7.3 gives a smooth isotopy between the manifolds $f^{-1}(b)$ and $f^{-1}(a)$. More generally a time dependent vector field $\xi_{x,t}$ yields a smooth isotopy provided the solutions exist for t in an interval $[0, a]$, $a > 0$. In fact, consider the time independent vecctor field $(\xi_{x,at}, \frac{\partial}{\partial t})$ on the manifold $M \times \mathbb{R}$, then the corresponding one parameter group ϕ_s defines the required smooth isotopy by $F(x, s) = \phi_s(x, 0)$. If M is compact, then it is easily verified, as in the time independent case, that the solutions $\phi_{x,t}$ exist for all $t \in \mathbb{R}$. A sufficient condition for the existence of ϕ_s when M is not necessarily compact is given in chapter 2. Time dependent vector fields and isotopies are useful tools for the construction of differentiable manifolds. For a continuous mapping $\psi : M \rightarrow M$ we define the *support* of ψ as the closed set

$$\text{supp}\psi = \overline{\{x \in M \mid \psi(x) \neq x\}}.$$

The following simple lemma or more precisely its corollaries will be used in the next subsection:

Lemma 0.4.7.1 *Let M be a manifold (without boundary) and $X \subset M$ a compact subset. Let U be an open neighborhood of X and $F : U \times I \rightarrow M$ a smooth isotopy such that*

1. *$F(.,0)$ is the inclusion of U in M.*
2. *$F(U \times I)$ is open in $M \times I$.*
3. *$\overline{\cup_{t\in I}\mathrm{supp}F(.,t)}$ is a compact subset of $F(U \times I)$.*

Then there is a smooth isotopy $F' : M \times I \rightarrow M$ and a compact subset $K \subset M$ such that

1. *F' is an extension of the restriction of F to a neighborhood of $X \times I$.*
2. *For every $t \in I$, $\mathrm{supp}F'(.,t) \subset K$.*

Proof - Let $\tilde{F}(x,t) = (F(x,t),t)$. The tangent vectors to curves $t \rightarrow \tilde{F}(x,t)$, $x \in U$, define a time dependent vector field ξ' in U. Let $\phi = \phi(x)$ be a C^∞ function identically 1 in a neighborhood of $\overline{\cup_{t\in I}\mathrm{supp}F(.,t)}$ and vanishing outside of a compact neighborhood of $\overline{\cup_{t\in I}\mathrm{supp}F(.,t)}$ in $F(U \times I)$. Then the vector field

$$\xi = \phi\xi' + \frac{\partial}{\partial t}$$

in $M \times I$ generates the desired smooth isotopy. ♣

Remark 0.4.7.1 below shows that some technical assumption, such as the hypotheses of lemma 0.4.7.1, is necessary for the validity of the conclusion of the lemma.

Corollary 0.4.7.1 *Let $f, h : D^k \rightarrow M$ be embeddings of the disc where $k \leq m-1$. Then f and h are smoothly isotopic.*

Corollary 0.4.7.2 *Let M be an oriented manifold, and $f, h : D^m \rightarrow M$ be embeddings of the disc. Assume both f and h are either orientation preserving or orientation reversing. Then f and h are smoothly isotopic.*

Proof of corollaries 0.4.7.1 and 0.4.7.2 - Let $k = 0$, then there is a smooth embedding $\gamma : [0,1] \rightarrow M$ with $\gamma(0) = f(0)$ and $\gamma(1) = h(0)$. Denote the image of γ by X, and let U be a small (tubular) neighborhood of X. Clearly there is a smooth isotopy $F : U \times I \rightarrow M$ such that between f and h and lemma 0.4.7.1 gives the desired isotopy. From the case $k = 0$ we deduce that for all $k \leq m$ we can assume $f(0) = h(0)$. Replacing f and h by smoothly isotopic maps $x \to f(\alpha x)$ and $x \to h(\alpha x)$, for some $0 < \alpha < 1$ if necessary, we can assume that images of f and h are contained in one coordinate neighborhood. Now example 0.4.7.5 is applicable to show that f and h are smoothly isotopic linear maps. The conclusions of both corollaries follows immediately. ♣

A slight modification of the proof of lemma 0.4.7.1 establishes the following analogue of it for manifolds with boundary:

Lemma 0.4.7.2 *Let M be a manifold with boundary $N \subset M$ be a compact submanifold and $F : N \times I \to M$ be a smooth isotopy with $F(x, 0) = x$ and such that*

1. *Either $F(N \times I) \subset \partial M$;*
2. *Or $\overline{\cup_{t\in I}\text{supp}F(., t)} \cap \partial M = \emptyset$.*

Then F extends to a smooth isotopy $F' : M \times I \to M$.

The relevance of groups of diffeomorphisms of manifolds to problems of geometry, topology and physics has already been established in a number of problems. We will make occasional references to these groups. Let us briefly consider the group $\text{Diff}(S^m)$ of diffeomorphisms of S^m and its subgroup $\text{Diff}_+(S^m)$ consisting of orientation preserving diffeomorphisms. Let $\text{Diff}_+(B^{m+1})$ be the group of orientation preserving diffeomorphisms of the closed unit ball $B^{m+1} \subset \mathbb{R}^{m+1}$. By restriction we obtain a homomorphism

$$\rho : \text{Diff}_+(B^{m+1}) \longrightarrow \text{Diff}_+(S^m).$$

Denote the image of ρ by G.

Lemma 0.4.7.3 *With the above notation, G is a normal subgroup of $\text{Diff}(S^m)$ and consequently $\Gamma_{m+1} = \text{Diff}_+(S^m)/G$ is a group.*

Proof - Let $g \in G$ and $\psi \in \text{Diff}(S^m)$. By corollary 0.4.7.2 there is a smooth isotopy $F : S^m \times I \to S^m$ such that $F(., 0) = \text{id.}$ and $F(., 1) = g$. Consequently $\psi F \psi^{-1}$ gives a smooth isotopy between the identity map and $\psi g \psi^{-1}$. By lemma 0.4.7.2 $\psi g \psi^{-1}$ extends to a diffeomorphism of B^{m+1} which proves the assertion. ♣

Proposition 0.4.7.1 Γ_{m+1} *is an abelian group.*

Proof - Let $\psi_g \in \text{Diff}_+(S^m)$ represent an element $g \in \Gamma_{m+1}$, and $p \in S^m$ be the north pole. Since $SO(m) \subset \text{Diff}_+(B^{m+1})$ we may assume $\psi_g(p) = p$. Let $\bar{U} \neq S^m$ be the closure of any open subset, then $\psi_g(\bar{U})$ misses an open set in S^m and after composing ψ_g with an element of $\mathrm{Diff}_+(D^{m+1})$ we may assume $\psi_g(\bar{U})$ misses a neighborhood of the south pole q. Let U be a (small) neighborhood of the closed northern hemisphere N_+. Identifying $S^m\backslash\{q\}$ with $\mathbb{R}^m$ It follows from corollary 0.4.7.2 that the restriction of ψ_g to $\bar{U}$ is smoothly isotopic to the identity map. Let $F(.,.)$ be such a smooth isotopy. By lemma 0.4.7.1 $F(.,.)$ extends to a smooth isotopy of S^m which agrees with F on N_+. We denote this extension again by F. $F(.,.)$ extends to a smooth isotopy of B^{m+1} by lemma 0.4.7.2 with $F(.,1) = \psi_g$ on N_+. Consequently $F(.,1)^{-1}\psi_g \equiv \psi_g$ in Γ_{m+1} and we may assume ψ_g is the identity on the northern hemisphere N_+. Similarly if ψ_h represents an element of $h \in \Gamma$, then we can assume ψ_h is the identity on the southern hemisphere N_-. For these representatives of g and h it is trivial that $\psi_g\psi_h = \psi_h\psi_g$ proving the proposition. ♣

It is a consequence of exercise 0.4.7.3 that $\Gamma_2 = 0$. It is known that Γ_m is finite, and the smallest m for which $\Gamma_m \neq 0$ is $m = 7$. Vanishing of Γ_3 is shown in example 0.4.7.6 below. The fact that $\Gamma_4 = 0$ is considerably more difficult (see e.g. [Ce2]). In [Ce1] one finds a number of foundational results.

Remark 0.4.7.1 An examination of the proof of proposition 0.4.7.1 shows that any diffeomorphism ψ of S^m is smoothly isotopic to one supported in a hemisphere and therefore can be regarded as a diffeomorphism with compact compact support of $\mathbb{R}^m$. Assume the diffeomorphism is orientation preserving. Such a diffeomorphism is isotopic to an affine isomorphism and therefore to the identity map, however, it may not be possible to find such a compactly supported isotopy. (The isotopy in example 0.4.7.5 is not compactly supported.) In fact, the existence of such a compactly supported isotopy implies ψ, as a diffeomorphism of S^m, is isotopic to the identity map. Therefore it extends to a diffeomorphism of B^{m+1} and ψ vanishes in Γ_{m+1}. Using $\Gamma_7 \neq 0$, we can establish the necessity of some technical hypotheses for the validity of the conclusion of lemma 0.4.7.1. To understand this point let $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a diffeomorphism with $\mathrm{supp}\psi \subset B^m_{\frac{1}{2}}$ where $B^m_{\frac{1}{2}}$ denotes the open ball of radius $\frac{1}{2}$. We furthermore assume that $\psi(0) = 0$ and $D\psi(0)$ is the identity map. Now consider the diffeomorphism $\Phi : \mathbb{R}^m \simeq B^m_1$ defined by

$$\Phi(r, \theta) = (\phi(r), \theta),$$

where (r, θ) denotes spherical polar coordinates, and ϕ is C^∞ decreasing function of $r > 0$ with

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Now let $X = \overline{B^m_{\frac{1}{2}}}$, $U = B^m_1$, $\Psi = \Phi^{-1}\psi\Phi$ and F' be the isotopy between ψ and the identity map constructed in example 0.4.7.5. Then $F = \Phi^{-1}F'\Phi$ gives gives an isotopy between $\Phi^{-1}\psi\Phi$ and the identity map. Clearly $F(U \times I) = U$ is open $\mathbb{R}^m \times I$. However

$$\overline{\cup_{t\in I}\mathrm{supp}F(.,t)} = \overline{U}$$

is not a compact subset of $F(U \times I) = U$. In view of the arbitrariness of ψ, F does not admit of any extension of the form specified in the conclusion of lemma 0.4.7.1, for otherwise every compactly supported orientation preserving diffeomorphism would be smoothly isotopic through a compactly supported isotopy, to the identity map. ♡

Example 0.4.7.6 We show that $\Gamma_3 = 0$. Let $\psi_g \in \mathrm{Diff}_+(S^2)$ be a representative for $g \in \Gamma_3$. The proof of proposition 0.4.7.1 shows that we can assume ψ_g is the identity on the northern hemisphere N_+. Therefore we can regard ψ_g as a diffeomorphism of $\mathbb{R}^2$ which is the identity outside a compact subset K which we may take to be a rectangle with vertices A, B, C, D. Consider lines parallel to the side AB. Then the diffeomorphism ψ_g distorts the portions of these line segments in the interior of K while keeping their end points fixed. It is a standard result in the theory of first order ordinary differential equations in the plane that through a smooth isotopy fixing the boundary one can straighten out all the lines. (This is strictly a two dimensional theorem.) The composition of this isotopy with a reparametrization of the the straightened out curves, which can be implemented by a smooth isotopy, one obtains a smooth isotopy between ψ_g and the identity which is identity outside of K. It follows from lemma 0.4.7.2 that F extends to a smooth isotopy of the disc B^3 and therefore so does ψ_g which proves $\Gamma_3 = 0$. ♠

0.5 Representations of Groups

0.5.1 Representations of Groups and Lie Algebras

Let $V = K^N$ where $K = \mathbb{R}$, or $\mathbb{C}$ and $\rho : G \to GL(N, K)$ be a representation of G. We say ρ (or V) is *irreducible* (or V is an *irreducible G-module*) if V has no nontrivial proper $\rho(G)$-invariant subspaces. ρ is completely reducible if V admits of a decomposition $V = V_1 \oplus \cdots \oplus V_l$ where each subspace V_j is irreducible. In matrix notation, complete reducibility means that the matrices $\rho(g)$ can be simultaneously block diagonalized with each diagonal block corresponding to one irreducible representation. For example, for $l = 2$ this means that for a proper choice of basis, all matrices $\rho(g)$ are of the form

$$\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix},$$

and ρ_i's are representations of G. It is also customary to write $\rho = \rho_1 \oplus \cdots \oplus \rho_l$ when a representation ρ can be so block diagonalized even if ρ_i's are not irreducible. By a *complementary subspace* $W \subset V$ to an invariant subspace $V' \subset V$ we mean an invariant subspace W such that $V = V' \oplus W$. It is a simple matter to see that ρ (or V) is completely reducible if and only if every invariant subspace admits of a complementary subspace. Two representations ρ and ρ' of a group G are *equivalent* if there is an invertible linear transformation T such that for all $g \in G$ we have $\rho(g)T = T\rho'(g)$. For a Lie subgroup $G \subset GL(m, K)$ we have the *natural representation* ρ_1 of G given $\rho_1(g) = g$. The representation mapping every $g \in G$ to the identity matrix is called the *trivial representation* of G. The underlying field K is specified by referring to the representation as real or complex. The field K is generally clear from the context.

Exercise 0.5.1.1 *Let $G = \mathbf{Z}$ and T and S be fixed $m \times m$ complex matrices. Show that the representations $\rho, \rho' : G \to GL(m, \mathbb{C})$ given by $\rho(1) = T$ and $\rho'(1) = S$ are equivalent if and only if the matrices T and S have identical Jordan decompositions. Two representation $\rho, \rho' : \mathbf{Z} \to GL(m, K)$ are equivalent if and only if the matrices $\rho(1)$ and $\rho'(1)$ have identical rational forms over the field K.*

Example 0.5.1.1 Let $G = S^1 = \{e^{2\pi i\theta}\}$. Then for every $n \in \mathbf{Z}$

$$\rho_n(e^{2\pi i\theta}) = e^{2\pi i n\theta}$$

is an irreducible representation of G. The representations ρ_n and ρ_m are inequivalent for $n \neq m$. The mappings

$$(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_k}) \longrightarrow e^{2\pi i \sum_j n_j\theta_j},$$

for $n_1, \cdots, n_k \in \mathbf{Z}$ are irreducible representations of $S^1 \times \cdots \times S^1$. Two such representation are equivalent if and only if the corresponding integers n_j are identical. ♠

Example 0.5.1.2 Consider the representation β of $\mathcal{S}_3$ given by

$$\beta((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \beta((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \text{etc.},$$

By computing the eigenspaces for the matrices $\beta((12))$, $\beta((123))$ etc., one verifies that β is irreducible. The geometric meaning of β can be easily described. Let v_1, v_2 be a (positively oriented) basis for $\mathbb{R}^2$ such that the angle between v_1 and v_2 is $\frac{2\pi}{3}$. Then $\beta((12))$ and $\beta((23))$ are reflections with respect to the orthogonal complements of v_1 and v_2 respectively. In view of exercise 0.2.3.5, we have a semi-direct product decomposition $\mathcal{S}_4 \simeq N.\mathcal{S}_3$ with N a normal subgroup. Therefore β extends to a two dimensional irreducible representation of $\mathcal{S}_4$ by defining $\beta(\sigma) = I$ for $\sigma \in N$. The mapping which assigns to every permutation $\sigma \in \mathcal{S}_n$ its sign ϵ_σ is also a representation of $\mathcal{S}_n$. ♠

Exercise 0.5.1.2 *Let G be the group of order 8 of symmetries of the square (i.e., generated by reflections relative to the coordinate axes and rotation by $\frac{\pi}{2}$.) Construct an irreducible representation of degree 2 of G.*

Example 0.5.1.3 The notion of irreducibility depends on whether $K = \mathbb{R}$ or $\mathbb{C}$. For example, consider the group $SO(2) = \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. It is clear that $\mathbb{R}^2$ is an irreducible $SO(2)$-module. On the other hand, since the matrices $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ can be simultaneously diagonalized into $\begin{pmatrix} e^{i\theta} & 0 \\ o & e^{-i\theta} \end{pmatrix}$, the complexification $\mathbb{C}^2$ of $\mathbb{R}^2$ is not irreducible under $SO(2)$. ♠

Example 0.5.1.4 Let $G \subset GL(m, \mathbb{C})$ acting on $V = \mathbb{C}^m$ as a group linear transformations via the natural representation of $GL(m, \mathbb{C})$. Writing $z = (z_1, \cdots, z_m) = ((x_1, y_1), \cdots, (x_m, y_m)) \in \mathbb{R}^{2m}$ we obtain a representation $\rho : G \to GL(2m, \mathbb{R})$ where every entry g_{jk} of $g \in G$ is replaced by the 2×2 matrix $\begin{pmatrix} \Re g_{jk} & -\Im g_{jk} \\ \Im g_{jk} & \Re g_{jk} \end{pmatrix}$, and $\Re z$ and $\Im z$ denote the real and imaginary parts of the complex number z. Now assume V is an irreducible G-module. It is not difficult to see that $\mathbb{R}^{2m}$ is an irreducible $GL(m, \mathbb{C})$-module, i.e, ρ is irreducible. On the other hand, $\mathbb{C}^{2m}$ decomposes into the direct sum of irreducible subspaces, namely,

$$V_1 = \{(z_1, -iz_1, \cdots, z_m, -iz_m)\}, \quad V_2 = \{(z_1, iz_1, \cdots, z_m, iz_m)\}.$$

Note that $V_i \simeq V$ and $V \otimes \mathbb{C} \simeq V_1 \oplus V_2$. This can be restated by saying that if the complex vector space is irreducible as a G-module, then regarding V as a real linear space $\tilde{V}$, its complexification $\tilde{V} \otimes \mathbb{C}$ decomposes into equivalent irreducible G-modules V_1 and V_2. ♠

Exercise 0.5.1.3 *Let T be the group of $m \times m$ upper triangular matrices and ρ be its natural representation. Let $e_1, \cdots, e_m$ be the standard basis for K^m and V_j be the subspace spanned by $e_1, \cdots, e_j$. Show that V_j is a invariant under T, however, it does not admit of a complementary subspace. Conclude that ρ is not completely reducible.*

Exercise 0.5.1.4 *Show that the group T of exercise 0.5.1.3 is solvable by exhibiting a sequence of closed normal subgroups*

$$\{e\} \subset T_1 \subset T_2 \subset \cdots \subset T_m = T$$

such that every successive quotient T_i/T_{i-1} is abelian. Show that there is more than one way of exhibiting such a sequence. Find a sequence such that each T_i has the semi-direct product decomposition $T_i \simeq T_{i-1}.(T_i/T_{i-1})$ and describe the action of (T_i/T_{i-1}) on T_{i-1}.

Exercises 0.5.1.3 0.5.1.4 show that representations of solvable groups are not necessarily completely reducible. For compact groups the situation is completely different. An important general tool for the study of compact groups is the following lemma, sometimes called *Weyl's unitary trick*:

If $W \subset V$ is invariant under $\rho(g)$ for all $g \in G$, then from lemma 0.3.6.1 we have

$$< \rho(g)^{-1}(w), v >=< w, \rho(g)(v) >, \tag{0.5.1.1}$$

which implies that the orthogonal complement $W^\perp$ of W is also invariant under $\rho(g)$. Hence, in sharp contrast to the case of solvable analytic groups, we have

Proposition 0.5.1.1 *Every representation of a compact Lie group is completely reducible.*

Proposition 0.5.1.2 (Schur's Lemma) *Let $\rho : G \to GL(m, \mathbb{C})$ be a completely reducible representation of a group G. Then G is irreducible if and only if the only matrices commuting with all $\rho(g)$'s are multiples of identity.*

Example 0.5.1.5 By example 0.5.1.3, $\mathbb{C}$ cannot be replaced with $\mathbb{R}$ in the statement of Schur's lemma. An immediate consequence of Schur's Lemma is that an irreducible representation of an abelian group over a complex vector space is necessarily one dimensional. The hypothesis of complete reducibility cannot be removed since the only matrices commuting with the group $G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a \in \mathbb{C}^\times, b \in \mathbb{C}\}$ are multiples of identity. ♠

Remark 0.5.1.1 The proof of Schur's lemma follows easily from basic structure theorems for algebras. Let $\mathcal{A}_\rho(G)$ be the algebra over a field $K \supset \mathbf{Q}$ generated by the matrices $\rho(g)$, $g \in G$. The key point is the observation that if ρ is irreducible, then the set $C(\rho)$ of matrices commuting with $\mathcal{A}_\rho(G)$ is a division algebra. In fact, the kernel of such a matrix is an invariant subspace and therefore the matrix is either $\mathbf{0}$ or invertible. Since the only division algebra over $\mathbb{C}$ is $\mathbb{C}$ itself, this proves part of proposition 0.5.1.2. It follows from general theorems on structure of algebras and complete reducibility that the algebra $\mathcal{A}_\rho(G)$ is a direct sum of full matrix algebras over division algebras over K. These division algebras are isomorphic to $C(\rho)$'s for each irreducible summand. Schur's lemma now follows easily. Note that we have also obtained some understanding of the structure of $C(\rho)$ when fields are other than complex numbers. Example 0.5.1.3 shows that $C(\rho)$ for $SO(2)$ acting on $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$. Let $G = SU(2)$ be the group of unit quaternions as introduced in example 0.3.5.2. G acts on $\mathbb{R}^4$, identified with $\mathbf{H}$, by left multiplication as quaternions. $C(\rho)$ is isomorphic to $\mathbf{H}$ with $\mathbf{H}$ acting on itself by right multiplication. These examples illustrate how the situation will change if complex numbers are replaced by real numbers. We will concentrate on some consequences of Schur's Lemma. ♡

Example 0.5.1.6 Let ρ and ρ' be irreducible representations of groups G and G' on vector spaces V and V' of dimensions d and d'. Then $\rho \otimes \rho'$ is a representation of $G \times G'$ on the vector space $V \otimes V'$ given by the tensor product of the linear transformations $\rho(g)$ and $\rho'(g')$. Fix bases $\{e_i\}$ and $\{e'_p\}$ for V and V'. Then $\mathcal{B} = \{e_i \otimes e'_p\}$ is a basis for $V \otimes V'$, and we order them by the condition that $e_i \otimes e'_p$ precedes $e_j \otimes e'_q$ if $i < j$, and if $i = j$ then $p < q$. The matrices representing $\rho(g) \otimes \rho'(g')$, relative to $\mathcal{B}$, are obtained by multiplying the $dd' \times dd'$ commuting matrices

$$\begin{pmatrix} \rho(g) & 0 & 0 & \cdots & 0 \\ 0 & \rho(g) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \rho(g) \end{pmatrix}, \quad \begin{pmatrix} [\rho'_{11}(g'))] & [\rho'_{12}(g')] & \cdots & [\rho'_{1d'}(g')] \\ [\rho'_{21}(g'))] & [\rho'_{22}(g')] & \cdots & [\rho'_{2d'}(g')] \\ \vdots & \vdots & \ddots & \vdots \\ [\rho'_{d'1}(g'))] & [\rho'_{d'2}(g')] & \cdots & [\rho'_{d'd'}(g')] \end{pmatrix}, \tag{0.5.1.2}$$

where $[\rho'_{jk}(g'))]$ is the $d \times d$ matrix which is $\rho'_{jk}(g')$ times the identity. If V and V' are complex vector spaces and ρ and ρ' are irreducible, then by remark (0.5.1.1), $\mathcal{A}_\rho$ and $\mathcal{A}_{\rho'}$ are full matrix algebras $M_d(\mathbb{C})$ and $M_{d'}(\mathbb{C})$ and so $\mathcal{A}_\rho \otimes \mathcal{A}_{\rho'}$ is the full matrix algebra $M_{dd'}(\mathbb{C})$. Consequently, $\rho \otimes \rho'$ is an irreducible representation of $G \times G'$. Note that if $G = G'$, then $g \to \rho(g) \otimes \rho'(g)$ is a representation of G. Generally, this representation is *not* irreducible even if ρ and ρ' are. Normally $\rho \otimes \rho'$ refers to this representation of G. The cases where $\rho \otimes \rho'$ refers to a representation of $G \times G'$ will be clear from the context. ♠

Let ρ be a representation of a group G on a vector space $V \simeq K^N$. By a *G-invariant* of V we mean a vector $v \in V$ such that $\rho(g)(v) = v$ for all $g \in G$. The set of G-invariants is denoted by V^G. From the representation ρ we can construct its *contragredient representation* defined by $\rho^\star(g) = (\rho(g^{-1}))'$ where $'$ denotes transpose[12]. Naturally, $\rho^\star$ is a representation on the dual vector space $V^\star$. Let τ be a representation of G on a vector space $V' \simeq K^{N'}$, and $\mathrm{Hom}(V, V')$ denote the set of linear maps of V into V'. Then we have a representation of G on $\mathrm{Hom}(V, V')$ defined by $T \to \tau(g)T\rho^\star(g)$. Now $\mathrm{Hom}(V, V') \simeq V^\star \otimes V'$ and the representation on $\mathrm{Hom}(V, V')$ is equivalent to the representation $\rho^\star \otimes \tau$. Let U and V be G-modules, and $U^\star$ denote the dual of U with G acting on it via the contragredient representation. Then we have the useful and simple identity

$$(U^\star \otimes V)^G \simeq \mathrm{Hom}_G(U, V), \tag{0.5.1.3}$$

where superscript ^{G} denotes G-fixed elements. An immediate consequence of Schur's Lemma and (0.5.1.3) is the useful statement

Corollary 0.5.1.1 *Let ρ be an irreducible representation of G on the complex vector space V. Then*

$$(V^\star \otimes V)^G = \{\lambda I | \lambda \in \mathbb{C}\}.$$

Now assume ρ is an irreducible representation and τ is a completely reducible representation of G. Then it is customary to write $\tau = \tau_1 \oplus \cdots \oplus \tau_k$ and $V' = V'_1 \oplus \cdots \oplus V'_k$ with the representation τ_j irreducible and acting on the vector space V'_j. Some of the representations τ_j may be equivalent to ρ. If l of them are equivalent to ρ we say the *multiplicity* of ρ in τ is l (or ρ *occurs* l *times in* ρ'), and write $n(\rho, \tau) = l$. Completely analogous to corollary 0.5.1.1 is

Corollary 0.5.1.2 *Let ρ be an irreducible and τ a completely reducible representation of the group G on the complex vector spaces V and V'. Then*

$$\dim(V^\star \otimes V')^G = n(\rho, \tau), \qquad \dim(V'^\star \otimes V')^G = \sum n(\eta, \tau)^2,$$

where the summation is over all irreducible representations η of G occuring in τ.

As an application of corollary 0.5.1.2 we determine a distinguished Riemannian metric on certain homogeneous spaces. Let G be an analytic group and K a closed subgroup. Assume $\mathcal{G}$ admits of a decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M}, \tag{0.5.1.4}$$

with the following properties

[12] It should be noted that the reason for taking transpose is that we assuming that matrices act on the left on (column) vectors. If the vectors in the dual space are written as row vectors so that matrices are written on the right, then transpose sign would be unnecessary.

1. $\mathcal{M}$ is invariant under the adjoint action of K;

2. There are inner products $< .,. >_{\mathcal{K}}$ and $< .,. >_{\mathcal{M}}$ on $\mathcal{K}$ and $\mathcal{M}$ relative to which adjoint action of K is by orthogonal transformations.

Let $\mathrm{Ad}^\star_{\mathcal{G}} = \mathrm{Ad}^\star$ denote the representation contragredient to the adjoint representation of G. $\mathrm{Ad}^\star$ is called the *co-adjoint representation.* For $k \in K$, $\mathrm{Ad}^\star(k)$ leaves the subspaces $\mathcal{K}^\star$ and $\mathcal{M}^\star$ invariant. Let $\omega_1, \cdots, \omega_m$ and $\omega_{m+1}, \cdots, \omega_n$ be orthonormal bases for on $\mathcal{M}^\star$ and $\mathcal{K}^\star$ respectively. Then the left invariant symmetric 2-tensor $ds^2 = \omega_1^2 + \cdots + \omega_m^2$ is defined on G. It follows that ds^2 is defined on $M = G/K$ if and only if ds^2 is invariant under the co-adjoint representation of K on $S^2\mathcal{M}^\star$ (second symmetric power of $\mathcal{M}^\star$). If the representation of K on $\mathcal{M}^\star$ is absolutely irreducible, then by corollary 0.5.1.2 ds^2 is the unique vector, up to scalar multiplication, in $S^2\mathcal{M}^\star$ invariant under K. Clearly ds^2 is positive definite as symmetric bilinear form on $\mathcal{M}$. Summarizing

Proposition 0.5.1.3 *With the above notation and hypothesis, the homogeneous space $M = G/K$ admits of a G-invariant Riemannian metric which is unique, up to multiplication by a positive scalar, if the co-adjoint representation of K on $\mathcal{M}^\star$ is absolutely irreducible.*

Example 0.5.1.7 We present some examples where the hypotheses of proposition 0.5.1.3 are fulfilled. We let $e_1, \cdots, e_N$ be the standard basis for $\mathbb{R}^N$. Let $k \leq n$, $k \neq 2$, $G = SO(k+n)$ and $K = SO(k) \times SO(n)\imath G$ be the subgroup leaving the subspace spanned by $e_1, \cdots, e_k$ invariant. Then G/K is the Grassmann manifold $\mathbf{G}^\circ_{k,n}(\mathbb{R})$ of oriented k-planes in $\mathbb{R}^{k+n}$. We have the decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{M}$ is the set of skew symmetric matrices of the form

$$\begin{pmatrix} 0 & A \\ -A' & 0 \end{pmatrix},$$

and A is an arbitrary $k \times n$ matrix. The representation of K on $\mathcal{M}$ is the tensor product of the natural representations of $SO(k)$ and $SO(n)$ on $\mathbb{R}^k$ and $\mathbb{R}^n$ and is therefore absolutely irreducible[13]. Therefore $\mathbf{G}^\circ_{k,n}(\mathbb{R})$ has a unique up to scalar multiplication G-invariant metric. For $k = 1$, $\mathbf{G}^\circ_{k,n}(\mathbb{R}) = S^n$ and the standard metric on S^n which is invariant under $SO(n+1)$ is this distinguished metric. Similarly the homogeneous spaces $\mathbf{G}_{k,n}(\mathbb{R})$, $\mathbf{G}_{k,n}$ carry a unique, up to scalar multiplication, $O(k+n)$ or $U(k+n)$ invariant metric. For flag manifolds the situation

[13] In case $k = 2$ (or $k = 1 < n = 2$) the representation is irreducible but not absolutely irreducible. It follows easily from example 0.3.5.2 that the space of $SO(4)$ invariant metrics on $SO(4)/SO(2) \times SO(2)$ is one dimensional. For the remaining cases where $k = 2$ or $n = 2$ the required uniqueness of the G-invariant Riemannian metric can be checked directly.

is a little different. For instance, let $0 < k < l < n$ and consider the flag manifold $\mathbf{F}_{k,l}$ of pairs of subspaces $V_1 \subset V_2$ of dimensions k and l in $\mathbb{C}^n$. Here $K = U(k) \times U(l-k) \times U(n-l)$ and $\mathcal{M}$ is the set of skew hermitian matrices of the form

$$\begin{pmatrix} 0 & A_{12} & A_{13} \\ -\bar{A}'_{12} & 0 & A_{23} \\ -\bar{A}'_{13} & -\bar{A}'_{23} & 0 \end{pmatrix},$$

where A_{12} is a $k \times (l-k)$ matrix etc. It is clear that $\mathcal{M}$ is not irreducible under K and therefore the uniqueness part of proposition 0.5.1.3 does not hold. In fact representing a basis for $\mathcal{M}^\star$ as $\omega_{ij}, \bar{\omega}_{ij}$ we obtain the general expression for a $U(n)$-invariant Riemannian metric on $\mathbf{F}_{k,l}$ as

$$c_1 \sum_{i=1,j=k+1}^{i=k,j=l} \omega_{ij}\bar{\omega}_{ij} + c_2 \sum_{i=1,j=l+1}^{i=k,j=n} \omega_{ij}\bar{\omega}_{ij} + c_3 \sum_{i=k+1,j=l+1}^{i=l,j=n} \omega_{ij}\bar{\omega}_{ij},$$

where $c_1, c_2, c_3 > 0$. ♠

Since irreducibility under the sets of linear transformations $\{T\}$ and $\{\exp(T)\}$ are the same, irreducibility (or complete reducibility) of a representation ρ of an analytic group G and the corresponding representation of its Lie algebra are identical. (Of course, if G has more than one connected component, this assertion is no longer true.) In particular Schur's Lemma is valid for complex irreducible representation of the Lie algebra of an analytic group. Let ρ and ρ' be representations of the analytic group G, then writing

$$\begin{aligned}(\rho(\exp(tA)) \otimes \rho'(\exp(tA)))(v \otimes v') - (v \otimes v') &= (\rho(\exp(tA))(v) - v) \otimes (\rho'(\exp(tA)))(v') + \\ & \quad v \otimes ((\rho'(\exp(tB)))(v') - v'),\end{aligned}$$

we obtain (just as in the proof of Leibnitz' rule)

$$(\rho \otimes \rho')(A)(v \otimes v') = \rho(A)(v) \otimes v' + v \otimes \rho'(A)(v'). \tag{0.5.1.5}$$

Exercise 0.5.1.5 *Let $G = SU(2)$ and ρ_1 be the natural representation of $SU(2)$ on $\mathbb{C}^2$. Show that the representation $\rho_1 \otimes \cdots \otimes \rho_1$ (k-times) maps the space of symmetric tensors $S^k(\mathbb{C}^2)$ into itself. Let ρ_k denote the representation of $SU(2)$ on $S^k(\mathbb{C}^2)$. By computing the representation ρ_k of the Lie algebra $\mathcal{SU}(2)$ and using Schur's Lemma, show that ρ_k is irreducible.*

Exercise 0.5.1.6 *Let $G = SU(n)$ and ρ_1 be its natural representation on $\mathbb{C}^n$. Show that $\rho_1 \otimes \cdots \otimes \rho_1$ (k times, $k \leq n$) leaves the space of skew-symmetric tensors $\wedge^k\mathbb{C}^n$ invariant, and let λ_k be this representation of $SU(n)$ on $\wedge^k\mathbb{C}^n$. By computing the matrices $\lambda_k(A)$ for some simple matrices $A \in \mathcal{SU}(n)$, show that λ_k is irreducible.*

On $\mathcal{GL}(m, \mathbb{C})$ we have the symmetric bilinear form

$$< A, B >= -\frac{1}{2}\mathrm{Tr}(AB) \qquad (0.5.1.6)$$

The symmetric pairing $<,>$ is a positive definite inner product on the real subspace of skew hermitian matrices (the Lie algebra $\mathcal{U}(m)$ of the unitary group $U(m)$.) Furthermore for a unitary matrix $g \in U(m)$, we have

$$< \mathrm{Ad}(g)A, \mathrm{Ad}(g)B >=< A, B > .$$

Therefore $<,>$ is the inner product provided by the Weyl unitary trick. Notice that in terms of an orthonormal basis relative to $<,>$, the matrices $\mathrm{Ad}(g)$, $g \in U(m)$, are orthogonal matrices. In dealing with the adjoint representation of a closed Lie group of $U(m)$ we always assume that the inner product $<,>$ has been fixed.

Exercise 0.5.1.7 *Show that the adjoint representation of $G = SU(2)$ is equivalent to the representation ρ_2 of exercise 0.5.1.5.*

Exercise 0.5.1.8 *Show that the adjoint representation of $SO(4)$ decomposes into two irreducible and inequivalent representations.* (See example 0.3.5.2 above.)

Exercise 0.5.1.9 *Show that the adjoint representations of $SO(n)$ ($n \neq 4$) and of $SU(n)$ are irreducible.*

Example 0.5.1.8 It is not true that every representation of the Lie algebra $\mathcal{G}$ of an analytic group G comes from a representation of G. In fact, let $G = SO(3)$ so that $\mathcal{G}$ is the space of anti-symmetric 3×3 matrices. Now the adjoint representation Ad maps $SU(2)$ onto $SO(3)$ and the kernel is $\pm I$. Then the adjoint represention ad is an isomorphism $\mathcal{SU}(2) \simeq \mathcal{SO}(3)$. Now consider the representation $\rho_1.\mathrm{ad}^{-1} : \mathcal{SO}(3) \rightarrow \mathcal{GL}(2, \mathbb{C})$ where ρ_1 is the natural representation of $\mathcal{SU}(2)$. It is a simple exercise to see that this representation of $\mathcal{SO}(3)$ does not come from a representation of $SO(3)$. The difficulty is that using exp to obtain a representation of $SO(3)$, leads to a double valued representation reflecting the fact that $\ker \mathrm{Ad} = \pm I$. More generally, all the representations ρ_{2k} of exercise 0.5.1.5 lead to double valued representations of $SO(3)$. These and similar double valued representations for orthogonal groups in higher dimensions led to the discovery of spinors by É. Cartan (and independently later by P. A. M. Dirac). This subject will be revisited later in connection with spin groups. ♠

Finally in this subsection we derive Schur's orthogonality relations as an application of Schur's Lemma and consider some of their consequences. Let ρ and τ be two irreducible representations of the compact Lie group $G \subset GL(m,\mathbb{R})$ of degrees d and d'. As noted earlier we may assume all matrices $\rho(g)$ and $\tau(g)$ are unitary. Let $T : \mathbb{C}^d \to \mathbb{C}^{d'}$ be any linear mapping, and consider

$$\Phi = \int_G \tau(g) T \rho(g^{-1}) dv_G(g) : \mathbb{C}^d \longrightarrow \mathbb{C}^{d'}, \tag{0.5.1.7}$$

Clearly $\tau(g)\Phi = \Phi\rho(g)$ for all $g \in G$ by (0.2.3.3), and consequently $\ker \Phi$ is an invariant subspace. By irreducibility $\ker \Phi = \mathbf{0}$ or $\mathbb{C}^d$. Similarly, $\mathrm{Im}\Phi = \mathbf{0}$ or $\mathbb{C}^{d'}$. Therefore Φ either establishes equivalence of the representations ρ and τ or is the zero map. If ρ and τ are inequivalent, then Φ is necessarily the zero map. Set $T = E_{jk}$ (E_{jk} is the $d' \times d$ matrix with a single nonzero entry, viz., a 1 at $(j,k)^{\text{th}}$ spot) to obtain

$$\Phi = \int_G \begin{pmatrix} \tau_{1j}(g)\rho_{k1}(g^{-1}) & \tau_{1j}(g)\rho_{k2}(g^{-1}) & \cdots & \tau_{1j}(g)\rho_{kd}(g^{-1}) \\ \tau_{2j}(g)\rho_{k1}(g^{-1}) & \tau_{2j}(g)\rho_{k2}(g^{-1}) & \cdots & \tau_{2j}(g)\rho_{kd}(g^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{d'j}(g)\rho_{k1}(g^{-1}) & \tau_{d'j}(g)\rho_{k2}(g^{-1}) & \cdots & \tau_{d'j}(g)\rho_{kd}(g^{-1}) \end{pmatrix} dv_G. \tag{0.5.1.8}$$

Therefore if ρ and τ are inequivalent representations of G, then

$$\int_G \tau_{ij}(g)\rho_{kl}(g^{-1}) dv_G(g) = 0. \tag{0.5.1.9}$$

If $\rho(g) = \tau(g)$, then the matrix Φ is a multiple $\lambda = \lambda(T)$, possibly non-zero, of the identity. For $T = E_{jk}$, we obtain

$$\int_G \rho_{ij}(g)\rho_{kl}(g^{-1}) dv_G = 0, \quad \text{unless } i = l.$$

By (0.2.3.3) the integral is invariant under the transformation $g \to g^{-1}$ and therfore

$$\int_G \rho_{ij}(g)\rho_{kl}(g^{-1}) dv_G = 0, \quad \text{unless } i = l \text{ and } j = k. \tag{0.5.1.10}$$

Lemma 0.5.1.1 *Assuming $\rho = \tau$, we have in the above notation*

$$\lambda(E_{ii}) = \lambda(E_{jj}).$$

Proof - It suffices to show $\lambda(E_{11}) = \lambda(E_{22})$. From the integral representation for Φ and the invariance of the integral under the transformation $g \to g^{-1}$, we obtain

$$\begin{aligned}\lambda(E_{11}) &= \int \rho_{11}(g)\rho_{11}(g^{-1})dv_G \\ &= \int \rho_{21}(g)\rho_{12}(g^{-1})dv_G \\ &= \int \rho_{12}(g)\rho_{21}(g^{-1})dv_G \\ &= \lambda(E_{22}),\end{aligned}$$

proving the lemma. ♣

Now setting $T = I = E_{11} + \cdots + E_{dd}$, using lemma 0.5.1.1 and (0.5.1.10) we obtain

$$\int_G \rho_{ij}(g)\rho_{kl}(g^{-1})dv_G = \frac{1}{d}\delta_{il}\delta_{jk}\mathrm{vol}(G). \qquad (0.5.1.11)$$

Formulae (0.5.1.9) and (0.5.1.11) are known as *Schur's orthogonality relations.* Note that they generalize similar orthogonality properties of the exponential functions $e^{2\pi in\theta}$ in Fourier analysis. In fact, the subject of group representations may be justifiably regarded as a generalization of Fourier analysis.

The *character* of a representation ρ of a group G is defined as

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)).$$

A character is a class function, i.e., a function ϕ on G satifying $\phi(hgh^{-1}) = \phi(g)$ for all $g, h \in G$. On the space of continuous class function on the compact group G we define the inner product

$$\prec \phi, \psi \succ = \frac{1}{\mathrm{vol}(G)} \int_G \phi(g)\bar{\psi}(g)dv_G(g)$$

An immediate consequence of the Schur orthogonality relations is

Corollary 0.5.1.3 *Let ρ and τ be irreducible representations of the compact Lie group G. Then*

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Therefore a representation of a compact Lie group is completely determined (up to equivalence) by its character. It follows that corollary 0.5.1.2, in the case of compact groups, can be restated as (with the notation and hypotheses of the corollary)

$$n(\rho, \tau) = \int_G \chi_\rho(g)\bar{\chi}_\tau(g)dv_G, \quad \sum_\eta n(\eta, \tau)^2 = \int_G |\chi_\tau(g)|^2 dv_G. \qquad (0.5.1.12)$$

In particular, consider the case of a finite group G of order $|G|$, and let $\mathbb{C}[G]$ be the complex group algebra of G, i.e., complex vector space consisting of formal linear combinations of elements of G. The *left regular representation* $\mathcal{R} : G \rightarrow GL(|G|, \mathbb{C})$ is defined as

$$\mathcal{R}(g)(\sum a_h h) = \sum a_h gh.$$

From the basis $\{h|h \in G\}$ of $\mathcal{C}[G]$ we obtain

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From (0.5.1.12) and (**??**) it follows that if ρ is an irreducible representation of G of degree d (on a complex vector space), then

$$n(\rho, \mathcal{R}) = d. \tag{0.5.1.13}$$

Therefore we obtain

Corollary 0.5.1.4 *The multiplicity of an irreducible representation of degree d_ρ of a finite group G in its regular representation is d_ρ, and consequently*

$$\sum_\rho d_\rho^2 = |G|,$$

where the summation is over all complex irreducible representations of G.

Since the trace of the tensor product of two linear transformations is the product of their traces, the character of the representation $\rho \otimes \tau$ is

$$\chi_{\rho\otimes\tau} = \chi_\rho \chi_\tau. \tag{0.5.1.14}$$

This formula and (0.5.1.12) in principle tell us how the tensor product of two irreducible representations of a compact groups decomposes into a direct sum of irreducible ones.

Example 0.5.1.9 Let $G = SU(2)$, then the character χ_k of the representation ρ_k is

$$\chi_k(g) = \sum_{j=0}^{k} e^{(k-2j)i\theta};$$

where $e^{\pm i\theta}$ are the eigenvalues of $g \in SU(2)$. It is now easy to verify that

$$\chi_k \chi_l = \sum_{j=|k-l|}^{k+l} \chi_j.$$

We shall not pursue extensions of this formula here. ♠

As noted earlier, representations of a finite or compact group G is a generalization of the exponential fiunctions $e^{in\theta}$ on the circle. Let us exploit this analogy and derive some non-trivial consequences. The first fundamental theorem in the Fourier analysis is the Parseval or Plancherel theorem which is

$$\frac{1}{2\pi}\int_{\circ}^{2\pi}|f(\theta)|^2d\theta=\sum_{-\infty<n<\infty}|a_n|^2$$

where f is an L^2-function on S^1 with Fourier series expansion $f(\theta)=\sum a_ne^{in\theta}$. For an L^1 function on a compact Lie group G and representation ρ of G, the analogue of the Fourier coefficient a_n is the matrix

$$\rho(f)=\int_G f(x)\rho(x^{-1})dv_G.$$

An important property of $\rho(f)$ is that it converts convolutions to matrix multiplication

$$\rho(f\star h)=\rho(h)\rho(f), \tag{0.5.1.15}$$

which follows from the change of variable[14]:

$$\int_G\int_G f(xy^{-1})h(y)\rho(x^{-1})dv_G(y)dv_G(x)=\int_G\int_G f(z)h(y)\rho(y^{-1}z^{-1})dv_G(y)dv_G(z)=\rho(h)\rho(f).$$

To avoid some technical issues (e.g., convergence) we develop the analogous theory for a finite group G only although the main results (0.5.1.16) and (0.5.1.19) are valid for compact groups as well. For the left regular representation $\mathcal{R}$, we have

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Therefore for a function ϕ on G we have

$$\frac{1}{|G|}\mathrm{Tr}\mathcal{R}(\phi)=\frac{1}{|G|}\sum_{x\in G}\mathrm{Tr}\mathcal{R}(x^{-1})\phi(x)=\phi(e). \tag{0.5.1.16}$$

Define $f^\star(x)=\overline{f(x^{-1})}$. Then

$$(f\star f^\star)(e)=\sum_{x\in G}f(x^{-1})\bar{f}(x^{-1})=||f||^2_{L^2}. \tag{0.5.1.17}$$

[14] If we define the convolution of f and h as $\int f(y^{-1}x)h(y)dv_G(y)$ rather than $\int f(xy^{-1})h(y)dv_G(y)$, then the right hand side of (0.5.1.15) becomes $\rho(f)\rho(h)$. The same can be accomplished by modifying the definition of $\rho(f)$ as $\int f(x)\rho(x)dv_G$.

Therefore by (0.5.1.16)

$$\frac{1}{|G|}\mathrm{Tr}\mathcal{R}(f\star f^{\star}) = ||f||^2_{L^2}. \tag{0.5.1.18}$$

In view of (0.5.1.15) and the fact that the multiplicity of an irreducible representation ρ of G in $\mathcal{R}$ is equal to its dimension d_ρ, we can rewrite (0.5.1.18) as

$$||f||^2_{L^2} = \frac{1}{|G|}\sum_{\rho\in\hat{G}} d_\rho \mathrm{Tr}(\rho(f)\rho(f)^{\star}), \tag{0.5.1.19}$$

where $\hat{G}$ is the set of irreducible representations of G and $\rho(f)^{\star}$) is the transpose complex conjugate of the matrix $\rho(f)$. Note that $\mathrm{Tr}(\rho(f)\rho(f)^{\star})$ is simply the sum of the squares of the absolute values of the entries of the matrix $\rho(f)$ (generally called the *Hilbert-Schmidt* norm).

Example 0.5.1.10 Let A, B and C be conjugacy classes in a finite group G. As an application of the above analysis we obtain a formula for the number $\nu(A,B,C)$ of solutions of the equation

$$abc = e, \quad \text{with} \ \ a\in A,\ b\in B \text{ and } c\in C.$$

Define

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It follows that

$$\nu(A,B,C) = (f_A\star f_B\star f_C)(e) = \frac{1}{|G|}\mathrm{Tr}\mathcal{R}(f_A\star f_B\star f_C). \tag{0.5.1.20}$$

Now

$$\mathcal{R}(f_A\star f_B\star f_C) = \sum_{\rho\in\hat{G}} d_\rho \rho(f_C)\rho(f_B)\rho(f_A).$$

By Schur's lemma $\rho(f_A) = \lambda_\rho(f)I$ is a scalar multiple of the identity. It follows that

$$\lambda_\rho(f_A) = \frac{|A|}{d_\rho}\chi_\rho(A^{-1}),$$

where $|A|$ etc. is the cardinality of A etc. Substituting in (0.5.1.20) we obtain

$$\nu(A,B,C) = \frac{|A|\ |B|\ |C|}{|G|}\sum_{\rho\in\hat{G}}\frac{\chi_\rho(A^{-1})\chi_\rho(B^{-1})\chi_\rho(C^{-1})}{d_\rho}. \tag{0.5.1.21}$$

This formula is of interest in the study of of spaces of curves on surfaces and the evaluation of certain important integrals in theoretical physics. ♠

Exercise 0.5.1.10 *Show that (0.5.1.21) remains valid if we replace A^{-1}, B^{-1} and C^{-1} on the right hand side by A, B and C.*

Inducing a representation from a subgroup is a method for constructing representations of a group and has played an important role in the development of theory of group represetnations. Its construction resembles that of vector bundles from a principal one[15]. To avoid some technical complications we only consider finite (rather than compact) groups and complex representations. Let $\rho : H \rightarrow G\mathcal{L}(V)$ be a representation of a subgroup $H \subset G$, and $\mathcal{L}(G)$ denote the space of complex valued functions on G. Let $W = \mathcal{L}(G) \otimes_{\mathbb{C}[H]} V$ where $\mathcal{L}(G)$ is regarded as an H (or $\mathbb{C}[H]$) module via right action of H:

$$\psi \xrightarrow{h} \psi(xh^{-1}), \quad \psi \in \mathcal{L}(G).$$

Thus W is the quotient of the vector space $W = \mathcal{L}(G) \otimes_{\mathbb{C}} V$ by the subspace spanned by elements of the form

$$\psi(xh^{-1}) \otimes v - \psi(x) \otimes \rho(h)(v).$$

Regarding $\mathcal{L}(G) \otimes_{\mathbb{C}} V$ as the space of of V-valued functions on G, we may equivalently define W as the vector space of V-valued functions ψ on G which under right translation by $h \in H$ transform according to the representation ρ, i.e.,

$$\psi(xh^{-1}) = \rho(h)\psi(x).$$

Define the representation $\tilde{\rho}$ of G on W by

$$(\tilde{\rho}(g)\psi)(x) = \psi(gx).$$

$\tilde{\rho}$ is called the *representation induced from* ρ and is often denoted by $\mathrm{Ind}_H^G \rho$. Note that for $H = e$, ρ is necessarily the trivial representation and $\mathrm{Ind}_H^G \rho$ is the regular representation. It is clear that the degree of the representation $\mathrm{Ind}_H^G \rho$ is $\dim(V).|G/H|$. For a subset $A \subset G$ let

$$C_G(A) = \{g^{-1}ag \mid g \in G, a \in A\}.$$

Lemma 0.5.1.2 *The character $\chi_{\tilde{\rho}}$ of the induced representation $\tilde{\rho} = \mathrm{Ind}_H^G \rho$ is HEREWEARE*

[15]The connection between induced representations and homogeneous vector bundles can be made precise and has geometric implications, however, this will not be pursued here.

Proof - Follows immediately from the definitions. ♣

Note that $\chi_\rho(x^{-1}hx)$ appearing in lemma 0.5.1.2 may not be equal to $\chi_\rho(h)$ for $x \notin H$. For $x \notin H$, let $H_x = H \cap x^{-1} \cap Hx$. Then $\rho_x : h \to \rho(x^{-1}hx)$ is a representation of H_x which, in general, is not equivalent to the restriction of the representation ρ to H_x. A particularly important special case of lemma 0.5.1.2 is

Corollary 0.5.1.5 *Let ρ be the trivial* (*one dimensional*) *representation of H. Then*

$$\chi_{\tilde{\rho}}(g) = \frac{|G|}{|H|}\frac{|C_G(g) \cap H|}{|C_G(g)|}.$$

The following proposition is an important tool in analyzing induced representations.

Proposition 0.5.1.4 (Frobenius Reciprocity) *Let τ be an irreducible representation of the group G on a complex vector space W, and ρ a representation of H. Then the number of times τ appears in $\tilde{\rho} = \mathrm{Ind}_H^G \rho$ is equal to the number of times ρ appears in the restriction τ to H* (denoted by $\mathrm{Res}_H\tau$). *In terms of inner products of characters we have*

$$\prec \chi_\tau, \chi_{\tilde{\rho}} \succ_G = \prec \chi_{\mathrm{Res}_H\tau}, \chi_\rho \succ_H .$$

Proof - Let U be a right G-module. Applying the basic and elementary algebraic identity

$$U \otimes_{\mathbb{C}[G]} (\mathcal{L}(G) \otimes_{\mathbb{C}[H]} V) \simeq U \otimes_{\mathbb{C}[H]} V, \tag{0.5.1.22}$$

with $U = W^\star$ (and the dual of the representation τ acting on the right), and corollary 0.5.1.1 we obtain the first assertion. The second assertion follows from the first and corollary 0.5.1.3. ♣

The (left) regular representation is a special case of induced representations where H consists of the identity element. The fact that the number of times a representation occurs in the regulasr representation is equal to its dimension is also a special case of the Frobenius reciprocity.

Exercise 0.5.1.11 *Identify $\mathcal{S}/2$ with the subgroup of $\mathcal{S}_3$ generated by the transposition* (12). *Let ϵ be be the representation of $\mathcal{S}_2$ mapping* (12) *to* -1. *What is the decomposition of* $\mathrm{Ind}_{\mathcal{S}_2}^{\mathcal{S}_3}\rho$ *into irreducible $\mathcal{S}_3$-modules?*

Exercise 0.5.1.12 *Let $H = \mathcal{S}_2 \times \mathcal{S}_2 \subset \mathcal{S}_4 = G$ be the subgroup generated by the transpositions* (12) *and* (34), *and ρ be the trivial representation of H. What is the decomposition of* $\mathrm{Ind}_H^G\rho$ *into irreducible G-modules?*

Example 0.5.1.11 We specialize corollary 0.5.1.5 to the case $G = \mathcal{S}_n$ and $H = \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}$ where $n_1 + n_2 = n$ and as usual $\mathcal{S}_{n_1}$ permutes $\{1, 2, \cdots, n_1\}$ and $\mathcal{S}_{n_2}$ acts on the remaining integers. It is clear that $|H| = n_1!n_2!$. The conjugacy class of an element $\sigma \in \mathcal{S}_n$ is determined by specifying the cycle structure of the permutation. If there are α cycles of length 1, β cycles of length 2, γ cycles of length 3, etc., then it is customary to denote the cycle structure of the permutation σ as $1^\alpha 2^\beta 3^\gamma \cdots$. For instance the permutation $(345)(67)(8910) \in \mathcal{S}_{11}$ has cycle structure $1^3 2^1 3^2$. The non-negative integers $\alpha, \beta, \gamma, \cdots$ are subject to the obvious relation

$$\alpha + 2\beta + 3\gamma + \cdots = n.$$

It is not difficult to determine the centralizer in $\mathcal{S}_n$ of a permutation with cycle structure $1^\alpha 2^\beta 3^\gamma \cdots$ and in particular show that it has order $\alpha!2^\beta \beta!3^\gamma \gamma! \cdots$. It follows that

$$|C_G(\sigma)| = \frac{n!}{\alpha!2^\beta \beta!3^\gamma \gamma! \cdots}. \tag{0.5.1.23}$$

From (0.5.1.23) one easily calculates the order of $C_G(\sigma) \cap H$. In fact, we have

$$|C_G(\sigma) \cap H| = \sum \frac{n_1!}{\alpha_1!2^{\beta_1}\beta_1!3^{\gamma_1}\gamma_1! \cdots} \frac{n_2!}{\alpha_2!2^{\beta_2}\beta_2!3^{\gamma_2}\gamma_2! \cdots}, \tag{0.5.1.24}$$

where the summation is over all $\alpha_1, \alpha_2, \beta_1, \beta_2, \cdots$ such that

$\alpha_1 + 2\beta_1 + 3\gamma_1 + \cdots = n_1, \quad \alpha_2 + 2\beta_2 + 3\gamma_2 + \cdots = n_2,$

$\alpha_1 + \alpha_2 = \alpha, \quad \beta_1 + \beta_2 = \beta, \quad \gamma_1 + \gamma_2 = \gamma, \quad \cdots$

Substituting in corollary 0.5.1.5, we obtain a formula for characters of the representation $\mathrm{Ind}_{\mathcal{S}_{n_1} \times \mathcal{S}_{n_2}}^{\mathcal{S}_n} 1$, (1 denotes the trivial representation):

$$\chi_{\tilde{1}}(1^\alpha 2^\beta 3^\gamma \cdots) = \sum \frac{n_1!}{\alpha_1!\beta_1!\gamma_1! \cdots} \frac{n_2!}{\alpha_2!\beta_2!\gamma_2! \cdots}, \tag{0.5.1.25}$$

where the summation has the same range as in (0.5.1.24). ♠

Example 0.5.1.12 It is clear that the above example extends to the case where the subgroup $H = \mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_k}$ where $n = n_1 + \cdots + n_k$, $\mathcal{S}_{n_1}$ permutes the integers $\{1, \cdots, n_1\}$, $\mathcal{S}_{n_2}$ permutes the integers $\{n_1 + 1, \cdots, n_1 + n_2\}$, etc. In fact, the same argument gives

$$\chi_{\tilde{1}}(1^\alpha 2^\beta 3^\gamma \cdots) = \sum \frac{n_1!}{\alpha_1!\beta_1!\gamma_1! \cdots} \cdots \frac{n_k!}{\alpha_k!\beta_k!\gamma_k! \cdots}, \tag{0.5.1.26}$$

where the summation is over all $\alpha_1, \cdots, \alpha_k$, $\beta_1, \cdots, \beta_k$, $\gamma_1, \cdots, \gamma_k$, $\cdots$, such that

$\alpha_1 + 2\beta_1 + 3\gamma_1 + \cdots = n_1, \quad \cdots, \quad \alpha_k + 2\beta_k + 3\gamma_k + \cdots = n_k,$

$\alpha_1 + \cdots + \alpha_k = \alpha, \quad \beta_1 + \cdots + \beta_k = \beta, \quad \gamma_1 + \cdots + \gamma_k = \gamma, \quad \cdots$

The subgroups $\mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_k}$ are called *Young subgroups*[16]. ♠

[16]Frobenius used the the induced representations $\mathrm{Ind}_H^{\mathcal{S}_n} 1$ to obtain characters of irreducible representations

0.5.2 Young Diagrams

The material of this section will not be used extensively. It is relevant to some special topics in chapters 2, 5 and 6. In this subsection we briefly study some representation theory of $GL(m, K)$, $\mathcal{S}_n$ etc. ($K = \mathbb{R}$, or $\mathbb{C}$) which pertains to geometric considerations of later chapters. Let $V \simeq \mathbf{K}^m$, $T^n(V)$ denote the n^{th} tensor power of V, and $S^n(V)$ the n^{th} symmetric power of V. For $n = 2$ we have the decomposition $T^2(V) \simeq S^2(V) \oplus \wedge^2 V$. The group $G = GL(m, K)$ acts on $T^2(V)$ and leaves the subspaces $S^2(V)$ and $\wedge^2 V$ invariant. By experimenting with the effect of simple matrices, it is not difficult to see that each of the subspaces $S^2(V)$ and $\wedge^2 V$ is irreducible under the action of G. For $n > 2$, the decomposition of the tensor space generalizing $T^2(V) \simeq S^2(V) \oplus \wedge^2 V$ requires the notion of Young diagrams and tableaux defined below. While one can approach the subject from a more abstract and powerful view point of representation theory of Lie groups and algebras, it seems that the classical approach is more suitable for our modest goals. We only give a summary of the results of interest to us and demonstrate them by working out some examples in detail. The omitted proofs generally can be found in [W].

By a *Young diagram* we mean a partition of a positive integer $n = n_1 + n_2 + \cdots + n_k$ where $n_1 \geq n_2 \geq \cdots \geq n_k$. Normally one pictures a Young diagram as a collection of n squares arranged with n_1 squares in the first row, n_2 squares in the second etc. and left justified. We enumerate[17] the squares in a Young diagram by starting from upper left corner, moving along the first column from top to bottom, then along the second column etc. A *Young tableau* is a Young diagram where the squares are filled with integers $\{1, 2, \cdots, n\}$. For example, a Young tableau corresponding to the partition $7 = 4 + 2 + 1$ may be pictured as follows:

1	3	4	7
2	6		
5			

A permutation rearranges the integers in a Young tableau by moving the integers according to the enumeration of the squares of the Young diagram. For example, applying the permutation (2457) to the above Young tableau we obtain

1	2	4	6
7	3		
5			

of $\mathcal{S}_n$. The representations $\mathrm{Ind}_H^{\mathcal{S}_n} 1$ are not irreducible and the irreducible components were obtained by means of the orthogonality relations and symmetric functions. It is perhaps more appropriate to refer to subgroups $\mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_k}$ as *Frobenius subgroups*.

[17] Our enumeration is different from the conventional one which moves along rows rather than columns. The proposed enumeration appears to be more suitable for the study of the curvature tensor.

If a Young diagram T is filled with integers in two different ways, we denote the corresponding tableaux by $\{T\}$ and $\{T\}'$ or other self-explanatory notation. A Young diagram T, with the enumerations of squares as indicated above, specifies two subgroups of the symmetric group $\mathcal{S}_n$, namely, the subgroup $H = H_T$ consisting of all permutations preserving the rows, and $H' = H'_T$ consisting of all permutations preserving the columns. Let $\mathbf{Z}[\mathcal{S}_n]$ be the *integral group algebra* of the symmetric group, i.e., formal linear combinations with integers coefficients and multiplication inherited from the group law in $\mathcal{S}_n$. Define the *Young symmetrizer* $\mathbb{C} = \mathbb{C}_T \in \mathbf{Z}[\mathcal{S}_n]$ as

$$\mathbb{C} = \mathbb{C}_T = (\sum_{\tau\in H'} \epsilon_\tau \tau)(\sum_{\sigma\in H} \sigma) = \sum_{\sigma\in H,\tau\in H'} \epsilon_\tau \tau\sigma.$$

The symmetric group $\mathcal{S}_n$ and therefore its group algebra $\mathbf{Z}[\mathcal{S}_n]$ act on the tensor space $T^n(V)$. In fact, given a tensor $v_{i_1} \otimes \cdots \otimes v_{i_n}$, $v_{i_j} \in V$, and $\sigma \in \mathcal{S}_n$, the action of σ is given by

$$v_{i_1} \otimes \cdots \otimes v_{i_n} \xrightarrow{\sigma} v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(n)}}.$$

Notice that this action of the permutation group commutes with the induced action of $G = GL(m, K)$ on $T^n(V)$, and therefore we have a representation τ_n of $G \times \mathcal{S}_n$ on $T^n(V)$. It also follows that image of $T^n(V)$ under a Young symmetrizer is invariant under G. For example, for $n = 2$ there are two Young diagrams corresponding to the partitions $2 = 2$ and $2 = 1 + 1$. Denoting the transposition in $\mathcal{S}_2$ by ϵ, we see that the corresponding Young symmetrizers are $1 + \epsilon$ (for $2 = 2$) and $1 - \epsilon$ (for $2 = 1 + 1$). Therefore

$$T^2(V) = \mathrm{Im}(1 + \epsilon) \oplus \mathrm{Im}(1 - \epsilon) = S^2V \oplus \wedge^2 V, \tag{0.5.2.1}$$

yielding the decomposition of 2-tensors into symmetric and anti-symmetric ones.

By a *standard Young tableau* we mean a Young tableau such that the numbers are increasing along every row (from left to right) and along every column (from top to bottom). By a *semistandard Young tableau* for a partition of n we mean a Young diagram (with n squares) whose squares are filled with integers from $\{1, 2, \cdots, m\}$ ($m = \dim V$) in such a way that the numbers are are nondecreasing along each row from left to right and are increasing along each column from top to bottom.

Theorem 0.5.2.1 *Every partition $T : n = n_1 + \cdots + n_k$ with $n_1 \geq n_2 \geq \cdots \geq n_k$ determines a unique irreducible representation λ_T of $\mathcal{S}_n$, and every irreducible representation of $\mathcal{S}_n$ is of the form λ_T. The degree of λ_T is the number of standard Young tableaux whose underlying Young diagram is T.*

Remark 0.5.2.1 The standard construction of the representation λ_T and representation theory of compact classical groups can be found in [W]. While the construction will not be given here, the methods we develop on the basis of character theory and the duality between representations of $\mathcal{S}_n$ and $U(m)$ enable us to derive interesting information which may not be so easily available directly through the explicit construction. In particular we will present formulae for the degrees of representations and present a method for inductively computing characters in the subsection on Characters below. ♡

Since actions of $\mathcal{S}_n$ and G on $T^n(V)$ commute, $\mathrm{Im}(\mathbb{C}_T)$ is invariant under G, and we denote the representation of G on $\mathrm{Im}(\mathbb{C}_T)$ by ρ_T. Implicit in this notation is the fact that representations of G for distinct Young tableaux with the same underlying Young diagram T are equivalent.

Example 0.5.2.1 The decomposition $T^2(V)$ given by (0.5.2.1) is particularly simple. The reason for introducing Young diagrams is that we want to understand the decomposition of $T^n(V)$. Before explaining the general case let $n = 3$ which captures some of the spirit and complexities of the general case. For $n = 3$, then there are three Young diagrams corresponding to the partitions $T_1 : 3 = 3$, $T_2 : 3 = 2 + 1$, and $T_3 = 1 + 1 + 1$. Since there are two standard tableaux for T_2 we denote the Young symmetrizers by $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_{2'}$ and $\mathbb{C}_3$. It is straightforward to verify that

$$\mathbb{C}_i\mathbb{C}_j = \delta_{ij}t_i, \quad \text{where } i \neq j = 1, 2, 2', 3. \tag{0.5.2.2}$$

where $t_i \neq 0$ is an integer. Furthermore

$$\mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_{2'} + \mathbb{C}_3 = 4e. \tag{0.5.2.3}$$

$e \in \mathcal{S}_3$ is the identity. Let $Z_1 = \mathrm{Im}\mathbb{C}_1$, $Z_3 = \mathrm{Im}\mathbb{C}_3$ and $Z_2 = \mathrm{Im}\mathbb{C}_2 + \mathrm{Im}\mathbb{C}_{2'}$. Then relations (0.5.2.2) and (0.5.2.3) imply

$$T^3(V) = Z_1 \oplus Z_2 \oplus Z_3. \tag{0.5.2.4}$$

Z_1 and Z_3 are the spaces of symmetric and antisymmetric 3-tensors and the difference between (0.5.2.4) and (0.5.2.2) is the appearance of Z_2. Z_j's are invariant under the action of $\mathcal{S}_3$ and $GL(m, K)$. Now $Z_2 = \mathrm{Im}\mathbb{C}_2 + \mathrm{Im}\mathbb{C}_{2'}$ and the sum is direct since $\mathbb{C}_2\mathbb{C}_{2'} = 0$. Each summand of Z_2 is invariant under $GL(m, K)$ since the actions of $GL(m, K)$ and $\mathcal{S}_3$ commute. However neither summand is invariant under the action of $\mathcal{S}_3$ and it is not difficult to verify that Z_2 is irreducible under the action of $GL(V) \times \mathcal{S}_3$. In fact, the action of $GL(V) \times \mathcal{S}_3$ on Z_2 is isomorphic to the representation $\rho_T \otimes \lambda_T$. The general case is similar to this (see theorem 0.5.2.2 below). The proof involves certain subtleties since the relations between various Young symmetrizers are not quite so simple. ♠

The following theorem describes the decomposition of $T^n(V)$ under the product group $GL(m,K)\times\mathcal{S}_n$ and relates the representations λ_T and ρ_T:

Theorem 0.5.2.2 *The representation ρ_T of G is irreducible. For every Young diagram T corresponding to a partition of n, let $Z_T\subset T^n(V)$ be the minimal linear subspace containing* $\mathrm{Im}\mathbb{C}_T$ *and invariant under action of $GL(m,K)\times\mathcal{S}_n$. Z_T has dimension* $\deg(\rho_T)\deg(\lambda_T)$ *and is irreducible under the representation $\tau_T=\rho_T\otimes\lambda_T$ of $GL(m,K)\times\mathcal{S}_n$.* $\deg(\rho_T)$ *is equal to the number of semi-standard Young tableaux whose underlying diagram is T. Furthermore $T^n(V)$ admits of the decomposition, as a $G\times\mathcal{S}_n$-module (under τ_n),*

$$T^n(V)\simeq\sum_T Z_T,$$

where the summation is over all partitions of T of n. Let $\mathcal{A}_T(G)$ and $\mathcal{A}_T(\mathcal{S}_n)$ denote the algebras of linear transformations of Z_T generated by the matrices $\rho_T(g)\otimes I$, ($g\in G$), and $I\otimes\lambda_T(\sigma)$, ($\sigma\in\mathcal{S}_n$). Then the full matrix algebra on Z_T has the decomposition $\mathcal{A}_T(G)\otimes\mathcal{A}_T(\mathcal{S}_n)$.

Some comments will help understand the meaning of this theorem. As noted earlier the actions of G and $\mathcal{S}_n$ commute and as matrices, they are given by $\rho_T(g)\otimes I$ and $I\otimes\lambda_T(\sigma)$ for $g\in G$ and $\sigma\in\mathcal{S}_n$. Z_T regarded as a subspace of $T^n(V)$ is given by

$$Z_T=\sum_{\{T\}}\mathrm{Im}(\mathbb{C}_T),$$

where the summation is over all standard Young tableaux $\{T\}$ whose underlying Young diagram is T. As $G\times\mathcal{S}_n$-modules, the linear spaces Z_T are irreducible and inequivalent, so that no representation of $G\times\mathcal{S}_n$ occurs more than once in $T^n(V)$. Each Z_T decomposes under the action G into $\deg(\lambda_T)$ copies of the representation ρ_T, and similarly, under the action of $\mathcal{S}_n$, Z_T decomposes into $\deg\rho_T$ copies of the representation λ_T (see example 0.5.1.6).

Example 0.5.2.2 Let us consider the special case $n=4$. Then there are five partitions or Young diagrams, namely,

$$T_1:4=4;\ T_2:4=3+1;\ T_3:4=2+2;\ T_4:4=2+1+1;\ T_5:4=1+1+1+1.$$

It is easily verified that $\mathrm{Im}(c_{T_1})$ (resp. $\mathrm{Im}(c_{T_5})$) is the space of symmetric tensors $S^n(V)$ (resp. skew-symmetric tensors $\wedge^n V$); and the conclusion is valid for arbitrary n in the sense that the diagram with one row (resp. one column) yields the indicated space of tensors. The degrees of the representations of the symmetric group $\mathcal{S}_4$ are easily computed:

$$\deg\lambda_{T_1}=1;\ \deg\lambda_{T_2}=3;\ \deg\lambda_{T_3}=2;\ \deg\lambda_{T_4}=3;\ \deg\lambda_{T_5}=1.$$

The degrees of the representations ρ_{T_i} of G are also easily computed:

$$\deg\rho_{T_1} = \binom{m+3}{4};\ \deg\rho_{T_2} = \frac{m(m^2-1)(m+2)}{8};\ \deg\rho_{T_3} = \frac{m^2(m^2-1)}{12};$$
$$\deg\rho_{T_4} = \frac{m(m^2-1)(m-2)}{8};\ \deg\rho_{T_5} = \binom{m}{4}.$$

The validity of $\sum(\deg\lambda_{T_i})(\deg\rho_{T_i}) = m^4$ follows immediately. Consider the Young tableau $\{T\}$ given by

1	3
2	4

Then the Young symmetrizer is given by

$$\begin{aligned} \mathbb{C}_T &= ((1)-(12)-(34)+(12)(34))((1)+(13)+(24)+(13)(24)) \\ &= (1)-(12)-(34)+(12)(34)+(13)(24)-(1324)-(1423)+(14)(23)+ \\ &\quad (13)-(132)-(143)+(1432)+(24)-(124)-(234)+(1234). \end{aligned}$$

Of interest in the study of the curvature tensor (which is introduced in the next chapter) is the Young diagram $T = T_3$. Therefore it is useful to explicitly construct the representations of G and $\mathcal{S}_4$ corresponding to the partition $T : 4 = 2+2$. This is done in the next example. ♠

Example 0.5.2.3 The irreducible representation ρ_T of $G = GL(4,\mathbb{R})$ has degree 20. To construct this representation, let W' be the vector space of all 6×6 symmetric matrices $A = (A_{ij})$, $A_{ij} = A_{ji}$. Clearly $\dim W' = 21$ and $W' = S^2(\wedge^2 V)$ where $V = \mathbb{R}^4$. Let ρ_2 denote the second exterior power representation of G, and consider the representation ρ given by

$$\rho(g)(A) = \rho_2(g)' A \rho_2(g),$$

where $'$ denotes transpose. To understand the representation ρ and its relationship to ρ_T, it is convenient to introduce the basis $e_1\wedge e_2, e_1\wedge e_3, e_1\wedge e_4, e_2\wedge e_3, e_2\wedge e_4, e_3\wedge e_4$ for $\wedge^2 V$. Let $G' = SL(4,\mathbb{R})$ and denote the Lie algebras of G and G' by $\mathcal{G}$ and $\mathcal{G}'$ respectively. Let $L \subset S^2(\wedge^2 V)$ be the one dimensional subspace spanned by the symmetric matrix[18] $E = \begin{pmatrix} 0 & E_1 \\ E_1 & 0 \end{pmatrix}$ where

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

[18]The matrix E has an important interpretation. We regard E as a linear transformation of $\wedge^2 V = \mathbb{R}^6$ relative to the above described basis. Recalling that a nonzero $v\wedge v'$ represents an oriented 2-plane in $\mathbb{R}^4$, E maps the 2-plane $v\wedge v'$ to the 2-plane $w\wedge w'$ such that v, v', w, w' is a positively oriented basis for $\mathbb{R}^4$ relative to the standard orientation. As such, E is a special case of Weyl *complementary tensor* (see [W], p.156) or *Hodge $\star$-operator*.

We show that G' acts trivially on L and is invariant under G. A simple way of proving that G' acts trivially is by looking at the action of the Lie algebra $\mathcal{G}'$. Let E_{ij} be the 4×4 matrix whose sole nonzero entry is 1 at $(i,j)^{\text{th}}$ spot. For $i\neq j$, $E_{ij}\in\mathcal{G}'$. Now

$$\begin{aligned}\rho_2(E_{ij})(e_k\wedge e_l) &= E_{ij}(e_k)\wedge e_l + e_k\wedge E_{ij}(e_l),\\ \rho(E_{ij})(S) &= \rho_2(E_{ij})'S + S\rho_2(E_{ij}),\end{aligned}$$

where S is a symmetric 6×6 matrix. Computing a few of these matrices for $S=E$ one sees easily that $\rho(E_{ij})(E)=0$ for $i\neq j$. Since matrices E_{ij}, $i\neq j$, generate $\mathcal{G}'$ we have shown the triviality of the action of G' on L. For a diagonal matrix $D\in G$, $\rho_2(D)$ is also diagonal and we have $\rho(D)(L)=L$ and invariance of L under G follows. In view of this argument we have

$$SL(4,\mathbb{R})/\pm I \simeq SO(3,3)^\circ,$$

where $SO(3,3)$ is the special orthogonal group of a symmetric bilinear form of signature $(3,3)$, and $SO(3,3)^\circ$ is its connected component. Now let

$$W=\{S\in S^2(\wedge^2 V)|S_{16}-S_{25}+S_{34}=0\}$$

where $S=(S_{ij})$ is a symmetric 6×6 matrix. By an argument similar to one given above it is easily shown that W is invariant under G. It follows that we have the G-module decomposition

$$S^2(\wedge^2 V)\simeq L\oplus W.$$

$\dim W=20$, and the action of $\rho(g)$, $g\in G$ on W is the representation ρ_T of G. ♠

While W is irreducible under G, it is not so under the orthogonal group $K=O(4)$. Generally the irreducible representation ρ_T of $GL(m,K)$ decomposes further upon restriction to $O(m)$. To understand this phenomenon, consider the trace maps $\mathrm{Tr}_{ij}: T^n(V)\to T^{n-2}(V)$, $1\leq i<j\leq n$, which commute with the action of $O(m)$ where $V\simeq\mathbb{R}^m$. For example, for $F=\sum F_{i_1\cdots i_n}v_{i_1}\otimes\cdots\otimes v_{i_n}\in T^n(V)$, where $v_1,\cdots,v_m$ is an orthonormal basis fo $\mathbb{R}^m$, $\mathrm{Tr}_{12}(F)$ is defined by

$$\mathrm{Tr}_{12}(F)=\sum_{j_1,\cdots,j_{n-2}}\sum_j F_{jjj_1\cdots j_{n-2}}v_{j_1}\otimes\cdots\otimes v_{j_{n-2}}.$$

Similarly, one defines Tr_{ij}. Since the maps Tr_{ij} commute with the action of the orthogonal group, $\cap_{i<j}\ker(\mathrm{Tr}_{ij})$ is invariant under $O(m)$. Let $V_T=\mathrm{Im}(\mathbb{C}_T)\cap(\cap_{i<j}\ker(\mathrm{Tr}_{ij}))$. It is shown in ([W], chapter V) that

Proposition 0.5.2.1 *Each V_T is irreducible under the restriction of the representation ρ_T to $O(m)$. V_T remains irreducible under $SO(m)$ unless $m = 2n$ and the first column of T contains exactly n squares in which case it decomposes into two irreducible representations.*

Example 0.5.2.4 We consider the special case $m = 4$ with T the partition $4 = 2 + 2$, and explicitly describe the decomposition of W decomposes into into $O(4)$ and $SO(4)$ irreducible subspaces. In view of the definition of the Young symmetrizer where the indices along columns were antisymmetrized, the maps Tr_{ij} vanish if i and j correspond to squares in the same column of a Young diagram. In particular, for the diagram T corresponding to the partition $4 = 2 + 2$ the restriction of Tr_{12} and Tr_{34} to $\mathrm{Im}\mathbb{C}_T$ vanish. Similarly, one shows that Tr_{13} and Tr_{24} are the same maps on this space so that there is only one function Tr to consider. It is a simple exercise to show that this map $\mathrm{Tr}_{13} = \mathrm{Tr}_{24}$ is realized as the map κ from W to 4×4 symmetric matrices ($A = (a_{ij})$ symmetric 6×6 matrix):

$$\kappa(A) = \begin{pmatrix} a_{11} + a_{22} + a_{33} & a_{24} + a_{35} & -a_{14} + a_{36} & -a_{15} - a_{26} \\ a_{24} + a_{35} & a_{11} + a_{44} + a_{55} & a_{12} + a_{56} & a_{13} - a_{46} \\ -a_{14} + a_{36} & a_{12} + a_{56} & a_{22} + a_{44} + a_{66} & a_{23} + a_{45} \\ -a_{15} - a_{26} & a_{13} - a_{46} & a_{23} + a_{45} & a_{33} + a_{55} + a_{66} \end{pmatrix}$$

The kernel of κ is the set of matrices of the form

$$\ker \kappa : \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & -a_{15} \\ a_{13} & a_{23} & a_{33} & a_{34} & -a_{24} & a_{14} \\ a_{14} & a_{24} & a_{34} & a_{33} & -a_{23} & a_{13} \\ a_{15} & a_{25} & -a_{24} & -a_{23} & a_{22} & -a_{12} \\ a_{16} & -a_{15} & a_{14} & a_{13} & -a_{12} & a_{11} \end{pmatrix}$$

subject to the relations

$$a_{11} + a_{22} + a_{33} = 0, \quad a_{16} - a_{25} + a_{34} = 0.$$

Let $W_1 = \ker \kappa$, then $\dim W_1 = 10$ and by proposition 0.5.2.1, W_1 is irreducible under $O(4)$. However, W_1 decomposes into two irreducible subspaces[19] $W_1 = W_1' \oplus W_1''$. To understand this decomposition, note that the matrix E commutes with the action of $SO(4)$ (see e.g.,

[19] In general, under the action of a subgroup of index 2, an irreducible representation either remains irreducible or decomposes into two, not necessarily equivalent, irreducible representations.

the argument preceding exercise **??**), and $\mathbb{R}^6$ decomposes into three dimensional subspaces corresponding to eigenvalues ± 1 of E:

$$\text{eigenvalue} \quad +1: \quad \begin{pmatrix} v \\ E_1 v \end{pmatrix}; \qquad \text{eigenvalue} \quad -1: \quad \begin{pmatrix} v \\ -E_1 v \end{pmatrix},$$

where $v \in \mathbb{R}^3$. It follows that the decomposition $W_1 = W_1' \oplus W_1''$ is given by

$$W_1': \quad \begin{pmatrix} H & E_1\hat{H} \\ E_1 H & \hat{H} \end{pmatrix}; \quad W_1'': \quad \begin{pmatrix} H & -E_1\hat{H} \\ -E_1 H & \hat{H} \end{pmatrix} \tag{0.5.2.5}$$

where the matrix $\hat{H}$ is related to the 3×3 symmetric trace zero matrix $H = (h_{ij})$ by

$$\hat{H} = \begin{pmatrix} h_{33} & -h_{23} & h_{13} \\ -h_{23} & h_{22} & -h_{12} \\ h_{13} & -h_{12} & h_{11} \end{pmatrix}.$$

Using $E_1 H E_1 = \hat{H}$ we simultaneously block diagonalize W_1' and W_1'' by orthogonal transformations:

$$\begin{pmatrix} I & E_1 \\ -E_1 & I \end{pmatrix} \begin{pmatrix} H & E_1\hat{H} \\ E_1 H & \hat{H} \end{pmatrix} \begin{pmatrix} I & -E_1 \\ E_1 & I \end{pmatrix} = \begin{pmatrix} 4H & 0 \\ 0 & 0 \end{pmatrix}, \tag{0.5.2.6}$$

$$\begin{pmatrix} I & E_1 \\ -E_1 & I \end{pmatrix} \begin{pmatrix} H & -E_1\hat{H} \\ -E_1 H & \hat{H} \end{pmatrix} \begin{pmatrix} I & -E_1 \\ E_1 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4\hat{H} \end{pmatrix}. \tag{0.5.2.7}$$

Note that the Lie algebra $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) acts trivially on W_1'' (resp. W_1') so that the corresponding representations of $SO(4)$ are inequivalent. Finally in this example we note since $W_1 = \ker \kappa$ has dimension 10, $\mathrm{Im}(\kappa)$ is the 10 dimensional space of symmetric 4×4 matrices which we denote by $S^2(V)$. Under $O(4)$ and $SO(4)$, it further decomposes into irreducible subspaces

$$S^2(V) = S^2_\circ(V) \oplus \{\lambda I\}, \tag{0.5.2.8}$$

where $S^2_\circ(V)$ is the subspace of trace zero matrices and $\{\lambda I\}$ are the multiples of identity. ♠

Example 0.5.2.5 In the preceding example we showed that the kernel of κ can be block diagonalized. It is natural and useful for geometric reasons which will be explained in the next chapter, to understand the effect of this block diagonalization on the orthogonal complement of $\ker \kappa$ in W which is also invariant under $SO(4)$. (Here we are using the inner product on the space 6×6 symmetric matrices given by $\mathrm{Tr}(AB)$.) For this purpose, let $A = (a_{ij})$ be a 6×6 symmetric matrix satisfying $a_{16} - a_{25} + a_{34} = 0$, and compute the matrix

$$\begin{pmatrix} I & -E_1 \\ E_1 & I \end{pmatrix} A \begin{pmatrix} I & E_1 \\ -E_1 & I \end{pmatrix} = \begin{pmatrix} C & B \\ B' & D \end{pmatrix},$$

where B, C and D are 3×3 matrices and C and D are symmetric. Note that the matrix $\begin{pmatrix} I & -E_1 \\ E_1 & I \end{pmatrix}$ commutes with the action of $SO(4)$ via ρ_T, so that the above operation preserves the $SO(4)$-module structure of W. We obtain after a brief calculation

$$B = \begin{pmatrix} -\kappa_{13} - \kappa_{24} & -\kappa_{14} + \kappa_{23} & \frac{1}{2}(-\kappa_{11} - \kappa_{22} + \kappa_{33} + \kappa_{44}) \\ \kappa_{12} - \kappa_{34} & \frac{1}{2}(\kappa_{11} - \kappa_{22} + \kappa_{33} - \kappa_{44}) & -\kappa_{14} - \kappa_{23} \\ \frac{1}{2}(-\kappa_{11} + \kappa_{22} + \kappa_{33} - \kappa_{44}) & \kappa_{12} + \kappa_{34} & \kappa_{13} - \kappa_{24} \end{pmatrix},$$

where we have used the notation $\kappa(A) = (\kappa_{ij})$. Therefore the block diagonalization process maps $S^2_\circ(V)$ onto the set of symmetric matrices of the form $\begin{pmatrix} 0 & B \\ B' & 0 \end{pmatrix}$. Calculating the matrices C and D we see that the condition $a_{16} - a_{25} + a_{34} = 0$ translates into

$$\mathrm{Tr}(C) = \mathrm{Tr}(D). \tag{0.5.2.9}$$

Furthermore,

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} - \frac{\mathrm{Tr}(\kappa(A))}{6} I = \begin{pmatrix} 4H & 0 \\ 0 & 4\hat{H} \end{pmatrix} \tag{0.5.2.10}$$

where the 3×3 traceless matrix H is as given in the preceding example. This gives the complete decomposition of W as an $SO(4)$-module.

In the preceding discussion of Young diagrams and representation theory of the symmetric and general linear groups, the underlying field K could have been $\mathbb{R}$ or $\mathbb{C}$. It is important to note several points. Let $K = \mathbb{C}$, then the irreducible representation ρ_T of $GL(m, \mathbb{C})$ remains irreducible upon restriction of ρ_T to $GL(m, \mathbb{R})$ or $U(m)$. The general reason is that since ρ_T is given by polynomials and irreducibility is an algebraic condition, density (in the sense of Zariski) of $GL(m, \mathbb{R})$ or $U(m)$ in $GL(m, \mathbb{C})$ imply irreducibility under the former groups. Note that the complexification of the Lie algebra $\mathcal{U}(m)$ (skew hermitian matrices) is the full matrix algebra $M_m(\mathbb{C})$. The situation is different for $O(m)$ and $SO(m)$. These groups are not Zariski dense in $GL(m, \mathbb{C})$ since they are defined by a set of (quadratic) equations.

0.5.3 Characters

Representation theory of the symmetric group was originally investigated by Frobenius using character theory and symmetric functions. While this approach is no longer fashionable, yet it has merits and it appears that it may have interesting geometric applications[20]. For this reason we include a brief account of character theory of the symmetric and unitary groups.

[20] Some applications in connection with ramified coverings of Riemann surfaces were discovered by Adolf Hurwitz in late nineteenth and early twentieth centuries.

The Young diagram approach to representation theory of the symmetric and the general linear groups leads to interesting formulae involving characters and symmetric functions with applications and interpretations ranging from probability theory, combinatorics to geometry and topology. We shall touch upon some applications later in connection with intersection theory of Grassmann manifolds and Schubert's enumerative geometry and ramified coverings. For these reasons we give a brief discussion of character theory of $\mathcal{S}_n$ and $U(m)$.

Finally in this chapter we establish a simple relationship between characters of the unitary and symmetric groups and symmetric functions (example **??** below). To do so, we need some notation. For an element $\sigma \in \mathcal{S}_n$ let $[\sigma]$ denote its conjugacy class. Conjugacy classes in $\mathcal{S}_n$ are determined by partitions of n, however, it is judicious not to use the Young diagram notation T for denoting these partitions which parametrize conjugacy classes in $\mathcal{S}_n$. Let $X_1, \cdots X_m$ be indeterminates. For each conjugacy class $[\sigma]$ in $\mathcal{S}_n$ we define a homogeneous symmetric polynomial of degree n in X_j's which we denote by $\mathcal{Z}_{[\sigma]}(X)$. To do so we look at the cycle decomposition of $\sigma = \sigma_1\sigma_2\cdots\sigma_k$, and denote the length of the cycle σ_i by $|\sigma_i|$. Thus $|\sigma_1|+\cdots+|\sigma_k| = n$. For example, let $\sigma = (1)(2)(34)(56)(789) \in \mathcal{S}_9$, then $k = k(\sigma) = 5$, and $|(1)| = |(2)| = 1$, $|(34)| = |(56)| = 2$ and $|(789)| = 3$. Now set

$$\mathcal{Z}_{[\sigma]}(X) = \sum_{i_1,\cdots,i_k} X_{i_1}^{|\sigma_1|}X_{i_2}^{|\sigma_2|}\cdots X_{i_k}^{|\sigma_k|}, \tag{0.5.3.1}$$

where the summation is over all choices of $i_1, \cdots, i_k$ from $1, 2, \cdots, m$. In view of the summation $\mathcal{Z}_{[\sigma]}(X)$ depends only on the conjugacy class of σ. For example, for $\sigma = e = (1)(2)\cdots(n)$ we have

$$\mathcal{Z}_{[e]}(X) = \sum X_{i_1}\cdots X_{i_n} = (X_1 + X_2 + \cdots + X_m)^n.$$

For the other extreme case where $\sigma = (12\cdots n)$ we obtain

$$\mathcal{Z}_{[(12\cdots n)]}(X) = X_1^n + X_2^n + \cdots + X_m^n.$$

We can easily obtain a general formula for $\mathcal{Z}_{[\sigma]}$. To do so we introduce the power sum symmetric function

$$\mathcal{P}_k(X) = \sum_i X_i^k.$$

With the usual representation of the cycle decomposition of the conjugacy class $[\sigma]$ in form $1^\alpha 2^\beta 3^\gamma\cdots$ where $\alpha + 2\beta + 3\gamma + \cdots = n$, we obtain

$$\mathcal{Z}_{[\sigma]}(X) = (P_1(X))^\alpha(P_2(X))^\beta(P_3(X))^\gamma\cdots \tag{0.5.3.2}$$

Let $(g, \sigma) \in G \times \mathcal{S}_n$ where $G = U(m)$. As before let τ_n be the representation of $\mathcal{S}_n \times G$ on $T^n(V)$ discussed above, where $V = \mathbb{C}^m$. We compute trace of $\tau_n((g, \sigma))$ in two different ways. For simplicity of notation we write $\tau_n(g)$ for $\tau_n((g, e))$ and $\tau_n(\sigma)$ for $\tau_n(e, \sigma))$. Let g be a diagonal element of $U(n)$ with eigenvalues $e^{i\phi_1}, \cdots, e^{i\phi_n}$ relative to a fixed basis $e_1, \cdots, e_m$ for V and consider the basis for $T^n(V)$ consisting of all products of the form $e_{i_1} \otimes \cdots \otimes e_{i_n}$. Clearly $\tau_n(g)$ is diagonal relative to this basis. To compute $\mathrm{Tr}(\tau_n((g, \sigma)))$ we write $\tau_n((g, \sigma)) = \tau_n(\sigma)\tau_n(g)$. Now $\tau_n(\sigma)$ acts as a permutation matrix relative to this basis, and to compute $\mathrm{Tr}(\tau_n((g, \sigma)))$ (for g diagonal) we simply have to add the eigenvalues of $\tau_n(g)$ corresponding to eigenvectors which are left fixed by $\tau(\sigma)$. For example, for $\sigma = (1)(2)(34)(56)(789) \in \mathcal{S}_9$, the eigenvectors fixed by σ are precisely

$$\underbrace{e_{i_1}}_{1} \otimes \underbrace{e_{i_2}}_{1} \otimes \underbrace{e_{i_3} \otimes e_{i_3}}_{2} \otimes \underbrace{e_{i_4} \otimes e_{i_4}}_{2} \otimes \underbrace{e_{i_5} \otimes e_{i_5} \otimes e_{i_5}}_{3},$$

where i_j's are chosen arbitrarily from $1, 2, \cdots, m$ (repetitions allowed). Now let $X_1 = e^{i\phi_1}, \cdots, X_m = e^{i\phi_m}$, then it follows immediately that

$$\mathrm{Tr}(\tau_n((g, \sigma))) = \mathcal{Z}_{[\sigma]}(X). \tag{0.5.3.3}$$

This relation is valid for arbitrary $g \in U(m)$ by substituting the eigenvalues of g for $e^{i\phi_1}, \cdots, e^{i\phi_m}$. Next we express $\mathrm{Tr}(\tau_n((g, \sigma))$ by invoking theorem 0.5.2.2. Let $\wp_T$ denote the character of the irreducible representation ρ_T of G, and χ_T the character of the irreducible representation λ_T of $\mathcal{S}_n$. The functions $\wp_T$ are called *Schur functions*. Then theorem 0.5.2.2 implies

$$\mathrm{Tr}(\tau_n((g, \sigma))) = \sum_T \wp_T(g)\chi_T(\sigma), \tag{0.5.3.4}$$

where the summation is over all partitions of n (or Young diagrams T with n squares). In view of (0.5.3.3) and (0.5.3.4) we have proven the following proposition:

Proposition 0.5.3.1 *The following relation holds between characters of irreducible representation of $\mathcal{S}_n$ and those of the unitary group $U(m)$:*

$$\sum_T \wp_T(g)\chi_T(\sigma) = \mathcal{Z}_{[\sigma]}(X),$$

where for $X_1, \cdots, X_m$ on right hand side we substitute the eigenvalues of $g \in U(m)$.

For each multi-index $a = (a_1, \cdots, a_m) \in \mathbf{Z}_+^m$ and monomial $X_1^{a_1} \cdots X_n^{a_n}$, the expression

$$\sum_{\sigma \in \mathcal{S}_m} \epsilon_\sigma X_{\sigma(1)}^{a_1} X_{\sigma(2)}^{a_2} \cdots X_{\sigma(m)}^{a_m},$$

where $\epsilon_\sigma = \pm 1$ denotes the sign of the permutation σ, is an anti-symmetric polynomial in $X_1, \cdots, X_m$. It is clear that expressions of this form span the space of anti-symmetric polynomials. Therefore it vanishes unless all the integers $a_1, \cdots, a_m$ are distinct. It is no loss of generality to assume $a_1 > a_2 > \cdots > a_m$. It is convenient to introduce the multi-index $\delta = (m-1, m-2, \cdots, 1, 0)$ and define the partition (or Young diagram) $T : \nu = (n_1, \cdots, n_m)$, $n_1 + \cdots + n_m = n$, by

$$a = \nu + \delta, \quad n_k = a_k - m + k.$$

Then $n_j \geq n_{j-1}$. Define the anti-symmetric polynomial $\mathcal{W}^T$ as

$$\mathcal{W}^T = \sum_{\sigma \in \mathcal{S}_m} \epsilon_\sigma X_{\sigma(1)}^{n_1+m-1} X_{\sigma(2)}^{n_2+m-2} \cdots X_{\sigma(m)}^{n_m}. \tag{0.5.3.5}$$

For $n_1 = \cdots = n_m = 0$, we denote the resulting polynomial by $\mathcal{W}^\circ$. $\mathcal{W}^T$, being anti-symmetric in $X_1, \cdots, X_m$ is divisible by

$$\mathcal{W}^\circ = \prod_{j<k} (X_j - X_k).$$

Therefore the quotient $\frac{\mathcal{W}^T}{\mathcal{W}^\circ}$ is a symmetric polynomial in $X_1, \cdots, X_m$. The following property of the antisymmetric function $\mathcal{W}^\circ$ plays an important role in understanding the Schur functions $\wp_T$:

Lemma 0.5.3.1 *Setting $X_k = e^{i\phi_k}$ in $\mathcal{W}^\circ$ we obtain*

$$\int_\circ^{2\pi} \cdots \int_\circ^{2\pi} \mathcal{W}^\circ \overline{\mathcal{W}^\circ} d\phi_1 \cdots d\phi_m = (2\pi)^m m!$$

Proof - Expanding $\mathcal{W}^\circ$ after the substitution $X_k = e^{i\phi_k}$ we obtain

$$\mathcal{W}^\circ = \sum_{\sigma \in \mathcal{S}_m} \epsilon_\sigma e^{i[(m-1)\phi_{\sigma(1)} + (m-2)\phi_{\sigma(2)} + \cdots + \phi_{\sigma(m-1)}]},$$

which implies the required result. ♣

An immediate consequence of lemma 0.5.3.1 is the determination of the constant c in (0.2.3.9):

$$c = \frac{1}{2^m \pi^m m!}. \tag{0.5.3.6}$$

In preparation for relating $\mathcal{W}^T$ to Schur functions $\wp_T$, it is convenient to introduce a convention for the book-keeping of these expressions. The anti-symmetric expression $\mathcal{W}^T$ (0.5.3.5) may be written as

$$\mathcal{W}^T = X_1^{n_1+m-1} X_2^{n_2+m-2} \cdots X_m^{n_m} + \cdots$$

where only the lead term is displayed. When looking at a sum $\sum_{T'} a_{T'} \mathcal{W}^{T'}$, it is generally convenient to simply argue on the lead terms

$$\sum_{T'} a_{T'} X_1^{n_1+m-1} X_2^{n_2+m-2} \cdots X_m^{n_m},$$

since the coefficients of the remaining terms are determined by the anti-symmetry requirement. This convention is often helpful in keeping track of things.

The fundamental fact relating symmetric functions to the characters of the unitary group is the following special case of *Weyl Character Formula* (for $U(n)$):

Proposition 0.5.3.2 *The Schur function $\wp_T$ is given by*

$$\wp_T = \frac{\mathcal{W}^T}{\mathcal{W}^\circ},$$

where T is the Young diagram corresponding to the partition $n = n_1 + \cdots + n_m$.

Proof - Define $\mathcal{V}^T = \mathcal{W}^\circ \wp_T$. Then $\mathcal{V}^T$ is an antisymmetric function of $X_1, \cdots, X_m$. Therefore it is a linear combination of $\mathcal{W}^{T'}$ for Young diagrams T':

$$\mathcal{V}^T = \sum_{T'} c_{T'T} \mathcal{W}^{T'}.$$

Let $T^m \simeq U(1) \times \cdots \times U(1) \subset U(m)$ denote the subgroup of diagonal matrices with diagonal entries $e^{i\phi_k}$. The restriction of $\wp_T$ to T^m is a character. Since $\mathcal{S}_m \subset U(m)$ as permutation matrices, and conjugation by permutation matrices permutes the diagonal entries of T^m, $\wp_T$ is a symmetric function of $X_1, \cdots, X_m$. Substituting $e^{i\phi_k}$ for X_k in $\wp_T(X)$, we see that $\wp_T$ is a sum of exponentials $e^{il_k\phi_k}$ with positive coefficients, where $l_k \in \mathbf{Z}$, and $\wp_T$ is symmetric in the variables ϕ_k. Therefore $\mathcal{V}^T$ is an anti-symmetric polynomial in $e^{i\phi_k}$'s. Writing $\mathcal{V}^T$ in terms of lead terms we see that the coefficients $c_{T'T}$ are positive integers. Integrating and using (0.2.3.9) we obtain

$$\int_{U(m)} \wp_T \overline{\wp_T} dv_{U(m)} = \frac{1}{2^m \pi^m m!} \int_\circ^{2\pi} \cdots \int_\circ^{2\pi} \mathcal{V}^T \overline{\mathcal{V}^T} d\phi_1 \cdots d\phi_m = \sum_{T'} c_{T'T}^2.$$

The irreducibility of the representation corresponding to the Young diagram T implies

$$1 = \sum_{T'} c_{T'T}^2.$$

Therefore the sum has only one term and $\mathcal{V}^T = \mathcal{W}^T$ proving the proposition. ♣

Remark 0.5.3.1 In the current language of representation theory, the vector $\nu = (n_1, \cdots, n_m)$ is called the *highest weight* of the irreducible representation of $U(m)$ defined by the corresponding Young diagram. ♡

For any set of non-negative integers $b_1 > b_2 > \cdots > b_m$ we introduce the notation

$$\mathcal{D}(b_1, b_2, \cdots, b_m) = \prod_{j<k} (b_j - b_k) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{m-1} & b_2^{m-1} & \cdots & b_m^{m-1} \end{pmatrix}.$$

Corollary 0.5.3.1 *The dimension of the irreducible representation ρ_T of $U(m)$ determined by the Young diagram $T : n = n_1 + \cdots + n_m$ is*

$$\frac{\mathcal{D}(a_1, a_2, \cdots, a_m)}{\mathcal{D}(m-1, m-2, \cdots, 0)}$$

where $a_k = n_k + m - k$.

Proof - The dimension of a representation is equal to its character at e, however, the substitution $g = e \in U(m)$ (or $X_j = 1$) in proposition 0.5.3.2 leads to an indeterminacy $\frac{0}{0}$. To make sense out of the quotient $\frac{\mathcal{W}^T}{\mathcal{W}^\circ}$ we first make the substitution $\phi_k = (m-k)\phi$ and then let $\phi \to 0$. The substitution $\phi_k = (m-k)\phi$ leads to a van der Monde determinant and the required result follows immediately. ♣

Corollary 0.5.3.2 *The irreducible representation λ_T of $\mathcal{S}_n$ corresponding to the Young diagram $T : n = n_1 + n_2 + \cdots + n_m$ has dimension*

$$n! \frac{\mathcal{D}(a_1, a_2, \cdots, a_m)}{a_1! a_2! \cdots a_m!}$$

where $a_k = n_k + m - k$.

Proof - Substituting from proposition 0.5.3.2 in 0.5.3.1 we see that the dimension of the irreducible representation λ_T of $\mathcal{S}_n$ is the coefficient of $X_1^{a_1}X_2^{a_2}\cdots X_m^{a_m}$ in

$$\mathcal{W}^\circ\mathcal{Z}_{[e]}(X)=\sum_T \chi_T(e)\mathcal{W}^T,$$

or equivalently, in

$$(X_1+X_2+\cdots+X_m)^n\prod_{j<k}(X_j-X_k).$$

Now $\prod_{j<k}(X_j-X_k)$ is a homogeneous polynomial and a typical term is of the form

$$\epsilon X_1^{b_1}X_2^{b_2}\cdots X_m^{b_m},$$

where $(b_1,b_2,\cdots,,b_m)$ is a permutation of $(m-1,m-2,\cdots,0)$ and ϵ is the sign of this permutation. Therefore to obtain the required term in $\mathcal{W}^\circ\mathcal{Z}_{[e]}(X)$ we should multiply this term by ther term

$$\frac{n!}{(a_1-b_1)!(a_2-b_2)!\cdots(a_m-b_m)!}X_1^{a_1-b_1}X_2^{a_2-b_2}\cdots X_m^{a_m-b_m}$$

from $(X_1+X_2+\cdots+X_m)^n$. Therefore

$$\deg\lambda_T=\sum\epsilon\frac{n!}{(a_1-b_1)!(a_2-b_2)!\cdots(a_m-b_m)!}=n!\det\begin{pmatrix}\frac{1}{(a_1-m+1)!} & \frac{1}{(a_1-m+2)!} & \cdots & \frac{1}{a_1!}\\ \frac{1}{(a_2-m+1)!} & \frac{1}{(a_2-m+2)!} & \cdots & \frac{1}{a_2!}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{(a_m-m+1)!} & \frac{1}{(a_m-m+2)!} & \cdots & \frac{1}{a_m!}\end{pmatrix}$$

It is a simple calculation that the above determinant is equal to

$$\frac{1}{a_1!a_2!\cdots a_m!}\det\begin{pmatrix}a_1\cdots(a_1-m+2) & a_1\cdots(a_1-m+3) & \cdots & a_1(a_1-1) & a_1 & 1\\ a_2\cdots(a_2-m+2) & a_1\cdots(a_2-m+3) & \cdots & a_2(a_2-1) & a_2 & 1\\ \cdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ a_m\cdots(a_m-m+2) & a_m\cdots(a_m-m+3) & \cdots & a_m(a_m-1) & a_m & 1\end{pmatrix}.$$

The latter determinant is equal to $\mathcal{D}(a_1,\cdots,a_m)$, whence the required result. ♣

From propositions 0.5.3.1 and 0.5.3.2 we obtain

$$\mathcal{W}^\circ\mathcal{Z}_{[\sigma]}(X)=\sum_T \chi_T(\sigma)\mathcal{W}^T, \tag{0.5.3.7}$$

which is a form useful for applications. In fact we can use (0.5.3.7) to give recursion fomulae and even explicitly calculate characters of irreducible representations of the symmetric group. Let $\sigma \in \mathcal{S}_n$ have cycle decomposition $1^{\alpha_1}2^{\alpha_2}\cdots$ and σ' be the permutation obtained from σ by removing one of the cycles. For example if we remove one cycle of length l from σ then σ' will have cycle structure $1^{\beta_1}2^{\beta_2}\cdots$ where

$$\beta_i = \alpha_i \quad \text{for} \quad i \neq l, \quad \text{and} \quad \beta_l = \alpha_l - 1.$$

We want to express $\chi_T(\sigma)$ in terms of $\chi_{T'}(\sigma')$'s. The symmetric function $\mathcal{Z}_{[\sigma]}(X)$ on the left hand side (0.5.3.7) can be written in the form

$$\mathcal{Z}_{[\sigma]}(X) = \mathcal{Z}_{[\sigma']}(X)(X_1^l + X_2^l + \cdots + X_m^l). \tag{0.5.3.8}$$

Now $\chi_T(\sigma)$ is the coefficient of the antisymmetric polynomial on the right hand side of (0.5.3.7) with lead term $X_1^{a_1}x_2^{a_2}\cdots X_m^{a_m}$ where we recall that $a_j = n_j + m - j$. By applying (0.5.3.7) to $\mathcal{Z}_{[\sigma']}$ on the left hand side of (0.5.3.7) and using (0.5.3.8) we conclude that $\chi_T(\sigma)$ is expressible in terms of $\chi_{T'}(\sigma')$'s with T' ranging over Young diagrams T' obtained from T by removing l squares from a row. To carry this out and obtain the desired recursive formula, we seek anti-symmetric polynomials on the right hand side of

$$\mathcal{W}^\circ \mathcal{Z}_{[\sigma']} = \sum_{T'} \chi_{T'}(\sigma')\mathcal{W}^{T'}$$

containing a monomial

$$X_1^{b_1}X_2^{b_2}\cdots X_m^{b_m}$$

where for all but one index i, $b_j = a_j$ and $b_i = a_i - l$. This monomial may not appear with + sign in $\mathcal{W}^{T'}$ since the integers $b_1, \cdots, b_m$ are not necessarily in decreasing order. To determine whether or not this monomial appears and its sign if it does, it is clear that we simply have to follow the following rule:

1. If for all j, $b_j > b_{j-1}$, then the monomial appears with + sign;
2. If $b_j = b_k$, $j \neq k$, then the monomial does not appear in view of the antisymmetry;
3. If $b_j < 0$, then the monomial does not appear;
4. If $b_j < b_{j-1}$ neither rule (2) nor (3) is applicable, then move j^{th} row of the diagram down k rows until the numbers appear in decreasing order. The sign is $(-1)^k$.

To illustrate this let us consider a simple example.

Example 0.5.3.1 Consider the representation of $\mathcal{S}_{14}$ corresponding to the partition $14 = 5+3+3+2+1$, σ a permutation containing a 3-cycle, and σ' differ from σ by the deletion of a 3-cycle. The integers $a_1, \cdots, a_5$ are

$$a_1 = 9, \quad a_2 = 6, \quad a_3 = 5, \quad a_2 = 2, \quad a_1 = 1.$$

There are five ways of subtracting 3 from $a_1, \cdots, a_5$, but according to rules (2) and (3) three of them do not make any contribution, viz.,

$$(b_1, b_2, b_3, b_4, b_5) = (6,6,5,3,1), \ (9,3,5,3,1), \ (9,6,5,3,-2).$$

The other two terms appear with negative sign since

$$(9,6,2,3,1) \to -(9,6,3,2,1), \quad (9,6,5,0,1) \to -(9,6,5,1,0).$$

The partitions corresponding to the the above values for $(b_1, \cdots, b_5)$ are $T' : 11 = 5+3+1+1+1$ and $T'' : 11 = 5+3+3$, and therefore

$$\chi_T(\sigma) = -\chi_{T'}(\sigma') - \chi_{T''}(\sigma')$$

which reduces the calculation of the character from $\mathcal{S}_{14}$ to $\mathcal{S}_{11}$. ♠

An immediate consequence of the above reduction procedure is the *Branching Law*:

Corollary 0.5.3.3 *Let $T : n = n_1 + \cdots + n_m$, and we may assume $n_m > 0$. The representation λ_T of $\mathcal{S}_n$ when restricted to $\mathcal{S}_{n-1}$ is the direct sum of representations $\lambda_{T'}$ of $\mathcal{S}_{n-1}$ where $T' : n-1 = n'_1 + \cdots n'_m$ and*

$$n'_1 \geq n'_2 \cdots \geq \cdots \geq n'_m, \quad 0 \leq n_j - n'_j \leq 1, \quad \text{for all } j \leq m.$$

Each such representation occurs exactly once.

Proof - If $\sigma \in \mathcal{S}_{n-1} \subset \mathcal{S}_n$, then the cycle decomposition of σ as an element of $\mathcal{S}_n$ contains at least one 1-cycle. The required result follows from the application of the the above procedure for the deletion of a 1-cycle. ♣

The Branching Law is easily remembered if one draws a picture of the Young diagram to see which squares can be eliminated. For instance, the restriction of the representation corresponding to the partition $9 = 4+2+2+1$ to of $\mathcal{S}_9$ to $\mathcal{S}_8$ is the direct sum of the representations of $\mathcal{S}_8$ corresponding to the partitions

$$8 = 3+2+2+1, \quad 8 = 4+2+1+1, \quad 8 = 4+2+2.$$

The reduction procedure can be effectively used to calculate characters of representations of the symmetric group. A complete discusssion is not relevant to this context, and we briefly indicate its application to the case of a 2-cycle to demonstrate the principle.

Example 0.5.3.2 To derive a formula for the characters of the irreducible representations of $\mathcal{S}_n$ at a transposition we make use of the reduction procedure by removing a cycle. Since the cycle structure of a transposition is $1^{n-2}2^1$, the result is in terms of the degrees of irreducible representations of $\mathcal{S}_{n-2}$ and it admits of simplifications. The left hand side of (0.5.3.7) is

$$\mathcal{W}^\circ(X_1+\cdots+X_m)^{n-2}(X_1^2+\cdots+X_m^2).$$

Applying the reduction for removal of a 2-cycle and corollary 0.5.3.2 we obtain

$$\chi_T(\sigma)=(n-2)!\sum_{j=1}^{m}\frac{\mathcal{D}(a_1,\cdots,a_j-2,\cdots,a_m)}{a_1!\cdots(a_j-2)!\cdots a_m!},\tag{0.5.3.9}$$

where certain terms may become 0 in accordance with rules (2) and (3). Therefore

$$\frac{\chi_T(\sigma)}{\chi_T(e)}=\frac{1}{n(n-1)}\sum_{j=1}^{m}\frac{a_j(a_j-1)\mathcal{D}(a_1,\cdots,a_j-2,\cdots,a_m)}{\mathcal{D}(a_1,\cdots,a_m)}.\tag{0.5.3.10}$$

Now we regard the integers $a_1,\cdots,a_m$ as indeterminates. The denominator in (0.5.3.10) is an anti-symmetric function of $a_1,\cdots,a_m$ and is independent of j. It is not difficult to see that the sum of the terms in the numerator is also an anti-symmetric function of $a_1,\cdots,a_m$. Therefore $\frac{\chi_T(\sigma)}{\chi_T(e)}$ is a symmetric polynomial of degree 2 in $a_1,\cdots,a_m$:

$$\chi_T(\sigma)=\chi_T(e)\Big[C_1\sum a_j^2+C_2\sum a_ja_k+C_3\sum a_j+C_4\Big].$$

It is straighforward, though somewhat tedious, to evaluate the constants C_i by looking at the coefficients of corresponding terms on both sides of the equation. We obtain after some algebraic manipulations

$$C_1=1,\ C_2=0,\ C_3=-(2m-1),\ C_4=\frac{m(m-1)(2m-1)}{3}.$$

This leads to the formula

$$\chi_T(\sigma)=\frac{\deg\lambda_T}{n(n-1)}\Big[\sum_{j=1}^{m}n_j(n_j+1)-\sum_{j=1}^{m}jn_j\Big]\tag{0.5.3.11}$$

for the character of the representation λ_T at a transposition σ. ♠

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