Chapter 3

(CO)HOMOLOGY AND CRITICAL POINT THEORY

3.1 Basic Notions of Homology

3.1.1 Simplicial Theory

Let $\{e_0, \cdots, e_m\}$ be the standard basis for $\mathbf{R}^{m+1}$. By the *standard m-simplex* $\Delta(m)$ we mean the convex closure of the finite set $\{e_0, \cdots, e_m\}$, and an *m-simplex* is any homeomorphic image of the standard m-simplex. Simplicial theory is based on the idea that many objects of geometric interest may be decomposed into a union of m-simplices, or vaguely speaking are simpicial complexes (formal definition is given below). Intuitively this means that the topological space X can decomposed as $X = \cup_\alpha \Delta_\alpha$ where Δ_α is an m_α-simplex and $\Delta_\alpha \cap \Delta_\beta$ is either empty or is a simplex. For example, it is geometrically evident that every surface can be decomposed as a union of small triangles such that the above condition is satisfied. Notice that the set of simplices for such a decomposition consists of the triangles, their edges and their vertices. More generally, every smooth manifold can be triangulated, i.e. is a simplicial complex. In fact according to a fundamental theorem of J. H. C. Whitehead not only every manifold possibly with boundary has the structure of a simplicial complex, any triangulation of the boundary can be extended to a triangulation of the entire manifold. We shall not discuss the proof of this basic theorem, but will make extensive use of it. Let us note the nontrivial fact that not every topological manifold can be triangulated and therefore the assumption of smoothness is essential.

Formally, a *simplicial complex* $X = X_{\mathcal{V}}$ consists of a set $\mathcal{V} = \{v\}$ called *vertices* and a class of finite subsets $\mathcal{S} = \{s\}$ called *simplices* with the following properties:

1. Any set consisting of a single vertex is a simplex;

2. A non-empty subset of a simplex is a simplex.

It is trivial that the intuitive notion of simplicial complex satisfies these axioms. In view of axiom 2, the intersection of any two simplices is a simplex or is empty.

It is important to remember that the product of two simplices is not a simplex, but has the structure of a simplicial complex (in many different ways). The simplest case is that of the square $I \times I$ which is not a simplex, but by drawing one diagonal it becomes a simplicial complex. It is not difficult to convince oneself that one can proceed inductively and endow the product $\Delta(p) \times \Delta(q)$ with the structure of a simplicial complex. Nevertheless, the following example gives a formal description of the a simplical complex to $\Delta(p) \times \Delta(q)$.

Example 3.1.1 We use our knowledge of the symmetric group $\mathcal{S}_m$, to assign the structure of a simplicial complex to $\Delta(p) \times \Delta(q)$ which is suitable for the analysis of the product structure. To do so it is convenient to start with the product $\Box(m) = \Delta(1) \times \Delta(1) \times \cdots \times \Delta(1)$ of m copies of $\Delta(1)$. Identifying $\Delta(1)$ with the unit interval $I \subset \mathbf{R}$, we see that $\Box(m)$ is the unit hypercube in $\mathbf{R}^m$. For $\sigma \in \mathcal{S}_m$ consider the subset $\Box(m,\sigma) \subset \Box(m)$ defined by the inequalities

$$0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(m)} \leq 1.$$

Each $\Box(m,\sigma)$ is an m-simplex and

$$\Box(m) = \bigcup_{\sigma \in \mathcal{S}_m} \Box(m,\sigma) \tag{3.1.1}$$

gives $\Box(m)$ the structure of a simplicial complex. Next we use (3.1.1) to obtain an explicit decomposition of $\Delta(p) \times \Delta(q)$ as a simplicial complex. Let $p+q=m$ then $\Delta(p) \times \Delta(q) \subset \Box(m)$ and we use the structure $\Box(m)$ as a simplicial complex to triangulate $\Delta(p) \times \Delta(q)$. We identify $\Delta(p)$ and $\Delta(q)$ with the simplices

$$\Delta(p) = \{(x_1, \cdots, x_m) | 0 \leq x_1 \leq \cdots x_p \leq 1 = x_{p+1} = \cdots = x_m\},$$
$$\Delta(q) = \{(x_1, \cdots, x_m) | x_1 = \cdots = x_p = 0 \leq x_{p+1} \leq \cdots x_m \leq 1\},$$

Therefore $\Delta(p) \times \Delta(q)$ is identified with the subset of $\Box(m)$ defined by

$$\Delta(p) \times \Delta(q) = \{(x_1, \cdots, x_m) \in \Box(m) | x_1 \leq \cdots \leq x_p; \text{ and } x_{p+1} \leq \cdots \leq x_m\}.$$

It follows that a simplex $\Box(m,\sigma) \subset \Delta(p) \times \Delta(q)$ if and only if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p); \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(m). \tag{3.1.2}$$

Since an open simplex $\mathring{\Box}(m,\sigma)$ is either contained in $\Delta(p) \times \Delta(q)$ or is disjoint from it, we have obtained the desired triangulation of $\Delta(p) \times \Delta(q)$. We noted earlier that permutations satisfying (3.1.2) form a complete set of coset representatives for $\mathcal{S}_p \times \mathcal{S}_q \subset \mathcal{S}_m$. Therefore

$$\Delta(p) \times \Delta(q) = \bigcup_{\sigma \in \mathcal{S}_m / \mathcal{S}_p \times \mathcal{S}_q} \Box(m,\sigma) \tag{3.1.3}$$

for the particular choice of coset representatives. Clearly this argument generalizes to give, for $p_1 + \cdots + p_r = m$, the triangulation

$$\Delta(p_1) \times \cdots \times \Delta(p_r) = \bigcup_{\sigma \in \mathcal{S}_m / \mathcal{S}_{p_1} \times \cdots \times \mathcal{S}_{p_r}} \Box(m,\sigma) \tag{3.1.4}$$

of $\Delta(p_1) \times \cdots \times \Delta(p_r)$. ♠

To a simplicial complex is associated a topological space in a natural and intuitive fashion. A simple way of topologizing a finite, (i.e. finitely many vertices), simplicial complex with N vertices is to regard the unit vectors along the co-ordinate axes in $\mathbf{R}^N$ as the vertices, joining $v \in \mathcal{V}$ and $v' \in \mathcal{V}$ by a straight line if $\{v, v'\}$ is one of the simplices, taking convex closure of $v, v', v'' \in \mathcal{V}$ if $\{v, v', v''\}$ is a simplex, etc. for higher dimensional simplices. This embeds the given simplicial complex as a subcomplex of a standard simplex (a *subcomplex* of a simplicial is a subset $\mathcal{V}' \subseteq \mathcal{V}$ such that if $s' \subseteq \mathcal{V}'$ is a simplex of $X_{\mathcal{V}'}$ then it is a simplex as a subset of $X_{\mathcal{V}}$. It is a simple matter to extend this method of topologizing a finite simplicial complex to one satisfying some local finiteness condition, but we shall not dwell on this point since the topologies of the simplicial complexes we will encounter are unambiguously defined. In practice we often use the same notation X or $X_{\mathcal{V}}$ to denote a simplicial complex and the associated topological space. When it is becomes convenient to distinguish between a simplicial complex and the associated topological space we denote the latter by $|X|$ or $|X_{\mathcal{V}}|$.

By a *simplicial map* $f : X_{\mathcal{V}} \rightarrow Y_{\mathcal{U}}$ we mean a mapping $f : \mathcal{V} \rightarrow \mathcal{U}$ taking simplices to simplices. A *subcomplex* $A \subset X$ is a simplicial complex A such that

1. The vertex set $\mathcal{V}_A \subset \mathcal{V}_X$;
2. Every simplex of A is also a simplex of X.

Let us explain the useful notion of *barycentric subdivision* $\mathsf{Sd}(X_{\mathcal{V}})$ of a simplicial complex. Take a point p in the interior of the standard 1-simplex $V = \Delta(1)$, e.g. $p = (1/2, 1/2)$ on the line segment joining $(1, 0)$ to $(0, 1)$. Define $\mathsf{Sd}(\Delta(1))$ to be the simplicial complex with the set vertices $\mathcal{V}' = \{(1, 0), (1/2, 1/2), (0, 1)\}$, and the 1-simplices to be the subsets $\{(1, 0), (1/2, 1/2)\}$ and $\{(1/2, 1/2), (0, 1)\}$. To obtain the barycentric subdivision of $V = \Delta(2)$ we first subdivide the boundary which consists of three 1-simplices by adding vertices p_1, p_2 and p_3 to the 1-simplices (v_1, v_2), (v_2, v_3) and (v_3, v_1) repectively, and obtaining six 1-simplices (v_1, p_1), (p_1, v_2) etc. as decribed above. Now add a point q to the interior of $\Delta(2)$, e.g., $q = (1/3, 1/3, 1/3)$. We obtain a simplicial complex with vertices $\{v_1, p_1, v_2, p_2, v_3, p_3, q\}$, 1-simplices

$$\{(v_1, p_1), (p_1, v_2), (v_2, p_2), (p_2, v_3), (v_3, p_3), (p_3, v_1), (q, v_1), (q, p_1), (q, v_2), (q, p_2), (q, v_3), (q, p_3)\},$$

and set of 2-simplices

$$\{(q, v_1, p_1), (q, p_1, v_2), (q, v_2, p_2), (q, p_2, v_3), (q, v_3, p_3), (q, p_3, v_1)\}.$$

Figure 1.1 demonstrates the procedure. It is now clear how to proceed to obtain barycentric subdivision of higher dimensional simplices and simplicial complexes in general. We start by 1-simplices, subdivide them, next go to 2-simplices subdivide them, then to 3-simplices etc. To give a more description to the barycentric subdivision $X^{(1)}_{\mathcal{V}^{(1)}}$, we let the vertex set to be in bijection with the simplices of X. It is convenient to enumerate the the vertices of $X^{(1)}$ as

$$\begin{aligned} v_1^\circ, \cdots, v_{n_\circ}^\circ; \\ v_1^1, \cdots, v_{n_1}^1; \\ \cdots; \\ v_1^m, \cdots, v_{n_m}^m, \end{aligned}$$

where $v_1^k, \cdots, v_{n_k}^k$ runs over k-simplices of X. We denote the k-simplex of X corresponding to $v_j^{(k)}$ by $\nu_j^{(k)}$ The l-simplices of $X^{(1)}$ are given by sequences

$$[v_{i_\circ}^{(p_\circ)}, v_{i_1}^{(p_1)}, \cdots, v_{i_l}^{(p_l)}],$$

such that

1. $p_\circ < p_1 < \cdots < p_l$;
2. There is a simplex a simplex s of X containing the simplices $\nu_{i_\circ}^{(p_\circ)}$, $\nu_{i_1}^{(p_1)}$, $\cdots$, and $\nu_{i_l}^{(p_l)}$.

It is clear that this construction is the formal description of the intuitive description given above. A more succint reformulation of barycentric subdivision is given in exercise **??** in the discussion of posets.

Besides, barycentric subdivision there are other ways of subdividing a simplicial complex into a *finer* simplicial complex in teh sense that every simplex of the old complex is a union of simplices of the new complex. For example, We can join a point in the interior of a triangle to its vertices to obtain a simplicial complex with four vertices, six edges and three 2-simplices. It is straightforward to cosntruct many other examples.

To define homology theory of simplicial complexes we need an additional structure. For example we can consider ordered simplices, i.e., distinguishing between a simplex $[v_\circ, \cdots, v_k]$ and one obtained by a non-trivial permutation of the vertices. However it is more convenient to consider oriented simplices. This means we distinguish between $[v_\circ, \cdots, v_k]$ and $[v_{\sigma(\circ)}, \cdots, v_{\sigma(k)}]$ if and only if σ is an odd permutation, in which case $[v_\circ, \cdots, v_k] = -[v_{\sigma(\circ)}, \cdots, v_{\sigma(k)}]$. Let $C_k(X; R)$ be the free R-module on the set of oriented k-simplices of $X = X_{\mathcal{V}}$ with the proviso that a simplex and one differing from it by odd permutation of the vertices differ by -1. We define the boundary operator $\partial = \partial_n : C_n(X; R) \to C_{n-1}(X; R)$ by

$$\partial_n[v_\circ, \cdots, v_n] = \sum (-1)^i [v_\circ, \cdots, \hat{v}_i, \cdots, v_n], \tag{3.1.5}$$

where $\hat{v}_i$ means v_i is omitted. We have

Lemma 3.1.1 *With the above notation*

$$\partial_{n-1}\partial_n = 0.$$

Proof - For a simplex $\Delta = [v_\circ, \cdots, v_n]$ let $\Delta_i = [v_\circ, \cdots, \hat{v}_i, \cdots, v_n]$ (called i^{th} *face of* Δ). Since the boundary of $\partial(\Delta(n))$ is $\sum(-1)^i \Delta_i(n)$, $\partial\partial\Delta(n)$ is a linear combination of the simplices $\Delta_{ij}(n)$ where Δ_{ij} is the convex closure of $\{v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n\}$. The terms in $\partial\Delta(n)$ contributing to Δ_{ij} in $\partial\partial\Delta(n)$ are $\Delta_i(n)$ and $\Delta_j(n)$. Therefore the coefficient of Δ_{ij} in $\partial\partial\Delta(n)$ is $(-1)^{i+j-1} + (-1)^{i+j} = 0$, proving (3.1.1). ♣

We have the following sequence of R-modules and (boundary) homomorphisms:

$$\cdots \longrightarrow C_n(X; R) \xrightarrow{\partial_n} C_{n-1}(X; R) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_\circ(X; R) \longrightarrow 0. \tag{3.1.6}$$

In view of (3.1.1) $\mathrm{Im}\partial_{n+1} \subseteq \ker \partial_n$. We now define the homology of X in dimension n by

$$H_n(X; R) = \frac{\ker \partial_n}{\mathrm{Im}\partial_{n+1}}. \tag{3.1.7}$$

The elements of $C_n(X;R)$ are called *(simplicial) chains*. It is customary to write $Z_n(X;R) = \ker \partial_n$ and $B_n(X;R) = \mathrm{Im}\partial_{n+1}$ and refer to their elements as *(simplicial) cycles* and *(simplicial) boundaries*.

If we replace a simplicail complex by a subdivision of it, the R-modules $C_n(X;R)$, $Z_n(X;R)$ and $B_n(X;R)$ will change. However, the homology groups $H_n(X;R)$ will not change. Even more generally, the homology groups of a space, realizable as a simplicial complex, do not depend on the particular choice of the simplicial structure (or triangulation). This is by no means obvious, and we shall return to this issue later. For the time being, we will make use of this basic fact. In particular if a space X admits of a triangulation, we simply write $H_n(X;R)$ regardless of the choice of triangulation for X.

Example 3.1.2 For a simplicial complex X, the R-module $H_\circ(X;R)$ has a simple geometric intrepretation. In fact we show

$$H_\circ(X;R) \simeq \underbrace{R \oplus \cdots \oplus R}_{k \text{ copies}}$$

where k is the number of (path) components of X as a topological space. Let $X = X_1 \cup \cdots \cup X_k$ be the decomposition of X into path components. Then a simplicial 0-chain (necessarily a zero cycle) is of the form $c = \sum_j \sum_i a_i^j c_i^j(1)$ where $c_i^j(1) \in X_j$ is a vertex. Since $\partial\Delta(1) = (0,1) - (1,0)$, the 0-chain c is a boundary if and only if for all j, $\sum_i a_i^j = 0$. The claim follows immediately. ♠

Remark 3.1.1 To simplify notation we write $H_j(X)$ instead of $H_j(X;\mathbf{Z})$, i.e., unless the coefficient ring (or module) is specified to the contrary, it is assumed to be $\mathbf{Z}$.

Example 3.1.3 Let us compute the homology of the circle where it is given the structure of a simplicail complex by taking $v_\circ, v_1, v_2$ to be three distinct points moving counterclockwise on the circle (see Figure 1.2), and the edges being $[v_\circ, v_1]$, $[v_1, v_2]$, and $[v_2, v_\circ]$. Therefore the chain groups $C_\circ(S^1)$ and $C_1(S^1)$ are isomorphic to $\mathbf{Z}^3$, and the homology groups are computed from the sequence

$$0 \longrightarrow \mathbf{Z}^3 \xrightarrow{\partial} \mathbf{Z}^3 \longrightarrow 0,$$

with the boundary homomorphism given by the matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

It is now a simple exercise in linear algebra over $\mathbf{Z}$ that the homology groups of the circle are $H_\circ(S^1) \simeq \mathbf{Z} \simeq H_1(S^1)$ and $H_j(S^1) = 0$ for $j \neq 0, 1$.

Exercise 3.1.1 *By explicit triangulation of the sphere S^2 show that its homology is $\mathbf{Z}$ in dimensions 0 and 2 and vanishes in all other dimensions.*

Exercise 3.1.2 *What is wrong with the triangulation of the torus T^2 given in Figure 1.3? Using a correct triangulation show that $H_\circ(T^2) \simeq \mathbf{Z} \simeq H_2(T^2)$, $H_1(T^2) \simeq \mathbf{Z}^2$, and $H_j(T^2) = 0$ for $j \geq 3$.* (We will explain efficient methods for calculating homology groups. The complexity of this elementary calculation explains the necessity for devising better machinery for such calculations.)

Exercise 3.1.3 *Show that the homology of two circles joined at a point (i.e. figure ∞) is $\mathbf{Z}\oplus\mathbf{Z}$ in dimension 1. More generally, show that if $X \vee Y$ is two simplicial complexes X and Y joined at one vertex, then for $j > 0$*

$$H_j(X \vee Y; R) = H_j(X; R) \oplus H_j(Y; R).$$

For reasons that will become clear later it is essential to extend the notion of homology to the *relative case*, i.e., define $H_n(X, A; R)$ whee X is a simplicial complex and $A \subset X$ is a subcomplex. Then by construction $C_n(A; R) \subseteq C_n(X; R)$, and we define

$$C_n(X, A; R) = \frac{C_n(X; R)}{C_n(A; R)}.$$

Since $\partial_{n+1}(C_{n+1}(A; R)) \subseteq C_n(A; R)$, we have the induced boundary map $\partial_{n+1} : C_{n+1}(X, A; R) \to C_n(X, A; R)$ which we are denoting by the same letter ∂_{n+1} It clearly satisfies (3.1.1). We refer to $C_n(X, A; R)$ as the R-module of *relative chains*. Similarly the R-modules *relative cycles* and *boundaries* are defined as

$$Z_n(X, A; R) = \ker\partial_n, \qquad B_n(X, A; R) = \mathrm{Im}\partial_{n+1}.$$

We define $H_n(X, A; R)$ as

$$H_n(X, A; R) = \frac{Z_n(X, A; R)}{B_n(X, A; R)}.$$

(3.1.7) as before.

Exercise 3.1.4 *Let X be the disc $B^2 \subset \mathbf{R}^2$ and $A = \partial B^2$ be its boundary. By realizing B^2 and a simplicial complex and A as a subcomplex show that*

$$H_1(X, A; R) \simeq R, \quad H_j(X, A; R) = 0 \ \text{ for } j \neq 1.$$

Let X and Y be simplicial complexes and $f : X \to Y$ be a simplicial map. Therefore it gives R-module homomorphisms

$$f_n : C_n(X; R) \longrightarrow C_n(Y; R).$$

Then f_n's commutes with the boundary operators ∂_n, i.e., $\partial_n^Y f_n = f_{n-1}\partial_n^X$ where the superscripts ^{X} or ^{Y} refer to the simplicial complex relative to which the boundary operator is defined. It follows that f induces a homomorphism

$$f_\star = f_{n\star} : H_n(X; R) \longrightarrow H_n(Y; R).$$

Similar considerations apply to simplicial maps of pairs of simplicial complexes $f : (X, A) \to (Y, B)$. This means that $A \subset X$ and $B \subset Y$ are subcomplexes, f is a simplicial map of X to Y mapping A to B, and we have induced maps

$$f_\star = f_{n\star} : H_n(X, A; R) \longrightarrow H_n(Y, B; R).$$

In reality continuous maps between manifolds or topological spaces (admitting of triangulations) are not given as simplicial maps. A general devise for attaching an induced map on homology groups (or modules) is by approximating the given continuous with a simplicial map. The following theorem explains this approximation method:

Theorem 3.1.1 *(*Simplicial Approximation Theorem*) - Let $f : |X_{\mathcal{V}}| \to |Y_{\mathcal{U}}|$ be a continuous map of topological spaces associated to simplicial complexes. Then after passing to a finer subdivisions of X and Y, we can approximate f by a simplicial map f' (see below for explanation). f and f' are homotopic.*

Let us explain the meaning of this theorem. We say a simplicial map f' *approximates* or is *simplicial approximation* to f if $f(x) \in |s|$, where s is a simplex of $Y_{\mathcal{U}}$, implies $|f'|(x) \in |s|$. The proof of theorem 1.1 actually gives more than the statement above. In fact, it shows that there is a homotopy $F : |X| \times I \to |Y|$ between f and the map $|f'|$ such that if $f(x) \in s$ then throughout the homotopy the image of x lies in s, i.e., $F(x,t) \in s$ for all $t \in I$. The fact that f and $|f'|$ are homotopic implies that at the level of homology the induced maps are identical. Therefore this theorem means that in dealing with homology we can treat continuous maps of topological spaces, with implicit structure of simplicial complexes, as if they were maps of simplicial complexes without further ado. We shall omit the proof of this basic theorem since its proof is not useful for our purposes.

3.1.2 The Axioms and Singular Theory

In order to make more efficient use of homology theory it is useful to introduce it axiomatically. First we need some definitions. Let R be a commutative ring with identity. For our purposes, it often suffices to take R to be $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}$, or $\mathbf{Z}/p$. Let $A_1,\ A_2, \cdots$ be R-modules and $g_i : A_i \to A_{i-1}$ a homomorphism. We say the sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{g_{i+1}} A_i \xrightarrow{g_i} A_{i-1} \xrightarrow{g_{i-1}} \cdots$$

is *exact* if $\ker g_i = \mathrm{Im} g_{i+1}$ for all i. In particular, the sequence $0 \to A \xrightarrow{g} B \to 0$ is exact if and only if $g : A \to B$ is an isomorphism. An exact sequence of the form $0 \to A \to B \to C \to 0$ is called a *short* exact sequence. For such a sequence we may regard A as a submodule of B, and C is isomorphic to B/A.

We let (X, A) denote a pair of Hausdorf topological spaces with $A \subseteq X$. By a *homology theory* we mean an assignment of abelian groups $H_i(X, A)$ to every pair (X, A) and integer $i \geq 0$, and homomorphism $H_i(f) = f_\star : H_i(X, A) \to H_i(Y, B)$ (called *induced homomorphism*) to every continuous map $f : (X, A) \to (Y, B)$ such that the following conditions are satisfied:

1. **Covariance** - If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$, then $H_i(g \cdot f) = H_i(g) \cdot H_i(f)$;
2. **Homotopy** - If $f, g : (X, A) \to (Y, B)$ are homotopic, then the induced maps $H_i(f)$ and $H_i(g)$ are identical;
3. **Long Exact Sequence** - There is the exact sequence

$$\cdots \to H_i(A) \to H_i(X) \to H_i(X, A) \xrightarrow{\delta_i} H_{i-1}(A) \to \cdots \to H_\circ(X) \to H_\circ(X, A),$$

where the maps $H_i(A) \to H_i(X)$ and $H_i(X) \to H_i(X,A)$ are induced by the inclusions $A \hookrightarrow X$ and $(X,\emptyset) \hookrightarrow (X,A)$. The map δ_i is called the *connecting homomorphism*;

4. **Excision** - If $U \subseteq X$ is an open subset such that $\bar{U} \subseteq \breve{A}$, then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X,A)$ induces isomorphisms $H_i(X \setminus U, A \setminus U) \simeq H_i(X,A)$. (For simplicial complexes this axiom can be rephrased as follows: If $f : (X,A) \to (Y,B)$ is a map of pairs of simplicial complexes such that the restriction of f to the closure of $|X|\backslash|A|$ is a homeomorphism onto the closure of $|Y|\backslash|B|$, then $f_\star : H_j(X,A) \to H_j(Y,B)$ is an isomorphism.)

5. **Naturality** - For a continuous map $f : (X,A) \to (Y,B)$ the following diagram commutes:

$$\begin{array}{ccccccccccc} \cdots & \to & H_i(A) & \to & H_i(X) & \to & H_i(X,A) & \to & H_{i-1}(A) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H_i(B) & \to & H_i(Y) & \to & H_i(Y,B) & \to & H_{i-1}(B) & \to & \cdots \end{array}$$

(*commutes* means composition of maps starting from the same abelian group and ending at same abelian group are identical.)

6. **Normalization** - For a single point $\{p\}$, $H_i(\{p\}) = 0$ for $i > 0$ and $H_\circ(\{p\}) = \mathbf{Z}$, and $H_i(\text{id.}) = \text{id.}$.

Remark 3.1.2 We have followed the tradition of stating the (Eilenberg-Steenrod) axioms for $\mathbf{Z}$. In practice it becomes important to allow $H_\circ(\{p\})$ (called *coefficient group*) to be an abelian group other $\mathbf{Z}$ or be an R-module. Later we discuss algebraic methods for relating homology theories with different coefficient groups. The multiplicative structure of the ring R enters into the multiplicative structure of cohomology which plays a very important role. To clarify the role of the coefficient group (or ring) we modify the normalization axiom by setting $H_\circ(\{p\}; R) = R$. All homomorphisms in this case will be R-module homomorphisms. The general construction of homology theory (both simplicail and singular described below) is for any commutative ring R with identity.

We will shortly discuss the validity of the axioms. For the time being we simply assume the existence and uniqueness of such theory.

Exercise 3.1.5 *Show that if A is a deformation retract of X, then the homology groups of X and A are isomorphic. If $f : X \to Y$ is a homotopy equivalence, then the induced maps $f_\star : H_i(X) \to H_i(Y)$ are isomorphisms.*

When $R = \mathbf{R}$, the homology groups are vector spaces, and we set $b_i(X) = \dim H_i(X; \mathbf{R})$ which are called *Betti numbers* of X. The alternating sum of the Betti numbers, whenever defined, is called the *Euler characteristic* of X and is denoted by

$$\chi(X) = \sum (-1)^i b_i(X). \tag{3.1.8}$$

While the simplicial decsription of homology is intuitive and geometrically appealing, it has the problem that it requires the realization of a space as a simplicial complex. In practice many spaces of interest can be more easily realized as cell complexes (which will be defined later) and it is desirable to develop a theory which is applicable to general topological spaces with few restrictions. Singular (co)homology has been the most successful such theory. A *singular n-simplex* of X we mean a continuous map $c : \Delta(n) \rightarrow X$. The free R-module with basis the set of all singular n-simplices will be denoted by $C_n(X;R)$ (or simply $C_n(X)$ if there is no confusion) so that an element is a formal finite linear combination of singular n-simplices with R coefficients. An element of $C_n(X;R)$ is called a *singular n-chain* of X. A continuous map $f : A \rightarrow X$ induces an R-module homomorphism $f_n : C_n(A;R) \rightarrow C_n(X;R)$ which is injective if f is so. In particular, if $A \subseteq X$, then $C_n(A;R)$ is a submodule of $C_n(X;R)$ and we set $C_n(X,A;R) = C_n(X;R)/C_n(A;R)$ call this the R-module of *relative singular n-chains*. The i^{th} face $\Delta_i(n)$ of $\Delta(n)$ we mean the convex closure of $\{e_0, \cdots, \hat{e_i}, \cdots, e_n\}$ where $\hat{e_i}$ means e_i is omitted. Define the R-module homomorphism $\partial = \partial_n : C_n(X) \rightarrow C_{n-1}(X)$, called *boundary* operator or map etc. by

$$\partial_n c = \sum (-1)^i c_i, \tag{3.1.9}$$

where c_i is the restriction of c to the i^{th} face of $\Delta(n)$. Just as in the simplicial case it is not difficult to prove that

$$\partial_{n-1}\partial_n = 0. \tag{3.1.10}$$

Note that the boundary map ∂ commutes with $f_n : C_n(A) \rightarrow C_n(X)$ for any continuous map $f : A \rightarrow X$. In particular if $A \subset X$ then $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ induces the *relative boundary* homomorphism $\partial_n : C_n(X,A) \rightarrow C_{n-1}(X,A)$ which satisfies (3.1.1). The kernel and image of ∂_n are denoted by $Z_n(X;R)$ (or $Z_n(X,A;R)$ in the relative case) and $B_n(X;R)$ (or $B_n(X,A;R)$ in the relative case) respectively, and are called the R-module of *singular cycles* (or *relative singular cycles*) and *singular boundaries* (or *relative singular boundaries*) respectively. The n^{th} homology group (or more precisely R-module) is defined by

$$H_n(X;R) = Z_n(X;R)/B_n(X;R), \quad \text{or} \quad H_n(X,A;R) = Z_n(X,A;R)/B_n(X,A;R) \quad \text{in the relative case.}$$

It is not obvious but true that for a simplicial complex the simplicial and singular homology groups are isomorphic. For the time being being we simply assume this basic fact.

To lend credibility to the axioms, we briefly discuss their validity in the context of homology theories which have introduced. The covariance condition is immediate from the construction. Let us verify the homotopy axiom for singular theory (similar argument works for simplicial theory). In fact, if $f, g : X \rightarrow Y$ are homotopic, then there is a homotopy $F : \Delta(n) \times I \rightarrow Y$ between the n-chains $c_1 \equiv f \cdot c$ and $c_2 \equiv g \cdot c$. Now $\Delta(n) \times I$ can be realized as a union of standard $n+1$-simplices as explained in the discussion of simplicial theory. Now $\partial F = \sum_b \pm F_{|b}$ where summation is over all simplices lying on the boundary of $\Delta(n) \times I$ with the proper $\pm$ sign. Now it is not difficult to see (draw a picture) that if $\sum_i a_i c_i \in Z_n(X;R)$ (i.e., represents an element of $H_n(X;R)$), then using the homotopy F we have

$$\sum_i a_i f \cdot c_i - \sum_i a_i g \cdot c_i = \partial \sum_i a_i F \cdot c_i.$$

(Geometrically, the terms on left hand side represent the linear combination of the tops and bottoms of the cylinders with a_i coefficients. The contributions of the lateral portions of the cylinders $c_i \times I$ sum to zero because $\partial \sum_i a_i c_i = 0$.) This establishes the homotopy axiom.

The crucial point in the long exact sequence is understanding the connecting homomorphism. We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \xrightarrow{j_n} & C_n(X) & \xrightarrow{p_n} & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow \partial_n & & \downarrow \partial_n & & \downarrow \partial_n & & \\
0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{j_{n-1}} & C_{n-1}(X) & \xrightarrow{p_{n-1}} & C_{n-1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

with exact rows. If $c \in Z_n(X,A)$ then there is $c' \in C_n(X)$ such that $p_n(c') = c$. By commutativity of the diagram $p\partial(c') = 0$, and so there is $c'' \in C_{n-1}(A)$ such that $j_n(c'') = \partial_n(c')$. From $\partial\partial = 0$ and the commutativity of the diagram it follows that $\partial(c'') = 0$. Now if $c = \partial_{n+1}(e)$ is a boundary, then it is trivial to see that we may assume $c' = \partial(e')$ is also a boundary. Consequently, $\partial_n(c') = 0$ and so $c'' = 0$. Therefore the assignment $c \rightarrow c''$ induces a homomorphism $H_n(X,A;R) \rightarrow H_{n-1}(A;R)$ which is the connecting homomorphism. (In the assignment $c \rightarrow c''$ we made a choice for c'. Therefore one has to verify that different choices of c' lead to the same element in $H_{n-1}(A;R)$ which is trivial to show.) Exactness of the of the long exact sequence is also proven using similar *diagram chase*. For example, to show exactness at $H_n(X,A;R)$, assume $\delta_n(\lfloor c \rfloor) = 0$, (the homology class represented by a cycle c will be denoted $\lfloor c \rfloor$). By assumption $c'' = \partial_n(f)$ and by commutativity of the diagram $\partial_n(c' - j_n(f)) = 0$, i.e., $c' - j_n(f) \in Z_n(X)$. Clearly $p_n(c' - j_n(f)) = c$, and therefore $\lfloor c \rfloor$ lies in the image of $p_{n\star} : H_n(X) \rightarrow H_n(X,A)$. Conversely, if $c' \in Z_n(X)$, then $c'' = 0$ from the construction which proves exactness at $H_n(X,A;R)$. Exactness at other places is proven by similar arguments. We also note here that the long exact sequence ends as follows:

$$\cdots \rightarrow H_\circ(A;R) \rightarrow H_\circ(X;R) \rightarrow H_\circ(X,A;R) \rightarrow 0.$$

The naturality axiom is a consequence of the commutativity of the the boundary operator and the induced maps $f_n : C_n(X,A;R) \rightarrow C_n(Y,B;R)$, and the normalization is trivial to verify. The excision axiom has significant geometric content and its validity is easily established in dealing with simplicial complexes. Here we assume further that $\bar{U}$ is a subcomplex of A which is a subcomplex of X. Therefore $X \setminus U$ and $A \setminus U$ are also simplicial complexes. We have the isomorphisms

$$C_n(X \setminus U;R)/C_n(A \setminus U;R) \simeq C_n(X \setminus U;R) + C_n(A;R)/C_n(A;R) \simeq C_n(X;R)/C_n(A;R).$$

Since the boundary operators ∂_n commute with these isomorphisms, the excision axiom is readily verified for simplicial complexes.

We have not said anything about the uniqueness of such a theory which will in particular establish the identity of the simplicial and singular homology groups for simplicial complexes. In case of simplicial complexes the uniqueness of such a theory follows from the fact that the axioms, in principle, allow us to calculate the homology of any simplicial complex. If the simplicial complex consists of a single point, then the normalization axiom gives the homology of X. From the excision axiom then we obtain the desired result for any finite set of points which shows that the homology of simplicial complexes of dimension zero follows from the axioms. Proceeding inductively, we assume X is obtained from Y by the addition of a simplex A

of dimension n. Here we assuming that all simplices of dimension $< n$ are already contained in Y. Then the closure of $|X|\backslash|A|$ is identical with $|Y|$ and therefore we have isomorphisms

$$f_\star : H_j(X, A) \longrightarrow H_j(Y, \partial A).$$

Substituting in the long exact sequence we easily calculate the homology of $H_j(X)$:

$$0 \to H_{j+1}(X) \longrightarrow H_{j+1}(X, A) \to 0, \quad \text{for } \; j \geq 0.$$

The case $j = 0$ requires the additional argument that $H_\circ(A) \to H_\circ(X)$ is injective which is left to the reader.

There is still the basic issue of the indepndence of the homolgy groups from the triangulation. This is essentially a categorical or purely algebraic matter and we will return to it later. For the rest of this subsection we concentrate on some examples.

It is clear from the homotopy axiom that that homology groups of a contractible space are identical to those of a point (see exercise 3.1.5). Furthermore,

Exercise 3.1.6 *Let $I = [0, 1]$, $M = I \times S^1$, and $N = M \setminus \{p\}$, $p \in M$. Show that N has the homotopy type of figure* **8** *(draw a picture and expand the point p). Construct a function on N with no critical value. Deduce that the assumption of compactness of $f^{-1}([a, b])$ in example 3.1.1 of chapter 1, cannot be removed.*

Example 3.1.4 Let M be a manifold of dimension m. If M is not compact, then $H_m(M; \mathbf{Z}) = 0$. In fact fix a triangulation of M and let $c = \sum a_i c_i$ (finite sum) where $a_i \in \mathbf{Z}$ and c_i's are the m-simplices. Since $\cup c_i$ is compact and is the closure of an open subset of a non-compact manifold, $\partial c \neq \emptyset$, and so there are no m-cycles. If M is a compact manifold with $\partial M \neq \emptyset$, then from the homotopy axiom we easily see that the homology groups of $\breve{M}$ and M are identical, and in particular, $H_m(M; \mathbf{Z}) = 0$. ♠

Example 3.1.5 Let us consider the real projective plane $\mathbf{RP}(2)$. Recall that $\mathbf{RP}(2)$ is the quotient of S^2 where antipodal points are identified. To triangulate $\mathbf{RP}(2)$ we consider a closed hemisphere and identify antipodal on its boundary S^1. In the triangulation of Figure 1.4, there are six vertices, fifteen 1-simplices and ten 2-simplices. We orient the 2-simplex $\tau_\circ$ by the ordered j-tuples $ABC = -ACB$ etc. The simplicial chain groups are $C_\circ(\mathbf{RP}(2); \mathbf{Z}) \simeq \mathbf{Z}^6$, $C_1(\mathbf{RP}(2); \mathbf{Z}) \simeq \mathbf{Z}^{15}$, and $C_\circ(\mathbf{RP}(2); \mathbf{Z}) \simeq \mathbf{Z}^{10}$. Therefore the homology of $\mathbf{RP}(2)$ is computed from the sequence

$$0 \longrightarrow \mathbf{Z}^{10} \xrightarrow{\partial_2} \mathbf{Z}^{15} \xrightarrow{\partial_1} \mathbf{Z}^6 \longrightarrow 0,$$

where the boundary homomorphisms are easily computed. It is not difficult (although laborious) to analyze the homomorphisms ∂_i to obtain

$$H_\circ(\mathbf{RP}(2); \mathbf{Z}) = \mathbf{Z}, \quad H_1(\mathbf{RP}(2); \mathbf{Z}) = \mathbf{Z}/2, \quad H_j(\mathbf{RP}(2); \mathbf{Z}) = 0 \;\text{ for } j \geq 2.$$

On the other hand, if we use $\mathbf{Z}/2$ instead of $\mathbf{Z}$ the same calculation yields

$$H_j(\mathbf{RP}(2); \mathbf{Z}/2) = \begin{cases} \mathbf{Z}/2, & \text{if } j = 0, 1, 2; \\ 0, & \text{otherwise.} \end{cases}$$

The important explanation for non-vanishing of $H_2(\mathbf{RP}(2);\mathbf{Z}/2)$ is given in connection with orientation. While in theory one can compute the homology groups of simplicial complexes, it should be clear that more elaborate methods are necessary for application to topological spaces of interest especially in higher dimensions. Cell complexes are useful tools for computational purposes as well as for proving general theorems, and will be discussed in the next section. ♠

Next we give an application of the simplicial approximation theorem:

Example 3.1.6 Let X be a simplicial of dimension k (by the *dimension* of a simplicial complex X we mean the largest integer k such that X contains a k-simplex). Then any continuous map $f : |X| \rightarrow S^n$, for $n > k$, is homotopic to a constant map. In fact, by approximating f with a simplicial map we may assume that f is a map of simplicial complexes. Therefore f misses at least one point of S^n since $k < n$, and so it can be regarded as a map into $\mathbf{R}^n$ which is contractible. Thus every continuous map $S^k \rightarrow S^n$, $k < n$, is homotopically trivial. ♠

3.1.3 Examples and Applications

The following simple algebraic lemma has important geometric content:

Lemma 3.1.2 *Let C_i, $i = -1, 0, \cdots, n+1$, be finite dimensional vector spaces with $C_{-1} = 0 = C_{n+1}$, and $\partial_i : C_i \rightarrow C_{i-1}$ linear maps with the property $\partial_{i-1}\partial_i = 0$. Set $H_i = \ker \partial_i / \mathrm{Im}\partial_{i+1}$, and let $f_k : C_k \rightarrow C_k$ be linear maps commuting with ∂_k's, i.e., the diagram*

$$\begin{array}{ccc} C_k & \xrightarrow{\partial_k} & C_{k-1} \\ f_k \downarrow & & \downarrow f_{k-1} \\ C_k & \xrightarrow{\partial_k} & C_{k-1} \end{array}$$

commutes for all k. Let $f_{i\star} : H_i \rightarrow H_i$ be the map induced by f_i. Then

$$\sum (-1)^i \mathrm{Tr}(f_i) = \sum (-1)^i \mathrm{Tr}(f_{i\star}).$$

Proof - We decompose C_i into a direct sum

$$C_i \simeq A_i \oplus B_i \oplus H_i,$$

where $B_i = \mathrm{Im}\partial_{i+1}$, and $A_i = C_i / \ker \partial_i$. The summands are invariant under the maps induced by f_i. Therefore

$$\mathrm{Tr} f_i = \mathrm{Tr} f_{i\star} + \mathrm{Tr}(f_i \text{ on } B_i) + \mathrm{Tr}(f_i \text{ on } A_i).$$

Taking alternating sums, $\mathrm{Tr}(f_i$ on $B_i)$ cancels out $\mathrm{Tr}(f_i$ on $A_{i+1})$, and the required result follows. ♣

Example 3.1.7 The geometric content of lemma 3.1.2 can be understood, for example for f = id., by recalling a theorem of Euler to the effect that for a compact polyhedron (homeomorphic to S^2) the number of vertices minus the number of edges plus the number of faces is equal to 2. In other words, the alternating sum of the dimensions of the chain groups is independent of the triangulation and is equal to the Euler characteristic. ♠

Example 3.1.8 As an application of this lemma we compute the homology of a finite connected graph. Recall that a graph is a simplicial complex consisting of only of vertices and edges. For example in Figure 2.1 the simplices are $\{v_i\}_{i=0,\cdots,10}$, $\{[v_i, v_{i+1}]\}_{i=0,\cdots,9}$, $[v_1, v_3]$, $[v_4, v_{10}]$, $[v_5, v_8]$, $[v_5, v_9]$, $[v_6, v_{10}]$, and $[v_7, v_{10}]$. The topology of X is the obvious one. Let $R = \mathbf{Z}$. There are no simplices of dimension 2 or higher, and so $H_j(X;\mathbf{Z}) = 0$ for $j \geq 2$. Since X is connected, $H_\circ(X;\mathbf{Z}) = \mathbf{Z}$. Furthermore, $H_1(X;\mathbf{Z})$ is free abelian since it is the subgroup of a free abelian group $C_1(X;\mathbf{Z})$. For the same reason, $\mathrm{rank}(H_1(X;\mathbf{Z})) = \dim H_1(X;\mathbf{R})$. Therefore the Euler characteristic of X is $\chi(X) = 1 - \mathrm{rank}(H_1(X;\mathbf{Z}))$. From lemma 2.1, $\chi(X) =$ number of vertices − number of edges, from which we see that $H_1(X;\mathbf{Z})$ is the free abelian group

$$1 + \text{number of vertices} - \text{number of edges} \tag{3.1.11}$$

generators. ♠

Exercise 3.1.7 *For a simplicial complex X, let $X(m) = \cup_{j\leq m}(j - \text{simplices of } X)$ ($X(m)$ is called the m-*skeleton *of X). Show that the inclusion $X(m) \hookrightarrow X$ induces isomorphisms $H_j(X(m)) \simeq H_j(X)$ for $j < m$. For X the standard n-simplex and $m > 0$, show that $H_m(X(m))$ is the free abelian group on*

$$(-1)^{m-1}[1 + \sum_{j=1}^{m+1} (-1)^j \binom{n+1}{j}],$$

generators, where $\binom{n}{j}$ is the binomial coefficient n choose j.

Let X be a finite simplicial complex, and $f : |X| \to |X|$ a continuous map. Let $R = \mathbf{R}$, so that the homology groups become finite dimensional vector spaces. Denote the trace of the endomorphism $H_i(f) : H_i(X;\mathbf{R}) \to H_i(X;\mathbf{R})$ by $L(i, f)$, and

$$L(f) = \sum (-1)^i L(i, f).$$

Proposition 3.1.1 (Lefschetz) - *If f has no fixed points, then $L(f) = 0$.*

Proof - By the simplicial approximation theorem we may assume that $f : X \to X$ is a simplicial map. Since f has no fixed points, by going to a sufficiently fine subdivision of X, we may assume $f(s)$ and s are distinct simplices. Hence the trace of the map induced by f on the chain groups (vector spaces) $C_i(X;\mathbf{R})$ is zero. The required result follows from lemma 3.1.2. ♣

Notice that this theorem implies that a continuous map of the disc into itself has necessarily a fixed point, since by the contractibility of the disc, $L(f) = 1$. This is the *Brouwer fixed point theorem.* Slightly more general is

Exercise 3.1.8 *Let f and h be mappings of a compact manifold (or finite simplicial complex) M to itself. Assume that f is homotopic to a constant map and h is a homeomorphism. Prove the existence of $x, y \in M$ such that $f(x) = x$ and $f(y) = h(y)$.*

Exercise 3.1.9 *Show that the map induced by the antipodal map on $H_n(S^n; \mathbf{Z})$ is multiplication by $(-1)^{n+1}$. Deduce that the antipodal map is not homotopic to the identity map if n is even. Explicitly construct a homotopy between the antipodal and the identity maps for n odd.*

The assumption of finiteness of the simplicial complex X in proposition 3.1.1 is essential. For instance, since the open unit disc is homeomorphic to $\mathbf{R}^m$ and translations are fixed point free diffeomorphisms of the latter space, Brouwer's fixed point theorem is not valid for the open disc. For the closed disc in an infinite dimensional Hilbert space we have the following counter-example:

Exercise 3.1.10 *Let B be the closed unit disc in the Hilbert space ℓ^2 of square summable sequences. Show that the continuous map $f : B \rightarrow \partial B \subset B$ given by*

$$f(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, \cdots)$$

where $x = (x_1, x_2, \cdots) \in \ell^2$ and $\|x\|^2 = \sum |x_i|^2$, has no fixed point. Show that $H_j(\partial B; R) = 0$ for $j > 0$. (Look at the composition

$$\partial B \hookrightarrow B \xrightarrow{f} \partial B \xrightarrow{\sigma} \partial B,$$

where $\sigma(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$ is the shift operator.)

Under more stringent requirements than continuity, an infinite dimensional version of the Brouwer fixed point theorem is valid. This version of the theorem is of genuine interest in analysis and has applications to geometric problems.

Example 3.1.9 A more remarkable application of proposition 3.1.1 is the following: Let M be a compact manifold and ξ a nowhere vanishing vector field on M. Let ϕ_t be the one parameter group associated to ξ. Then for $t > 0$ sufficiently small, the mapping ϕ_t has no fixed points. Therefore $L(\phi_t) = 0$. The maps ϕ_s, for $0 \leq s \leq t$, give a homotopy between ϕ_t and the identity map $\phi_\circ = \text{id}$. But $L(\text{id.})$ is the alternating sum of the traces of the identity maps on $H_j(M; \mathbf{R})$'s, i.e., the Euler characteristic $\chi(M)$ of M. Therefore the existence of a nowhere vanishing vector field implies the vanishing of the Euler characteristic. ♠

Example 3.1.10 Let K be a compact convex subset of $\mathbf{R}^2$ with piecewise smooth boundary. In example 1.2 of chapter 2 we considered the problem of estimating the average or expected number of intersections inside of K of n lines meeting K. Let $L_1, \cdots, L_n$ be n lines intersecting K, and $\Lambda = \cup L_j$. We now estimate the *average* $C(n, K)$ of the number of connected components of $K \setminus \Lambda$. The idea is that the lines L_j give almost a triangulation of the set K where some of the faces are quadrilaterals, pentagons etc. instead of triangles. Nevertheless, it is trivial to see that the conclusion of lemma 2.1 that

$$\chi(K) = \text{number of vertices } - \text{ number of edges } + \text{ number of faces}$$

remains valid. Each line L_i intersects ∂K at two points and therefore there are $N(n,K)+2n$ vertices. Through each vertex in the interior pass four edges, and through each vertex on the boundary three edges. Therefore the number of edges is $\frac{1}{2}(4N(n,K)+6n) = 2N(n,K)+3n$. Since $\chi(K)=1$ we obtain

$$C(n,K) = 1 + N(n,K) + n,$$

for the average of the number of connected components of $K\backslash\Lambda$. ♠

Combining lemma 3.1.2 with the Gauss-Bonnet theorem of chapter 2, we obtain the following:

Theorem 3.1.2 (Gauss-Bonnet) - *Fix a Riemannian metric on the compact orientable surface M, and denote the corresponding volume element and curvature by dv and K. Then*

$$\int_M K dv = 2\pi\chi(M).$$

Proof - We fix a triangulation of M and we may assume that all the edges are geodesics. Denote the number of vertices, edges and faces by v, e, and f respectively. Let Δ be a triangle of this triangulation. Then from the Gauss-Bonnet theorem of chapter 2, we know that

$$\int_\Delta K dv = 2\pi - \sum_i \alpha_{\Delta,i},$$

where $\alpha_{\Delta,i}$'s are the exterior angles of Δ. Summing over all triangles we obtain

$$\int_M K dv = 2\pi f - \sum_\Delta \sum_i \alpha_{\Delta,i}. \tag{3.1.12}$$

To evaluate the double sum we look at a vertex x and denote the number of edges through x by e_x. Let $\alpha_{x,j}$ be the exterior angles at x of the triangles having x as a vertex, and $\beta_{x,j}$ be the interior angles at x. Then $\beta_{x,j} = \pi - \alpha_{x,j}$ and

$$\sum_j \alpha_{x,j} = \sum_j (\pi - \beta_{x,j}) = \pi e_x - 2\pi.$$

Now the double sum in (3.1.12) can be written as

$$\begin{aligned} \textstyle\sum_\Delta \sum_i \alpha_{\Delta,i} &= \textstyle\sum_x \sum_j \alpha_{x,j} \\ &= \pi \textstyle\sum_x e_x \; - \; 2\pi v \end{aligned}$$

Since every edge connects two vertices, every edge is counted twice in $\pi\sum_x e_x$, (3.1.12) becomes

$$\int_M K dv = 2\pi f - 2\pi e + 2\pi v,$$

which implies the desired result by lemma 3.1.2. ♣

3.2 Cell Complexes and the Computation of Homology

3.2.1 Mayer-Vietoris Sequence

Next we introduce the Mayer-Vietoris sequence which is an important tool in algebraic topology. Its importance lies in the fact that it allows us to patch together local information into a global one. This will become evident in the examples below, the discussion of orientation and the Poincaré duality. We first discuss the following version of it which relates the homology of the union of two spaces to that of the subspaces:

Theorem 3.2.1 *Let $K \supset A$ and $L \supset B$ be subcomplexes of a simplicial complex X, then the following sequence (called the Mayer-Vietoris sequence) is exact:*

$$\cdots \longrightarrow H_k(K\cap L, A\cap B; R) \xrightarrow{i_\star} H_k(K, A; R) \oplus H_k(L, B; R) \xrightarrow{j_\star}$$
$$H_k(K\cup L, A\cup B; R) \xrightarrow{\delta_k} H_{k-1}(K\cap L, A\cap B; R) \longrightarrow \cdots$$

where $i_\star$ is induced by the obvious inclusions $(K\cap L, A\cap B) \to (K, A)$ and (L, B), and if $\imath : (K, A) \to (K\cup L, A\cup B)$ and $\jmath : (L, B) \to (K\cup L, A\cup B)$, then the homomorphism $j_\star$ is given by

$$j_\star(c, c') = \imath_\star(c) - \jmath_\star(c').$$

The connecting homomorphism δ_k is described below.

Remark 3.2.1 (a) The special case of particular importance is when $K \cup L = X$ and $A = B = \emptyset$. Then the Mayer-Vietoris sequence becomes:

$$\cdots \to H_k(K\cap L; R) \to H_k(K; R) \oplus H_k(L; R) \to H_k(X) \to H_{k-1}(K\cap B; R) \to \cdots$$

(b) In theorem 3.2.1 we may replace the hypotheses that the spaces under consideration are simplicial complexes by K, L, A and B are open and use singular homology. ♡

The connecting homomorphism δ_k is similar to the one in the long homology exact sequence. For simplicity consider the special case described in remark 3.2.1(a) above. Then we have the row exact commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_k(K\cap L; R) & \to & C_k(K; R)\oplus C_k(L; R) & \to & C_k(X; R) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_{k-1}(K\cap L; R) & \to & C_{k-1}(K; R)\oplus C_{k-1}(L; R) & \to & C_{k-1}(X; R) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

The construction of the connecting homomorphism and the exactness of the Mayer-Vietoris is similar to that of the long exact sequence which was discussed earlier. We now use theorem 3.2.1, or more precisely the version given in remark 3.2.1 to compute some homology groups.

The proof of theorem 3.2.1 is not difficult and not much will be gained by going through its detailed verification.

Example 3.2.1 We show by induction that $H_j(S^n; \mathbf{Z}) = \mathbf{Z}$ for $j = 0$ or n, and vanishes otherwise. For $n = 1$ this was shown in example 3.1.3. Assume the result is true for $n-1$. Let K and L be contractible open subsets containing the lower and upper hemispheres respectively, and such that the equator $S^{n-1} \subset S^n$ is a deformation retract of $K \cap L$. Substituting in the Mayer Vietoris sequence we obtain

$$\cdots \to H_m(S^{n-1}; \mathbf{Z}) \to H_m(K; \mathbf{Z}) \oplus H_m(L; \mathbf{Z}) \to H_m(S^n; \mathbf{Z}) \to H_{m-1}(S^{n-1}; \mathbf{Z}) \to \cdots$$

Since $H_m(K; \mathbf{Z}) = 0 = H_m(L; \mathbf{Z})$ for $m > 0$, we obtain the desired result using the induction hypothesis. ♠

Exercise 3.2.1 *Let $Y = I \times X$ and SX, called suspension of X, be the quotient of Y by identitifying all points of the form $\{(1, x)\}$ and all points of the form $\{(0, x)\}$, i.e., we collapse the top and bottom of the cylinder each to a point. Show that for $m > 1$, $H_m(SX; \mathbf{Z}) \simeq H_{m-1}(X; \mathbf{Z})$.*

Example 3.2.2 In section 2 we showed that the Euler characterisitic of a compact manifold admitting of a nowhere vanishing vector field is zero. By example 3.2.2, the Euler characteristic of an odd dimensional sphere is zero. Notice that the vector field

$$\xi_{(x_1, \cdots, x_{2n})} = (-x_2, x_1, -x_4, x_3, \cdots, -x_{2n}, x_{2n-1}),$$

is a nowhere vanishing tangent vector field to the sphere S^{2n-1}. This is a general fact in the sense that the vanishing of the Euler characterisitic of a compact orinetable manifold is equivalent to the existence of a nowhere vanishing vector field. The proof of this fact requires a technique which we have not yet introduced. ♠

Example 3.2.3 Let B^m denote the closed unit disc in $\mathbf{R}^m$, and $S^{m-1} = \partial B^m$. From the long exact sequence

$$\cdots \to H_j(S^{m-1}; R) \to H_j(B^m; R) \to H_j(B^m, S^{m-1}; R) \to H_{j-1}(S^{m-1}; R) \to \cdots,$$

and our knowledge of the homology of B^m and S^{m-1} we immediately obtain

$$H_m(B^m, S^{m-1}; R) = R, \quad \text{and} \quad H_j(B^m, S^{m-1}; R) = 0 \quad \text{for all} \quad j \neq m.$$

Similarly, one shows that $H_m(B^m, B^m \setminus \mathbf{0}; R) = R$ and $H_j(B^m, B^m \setminus \mathbf{0}; R) = 0$ for $j \neq m$. Now let M be a manifold of dimension m and M' the complement of a small disc (or a point) in M. We show that

$$H_j(M, M'; R) = \begin{cases} R, & \text{if } j = m; \\ 0, & \text{if } j \neq m. \end{cases}$$

We have already proven this for $M = \mathbf{R}^m$. For the general case, let $V \subset M'$ be an open subset such that $\bar{V} \subset \breve{M}'$, and $M \setminus V$ is homeomorphic to $\mathbf{R}^m$. By the excision axiom, $H_j(M, M'; R) \simeq H_j(M \setminus V, M' \setminus V; R)$ from which the assertion follows. ♠

Example 3.2.4 Let

$$0 \to A_n \to \cdots \to A_1 \to A_\circ \to 0,$$

be an exact sequence of finite dimensional vector spaces. From lemma 3.1.2 it follows that

$$\sum_j (-1)^j \dim A_j = 0. \tag{3.2.1}$$

Let M be a manifold of dimension m and M' be the manifold obtained from M by removing a small disc B^m of dimension m. By example 3.2.3

$$H_j(M, M'; R) = 0 \text{ for } j < m, \quad H_m(M, M'; R) = R.$$

Applying (3.2.1) to the long exact sequence for homology of the pair (M, M') we obtain

$$\chi(M) = \chi(M') + (-1)^m,$$

relating the Euler characteristics of M and M'. ♠

Example 3.2.5 In this example we compute the homology of the complex projective space $\mathbf{CP}(n)$ using the Mayer-Vietoris sequence. This example will serve as a motivation for the concept of cell-complex which will be introduced shortly. From the construction of $\mathbf{CP}(n)$ given in chapter 1, we know that that $\mathbf{CP}(n)\setminus\mathbf{CP}(n-1)$ is naturally represented as the set $\{[1, z_1, \cdots, z_n]\} \simeq \mathbf{C}^n$. Hence, in some sense to be made precise, $\mathbf{CP}(n)$ is obtained from $\mathbf{CP}(n-1)$ by attaching $\mathbf{C}^n \simeq \mathbf{R}^{2n}$. It is convenient to look at the process of attaching cells (by a *cell* we mean a closed disc) in the following equivalent way: Let $r^2 = |z_\circ|^2 + \cdots + |z_n|^2$ and consider the map $q : \mathbf{C}^{n+1} \setminus 0 \to B^{2n}$ given by

$$q : (z_\circ, \cdots, z_n) \longrightarrow \frac{1}{r}(z_1, \cdots, z_n).$$

Thus q maps the set $U_1 = \{(1, z_1, \cdots, z_n)\} \simeq \mathbf{C}^n$ homeomorphically onto the interior of the unit disc $B^{2n} \subset \mathbf{R}^{2n}$. Furthermore, q maps $U_1' = \{(0, z_1, \cdots, z_n)|\text{not all } z_j = 0\}$ onto $\partial B^{2n} = S^{2n-1}$. Since $q(re^{i\theta}(z_\circ, \cdots, z_n)) = e^{i\theta} q(z_\circ, \cdots, z_n)$, the quotient of $q(U_1') = S^{2n-1}$ under the action of $S^1 = \{e^{i\theta}\}$ is $\mathbf{CP}(n-1)$. Therefore we can say that $\mathbf{CP}(n)$ is obtained from $\mathbf{CP}(n-1)$ by attaching the cell B^{2n} by mapping the boundary $S^{2n-1} = \partial B^{2n} \xrightarrow{p} \mathbf{CP}(n-1)$ (note that this is a principal bundle with group S^1). To apply the Mayer-Vietoris sequence to compute the homology of $\mathbf{CP}(n)$, let $U = \breve{B}^{2n} = q(U_1)$, and V be a small tubular neighborhood of $\mathbf{CP}(n-1)$ such that the latter space is a deformation retract of V. We may assume $V \setminus \mathbf{CP}(n-1) = U \cap V \simeq \{z \in \mathbf{C}^n | \rho < \|z\| < 1\}$ for some $\rho > 0$. Therefore $U \cap V$ has the homotopy type of S^{2n-1}. Inductively we show

$$H_j(\mathbf{CP}(n)) = \begin{cases} \mathbf{Z}, & \text{if } j = 2k \leq 2n; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathbf{CP}(1) \simeq S^2$, the claim is valid for $n = 1$. Assuming the result for $n-1$ and substituting in the Mayer-Vietoris sequence we obtain

$$\cdots \to H_{2j-1}(U; \mathbf{Z}) \oplus H_{2j-1}(V; \mathbf{Z}) \to H_{2j-1}(\mathbf{CP}(n); \mathbf{Z}) \to H_{2j-2}(U \cap V; \mathbf{Z}) \to \cdots$$

Therefore $H_{2j-1}(\mathbf{CP}(n);\mathbf{Z}) = 0$. Similarly, from

$$0 \to H_{2k}(U;\mathbf{Z}) \oplus H_{2k}(V;\mathbf{Z}) \to H_{2k}(\mathbf{CP}(n);\mathbf{Z}) \to H_{2k-1}(U \cap V;\mathbf{Z})(\simeq \mathbf{Z}) \to 0,$$

we obtain $H_{2k}(\mathbf{CP}(n);\mathbf{Z}) \simeq \mathbf{Z}$ for $0 \leq k \leq n$. Notice that the proof also shows that the inclusion $\mathbf{CP}(n-1) \hookrightarrow \mathbf{CP}(n)$ induces isomorphisms on homology in dimensions $< 2n$. ♠

Exercise 3.2.2 *Let X_n be the union of n copies of* $\mathbf{CP}(1)$ *linearly embedded and in general position in* $\mathbf{CP}(2)$. (This means that every two copies of $\mathbf{CP}(1)$ intersect at exactly one point, and no three are concurrent.) *Show that (e.g. by induction on n) the homology of X_n is*

$$\begin{array}{ll} H_\circ(X_n;\mathbf{Z}) \simeq \mathbf{Z}, & H_1(X_n;\mathbf{Z}) \simeq \mathbf{Z}^{\frac{(n-1)(n-2)}{2}}, \\ H_2(X_n;\mathbf{Z}) \simeq \mathbf{Z}^n, & H_j(X_n;\mathbf{Z}) = 0, \ \text{for } j \geq 3. \end{array}$$

3.2.2 Cell Complexes

As suggested by example 3.2.5, to compute the homology of a space X it is very helpful to describe X by a process of adjunction of simpler spaces. To make this more precise and systematic, let (X, A) be a topological pair and let $f : A \to Y$ be a continuous map. From these data we construct a new space, to be denoted by $Z = X \cup_f Y$ which is obtained by identifying a point $z \in A$ with its image in Y under f. Notice that the map f is not necessarily one to one so that distinct points of A maybe identified with the same point in Y. A neighborhood of the image of $z \in A$ in Z is the union of a neighborhood of $f(z)$ in Y and the image of a neighborhood of z in X. The most important case is when $X = B^n$ is a cell and $A = \partial B^n = S^{n-1}$. The attached cells B^n are called the *n-cells* of Z. Spaces that can be constructed by successive attachments of cells via maps of their boundaries are called CW- or cell-complexes. More precisely, a *cell complex* consists of a topological space X together with a sequence of subspaces $X(0) \subset X(1) \subset \cdots X(n) \subset \cdots$ with the property $X = \cup X(k)$ and such that $X(0)$ consists of 0-cells (points) and $X(k)$ is obtained from $X(k-1)$ by attaching k-cells. We note that if $x \in X(k)$ then a neighborhood of x consists of a union of neighborhoods of x in the $X(k)$'s to which it belongs. If $X = X(m)$ for some m, we say X has *dimension* m. A cell complex is called *regular* if every attaching map $f : S^{n-1} \to X(n-1)$ is homeomorphism. It is clear that every simplicial complex is a also regular cell complex.

Example 3.2.6 Clearly the sphere S^m is a cell complex with $S^m = X(0) \cup X(m)$ where $X(0)$ consists of a single point and $X(m) = S^m$ is obtained by attaching the unit disc $D(m) \subset \mathbf{R}^m$ by mapping its boundary to the point $X(0)$. To realize the torus T^2 as a cell complex it is convenient to look at it as a square S with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ and opposite sides identified (or equivalently $T^2 = \mathbf{R}^2/\mathbf{Z}^2$). Let $X(0) = (0,0)$ and $X(1)$ be the union of the images of the line segments $[t,0]$ and $[0,t]$ $(0 \leq t \leq 1)$. Now $T^2 = X(2)$ is obtained by identifying the unit disc $D(2)$ with S and the natural identification of the boundary, i.e., for example $e^{2\pi it} \to (4t,0)$ for $0 \leq t \leq \frac{1}{4}$, $e^{2\pi it} \to (1, 4t-1)$ for $\frac{1}{4} \leq t \leq \frac{1}{2}$ etc. ♠

Exercise 3.2.3 *Generalize example 3.2.6 to realize T^m as a cell complex with $X(k)$ obtained from $X(k-1)$ by attaching $\binom{n}{k}$ cells of dimension k. Describe the attaching maps explicitly.*

Almost any familiar space can be constructed in this fashion, and therefore once the structure of a space as a cell-complex is known, we have a powerful tool for computing its homology groups as exemplified by the computation of the homology groups of $\mathbf{CP}(n)$. One should however be cognizant of the fact that there are esoteric examples of spaces which are not cell complexes. For example, in general, the product of two cell complexes need not be a cell complex unless some mild assumptions are made (see [D]). An important feature of the definition of cell complex is that the space is constructed incrementally in such a way that the dimensions of the attaching cells are nondecreasing. It is an important technical fact that it is possible to re-arrange the process of attaching cells, without changing the homotopy type of the space, so that the nondecreasing requirement of the dimensions is satisfied and the attaching maps take values in the union of cells of lower dimensions. To see the truth of this assertion simply use the idea of 3.1.6, and we note that a version of the Simplicial Approximation theorem (called *Cellular Approximation theorem*) is valid for cell complexes.

We now clarify the method of computation, but instead of using the Mayer-Vietoris sequence, we alternatively use the long exact sequence for homology. For a cell-complex X, let $X(m) = \cup_{j\leq m}(j\text{-cells of } X)$.

Proposition 3.2.1 *For a cell complex X we have*

$$H_j(X(m), X(m-1); R) = \begin{cases} 0 & \text{if } j \neq m; \\ R^{k(m)} & \text{if } j = m, \end{cases}$$

where $k(m)$ is the number of cells of dimension m. The inclusion $X(m) \hookrightarrow X(m+n)$ induces isomorphisms on homology for $j < m$ and $n \geq 1$. *Furthermore, $H_j(X(m); R) = 0$ for $j > m$.*

Proof - Let U_{m-1} and V_{m-1} be small neighborhoods of $X(m-1)$ in $X(m)$ with the property that $X(m-1)$ and V_{m-1} are deformation retracts of U_{m-1} and $\bar{V}_{m-1} \subset U_{m-1}$. Excising out V_{m-1} we get

$$H_j(X(m), X(m-1); R) \simeq \oplus H_j(B^m, S^{m-1}; R),$$

where the direct sum is over the m-cells. The first assertion follows immediately. The long exact sequence applied to $(X(m), X(m-1))$ yields

$$H_{j+1}(X(m+1), X(m); R) \to H_j(X(m); R) \to H_j(X(m+1); R) \to H_j(X(m+1), X(m); R)$$

which, by the first assertion, implies $H_j(X(m+1); R) \simeq H_j(X(m); R)$ for $j < m$. Continuing inductively we get the isomorphism $H_j(X(m+n); R) \simeq H_j(X(m); R)$ for $j < m$. Clearly $H_j(X(0); R) = 0$ for $j > 0$, and assume $H_j(X(m); R) = 0$ for $j > m$. Assuming $j > m+1$ and substituting in the long exact sequence for the pair $(X(m+1), X(m))$, we get $H_j(X(m+1); R) = 0$ as desired. ♣

Most of the familiar spaces have the homotopy type of a finite cell complex (i.e. only finitely many cells are attached). Even in cases where cells of arbitrarily high dimension are attached, it is still true that $H_j(X(n); R) \simeq H_j(X; R)$ for $j < n$, and our method of computation is still applicable to these spaces. Spaces where cells of arbitrarily high dimension are attached occur, for example, in connection with the space of paths on manifolds which will be discussed later.

Corollary 3.2.1 *Assume that no cells of dimension* $m-1$ *are attached. Then* $H_m(X(m);R) \simeq R^{k(m)}$, *where* $k(m)$ *is the number of cells of dimension* m. *If furthermore no cells of dimension* $m+1$ *are attached, then* $H_m(X;R) \simeq R^{k(m)}$.

Proof - Since there no cells of dimension $m-1$, $X(m-1) = X(m-2)$ and by the last assertion of proposition 3.2.1 $H_j(X(m-1);R) = 0$ for $j > m-1$. Therefore from the long exact sequence for the pair $(X(m), X(m-1))$ and the first assertion of proposition 3.1 we obtain

$$0 = H_m(X(m-1);R) \rightarrow H_m(X(m);R) \rightarrow R^{k(m)} \rightarrow 0,$$

which proves the first statement. If furthermore there are no cells of dimension $m+1$, then $X(m+1) = X(m)$, and we similarly obtain the exact sequence

$$0 \rightarrow H_m(X(m);R) \rightarrow H_m(X(m+1);R) \rightarrow 0,$$

i.e., $H_m(X(m);R) \simeq H_m(X(m+1);R)$. By proposition 3.1, $H_m(X(m+1);R) \simeq H_m(X(m+j);R)$ for $j \geq 1$ and consequently $H_m(X;R) \simeq R^{k(m)}$ as desired. ♣

Remark 3.2.2 If the structure of m-skeletons of a simplicial complex X are known, then by similar techniques one can compute the homology of X.

For spaces where cells of consecutive dimensions are attached, the application of the above technique requires understanding some of the maps in the Mayer-Vietoris and/or the long homology exact sequence. Some examples demonstrating this fact are given in exercises 3.2.4, 3.2.5 and 3.2.8 below.

Exercise 3.2.4 *Using the cell structure of the torus given in execise 3.2.3, compute the homology of* T^m *by computing the necessary homomorphisms in the long exact sequence.*

Exercise 3.2.5 *Let* X *be the space obtained by identifying two points on the sphere* S^2 (called *pinched sphere*). *Show that* $H_j(X;\mathbf{Z}) = \mathbf{Z}$ *for* $j = 0,1,2$ *and vanishes for* $j \geq 3$.

Exercise 3.2.6 *Let* S *be a smoothly embedded copy of* S^1 *in* S^3. *Show that the homology groups of the complement of* S *in* S^3 *are* $\mathbf{Z}$ *in dimensions 0 and 1, and vanish elsewhere.* (Let N be a small tubular neighborhood of S in S^3, and $V = S^3 \setminus S$. Clearly $N \cap V$ has the homotopy type of a torus. To prove $H_2(V;\mathbf{Z}) = 0$, show that the map $H_2(N \cap V;\mathbf{Z}) \rightarrow H_2(V;\mathbf{Z})$ in the Mayer-Vietoris sequence is the zero map.)

Exercise 3.2.7 *Let* S *be as in exercise 3.5, and* Y *be the space obtained by collapsing* S *to a point. Show that* $H_j(Y;\mathbf{Z}) = \mathbf{Z}$ *for* $j = 0,2,3$ *and zero otherwise. Generalize to the case when* S *is a smoothly embedded copy of* S^k *in* S^n *where* $n > k$.

Exercise 3.2.8 *Let* X *be a cell complex,* $f : S^{n-1} \rightarrow X$ *and* $Y = X \cup_f B^n$ *be the cell complex obtained from attaching the cell* B^n *via the map* f. *Show that* $H_j(X;R) \simeq H_j(Y;R)$ *for* $j \neq n-1, n$. *Furthermore, map* $H_{n-1}(X;R) \rightarrow H_{n-1}(Y;R)$ *(resp.* $H_n(X;R) \rightarrow H_n(Y;R)$*) induced by the inclusion* $X \hookrightarrow Y$ *is surjective (resp. injective).*

Exercise 3.2.9 *Let $S_1, \cdots, S_l$ be l circles smoothly embedded and disjoint in S^3, and $M = S^3 \backslash (S_1 \cup \cdots \cup S_l)$. Show that $H_\circ(M;R) \simeq R$, $H_1(M;R) \simeq R^l, H_2(M;R) \simeq R^{l-1}$,* and *$H_j(M;R) = 0$ for $j \geq 3$.* (The proof requires the computation of connecting homomorphisms in the Mayer-Vietoris sequence which in this case can be done by following through its construction.)

3.2.3 Morse Inequalities

The procedure described above for the computation of the homology of a space from its cell decomposition requires the calculation of certain homomorphisms if there are cells in consecutive dimensions. It is worthwhile to make this precise. By proposition 3.2.1 $H_{k-1}(X) = H_{k-1}(X(k))$ and consequently

$$H_{k-1}(X;R) \;\simeq\; H_{k-1}(X(k-1);R)/\mathrm{Im}(\delta_k). \tag{3.2.2}$$

Let $E_k = H_k(X(k), X(k-1); R)$. From the long homology exact sequence we obtain the following row exact the diagram (the coefficient group R is omitted for simplicity of notation):

$$\begin{array}{ccccccccccc} H_k(X(k)) & \rightarrow & E_k & \stackrel{\delta_k}{\rightarrow} & H_{k-1}(X(k-1)) & \rightarrow & H_{k-1}(X(k)) & \rightarrow & 0 & & \\ & & & & \mathrm{id.}\ \| & & & & & & \\ & & 0 & \rightarrow & H_{k-1}(X(k-1)) & \stackrel{p_{k-1}}{\rightarrow} & E_{k-1} & \stackrel{\delta_{k-1}}{\rightarrow} & H_{k-2}(X(k-2)) & \rightarrow & \cdots \end{array}$$

From (3.2.2) and the above diagram it follows that $H_{k-1}(X;R)$ is isomorphic to the $\ker(\delta_{k-1})/\mathrm{Im}(p_{k-1}\delta_k)$. Now p_j's are injective, and therefore, setting $\varrho_k = p_{k-1}\delta_k : E_k \longrightarrow E_{k-1}$, we obtain the following proposition whose significance will be demonstrated shortly:

Proposition 3.2.2 *The homology of the sequence*

$$\cdots \longrightarrow E_k \xrightarrow{\varrho_k} E_{k-1} \longrightarrow \cdots$$

is $H_k(X;R)$. (Notice that for a regular cell complex the map ϱ_k assigns to an oriented k-cell its boundary with the induced orientation.)

Corollary 3.2.2 (Morse Inequalities)- *Let X be a cell complex with $\mu_k < \infty$ the number of cells of dimension k in X, and b_k be the k^{th} Betti number of X. Then*

$$\mu_k - \mu_{k-1} + \cdots + (-1)^{k-1}\mu_1 + (-1)^k\mu_\circ \geq b_k - b_{k-1} + \cdots + (-1)^{k-1}b_1 + (-1)^k b_\circ.$$

If furthermore X is a cell complex of dimension m then equality holds for $k = m$.

Proof - Applying proposition 3.2.2 and lemma 3.1.2 to $Y = X(k)$ yields

$$\chi(Y) = (-1)^k b_k(Y) + \sum_{j=0}^{k-1}(-1)^j b_j(Y) = (-1)^k \mu_k(Y) + \sum_{j=0}^{k-1}(-1)^j \mu_j(Y),$$

where $b_j(Y)$ is the j^{th} Betti number of Y. The required result follows from the observation that $b_k(Y) \geq b_k(X)$, and $b_j(Y) = b_j(X)$ for $j \leq k-1$ since X is obtained from Y by attaching cells of dimension $\geq k+1$. ♣

Exercise 3.2.10 *Show that the Morse inequalities are equivalent to the statement*

$$M(f,t) - P(M,t) = (1+t)R(t),$$

where $M(f,t) = \sum \mu_k t^k$, $P(M,t) = \sum b_k(M)t^k$ *and* $R(t)$ *is a polynomial in* t *with non-negative coefficients.*

Notice that we stated the Morse inequalities for a cell complex rather than the conventional way of formulating it for critical points of a Morse function. There are some advantages to this formulation, but the relationship with critical points should kept in mind and is discussed below. As a simple application of corollary 3.4.1 to combinatorics of polytopes we have

Example 3.2.7 By a *polytope* P we mean the convex closure of a finite set $\{x_1, \cdots, x_n\} \subset \mathbf{R}^m$, and $\dim P$ is the dimension of the set P. Let $\{v_1, \cdots, v_l\} \subset \{x_1, \cdots, x_n\}$ be the extreme points of P. We refer to v_i's as *vertices* or 0-*faces* of P. The intuitive notion of a face of P can be formalized by saying that a closed convex subset $F \subset P$ is a *face* of P if whenever the interior (x, y) of the line segment $[x, y]$ joining $x \in P$ to $y \in P$ intersects F, then $[x, y]$ lies entirely in F. Naturally, by a k-*face* we mean a face F of dimension k. Let f_k be the number of k-faces of the convex polytope of the polytope P. Then corollary 3.4.1 applied to the boundary ∂P of the polytope P of dimension d yields

$$f_\circ - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} = 1 + (-1)^{d-1}. \tag{3.2.3}$$

Furthermore for $k \leq d-2$ we have the inequalities

$$f_k - f_{k-1} + f_{k-2} - \cdots + (-1)^k f_\circ \geq (-1)^k. \tag{3.2.4}$$

While non-topological proofs of (3.2.3) exist, the above proof appears to be the simplest and the most effective one. Besides giving the additional information in (3.2.4), the hypothesis that P is a convex polytope can be replaced by a much weaker one such as P is a cell complex of dimension d. Of course then (3.2.3) and (3.2.4) are simply re-statements of the Morse inequalities. There are other relations (Dehn-Sommerville relations) between the number of simplices of various dimensions for a triangulation of a compact manifold. These relations are most easily obtained in the context of simple, rather than simplicial, complexes which are discussed later. ♠

3.2.4 Product Spaces

Next we consider the general problem of computing the homology groups of a space. For product spaces (e.g., the torus T^n) homology can be computed effectively via the following theorem:

Proposition 3.2.3 *Assume that the homology groups of* X *and* Y *are free* R*-modules. Then*

$$H_n(X \times Y; R) = \sum_{i+j=n} H_i(X; R) \otimes H_j(Y; R).$$

(The generalization of this theorem to the case where there is torsion in homology requires additional algebraic machinery. This extension and in particular the proof of this proposition will be discussed later in this chapter.)

Exercise 3.2.11 *Show that the homology of T^n (the product of n circles) is*

$$H_i(T^n;R) \simeq R^{\binom{n}{i}}$$

Corollary 3.2.3 *The Euler characteristic of the product of two spaces is the product of their Euler characteristics.*

In order to clarify the geometric content of proposition 3.2.3 and for other applications we introduce several definitions. By a *complex* $\mathcal{E}$ we mean a sequence of R-modules and homomorphisms of the form $\cdots \to E_n \xrightarrow{f_n} E_{n-1} \to \cdots$ such that $f_n f_{n-1} = 0$. We also use the term complex if the homormorphisms are *ascending*, i.e., $f_n : E_n \to E_{n+1}$ rather than *descending* $f_n : E_n \to E_{n-1}$. By a *positive* we mean a complex such that $E_j = 0$ for $j < 0$ and *negative* complex is defined similarly. (Co)homology of a complex is defined in the familiar way as ker /Im, however, homology (resp. cohomology) assumes that the complex is descending (resp. ascending). Suppose we have two complexes $\mathcal{E}$ and $\mathcal{E}'$. By a *mapping* (or *morphism*) $\Phi : \mathcal{E} \to \mathcal{E}'$ we mean a collection of homomorphisms $\Phi_j : F_j \to F'_j$ such the diagram

$$\begin{array}{ccc} F_j & \xrightarrow{f_j} & F_{j-1} \\ \Phi_j \downarrow & & \downarrow \Phi_{j-1} \\ F'_j & \xrightarrow{f'_j} & F'_{j-1} \end{array}$$

commutes. Therefore a morphism of complexes induces a map on the homology of the complexes. Given two such morphisms Φ and Ψ, by a *chain homotopy* between Φ and Ψ we mean a collection of homomorphisms $s_j : F_j \to F'_{j+1}$ such that

$$f'_{j+1} s_j \;+\; s_{j-1} f_j \;=\; \Phi_j \;-\; \Psi_j .$$

Lemma 3.2.1 *Chain homotopic morphisms induce identical maps on homology.*

The proof is a staightforward checking of the definitions. For ascending chains the correspondings maps are $s_j : F_j \to F'_{j-1}$. The usefulness of this simple algebraic lemma will demonstrated later.

Let $\mathsf{F} : \cdots \to F_1 \to F_\circ \to 0$ and $\mathsf{F}' : \cdots \to F'_1 \to F'_\circ \to 0$ be complexes of R-modules and denote the maps $F_n \to F_{n-1}$ (resp. $F'_N \to F'_{n-1}$) by ∂_n (resp. ∂'_n). The *tensor product of complexes* F and F' is the complex

$$(\mathsf{F} \otimes \mathsf{F}')_n \;=\; \sum_{p+q=n} F_p \otimes F'_q, \qquad \text{direct sum}$$

with the boundary map the sum over $p + q = n$ of the maps

$$\partial''_{p,q} = \partial_p \otimes \mathrm{id.} \oplus (-1)^p (\mathrm{id.} \otimes \partial'_q) : F_p \otimes F'_q \longrightarrow F_{p-1} \otimes F'_q \oplus F_p \otimes F'_{q-1}.$$

(All tensor products are over R.) The factor $(-1)^p$, which also appears in the exterior derivative of product of forms, is essential in making sure that $\partial''_{n-1}\partial''_n = 0$. More significantly,

Example 3.2.8 It is important to understand the geometric meaning of the boundary operator $\partial''_{p,q}$. Let $F_p = C_p(X;R)$ and $F'_q = C_q(Y;R)$ be the the chain R-modules of the simplicial complexes X and Y. Then the set of products $\sigma \times \tau$ where σ (resp. τ) runs over the simplices of X (resp. Y) gives a decomposition of $X \times Y$ as a regular cell complex. (Recall that the product of two simplices is not a simplex and can be triangulated in many different ways.) Assigning orientations to σ's and τ's and product orientations to $\sigma \times \tau$'s, we easily see that $\partial''(\sigma \otimes \tau)$ is the boundary of $\sigma \times \tau$ with induced orientation. In view of proposition 3.2.2 the homology of $X \times Y$ is identical with that of the complex $\mathsf{F} \otimes \mathsf{F}'$. Computing the homology of the latter complex is a purely algebraic matter as shown later in this chapter (Künneth Formula). This example and some of its ramifications are known as the *Eilenberg-Zilber theorem.* ♠

Remark 3.2.3 From the construction of singular or simplicial homology it is not difficult to see that any homomorphism $\beta : R \to S$ of rings (or more generally of R-modules) induces homomorphisms $\beta_\star : H_j(X;R) \to H_j(X;S)$. The relationship between $H_j(X;R)$ and $H_j(X;S)$ will be studied later. Here we only mention the fact (without proof) that if $R = \mathbf{Z}$, and $S = \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$, then $H_j(X;S) = H_j(X;R) \otimes S$. In particular, if $H_j(X;R)$ is a finitely generated abelian group of rank k, then $H_j(X;S)$ is the vector space over S of dimension k. ♡

3.2.5 Orientation Class

It is important to introduce a homological definition of orientation which would be more easily applicable to problems in a topological framework. First we want to translate the definition of orientation into a homological statement. To do so we fix an orientation for $\mathbf{R}^m$ and a triangulation of $S^{m-1} \subset \mathbf{R}^m$. Let s_i's be the $m-1$-simplices and $\{v_\alpha\}$ the set of the vertices of the triangulation. We may write every $(m-1)$-simplex in the form $s_i = [v_{i_1}, \cdots, v_{i_m}]$ and assume that the vectors $v_{i_1}, \cdots, v_{i_m}$ form an ordered basis $\mathcal{B}_i$ for $\mathbf{R}^m$. To each simplex s_i we assign an orientation $\epsilon_i = \pm$ according as the basis $\mathcal{B}_i$ is positively/negatively oriented. Now set $\lfloor S^{m-1} \rfloor = \sum_i \epsilon_i s_i$.

Lemma 3.2.2 *$\partial \lfloor S^{m-1} \rfloor = 0$, and $\lfloor S^{m-1} \rfloor$ is a generator of $H_{m-1}(S^{m-1};\mathbf{Z}) \simeq \mathbf{Z}$.*

Proof - To prove the first assertion it suffices to prove that if $\epsilon_i s_i$ and $\epsilon_j s_j$ have a face (i.e., an $m-2$-simplex) f_{ij} in common, then f_{ij} occurs in $\partial \epsilon_i s_i$ and $\partial \epsilon_j s_j$ with opposite signs. This follows easily from the fact that if $s_1 = [v_1, \cdots, v_{m-1}, v_m]$ and $s_2 = [v_1, \cdots, v_{m-1}, v'_m]$, then the vectors v_m and v'_m lie on opposite sides of the linear subspace spanned by $v_1, \cdots, v_{m-1}$, and consequently the linear transformation defined by

$$T(v_i) = v_i \ \text{ for } i = 1, \cdots, m-1, \quad T(v_m) = v'_m,$$

has negative determinant. This proves the first assertion. Let $\lfloor S^{m-1} \rfloor = k\sigma$ where σ is a generator. But σ is a linear combination of s_i's with integer coefficients, and so $k = \pm 1$. ♣

We have shown that an orientation for $\mathbf{R}^m$ determines orientations for the simplices of a realization of S^{m-1} as a simplicial complex which determines a generator of $H_{m-1}(S^{m-1};\mathbf{Z})$. From the proof of lemma 3.2.2 we see that the process can be reversed and by fixing a generator of $H_{m-1}(S^{m-1};\mathbf{Z})$ we obtain an orientation for $\mathbf{R}^m$. From the long exact sequence for homology we have an isomorphism $H_m(\mathbf{R}^m, \mathbf{R}^m \setminus \mathbf{0};\mathbf{Z}) \simeq H_{m-1}(S^{m-1};\mathbf{Z})$, and therefore we have shown that an orientation for $\mathbf{R}^m$ is equivalent to the choice of a generator for $H_m(\mathbf{R}^m, \mathbf{R}^m \setminus \mathbf{0}; \mathbf{Z})$. Intuitively, the choice of a generator for $H_m(\mathbf{R}^m, \mathbf{R}^m \setminus \mathbf{0}; \mathbf{Z})$ amounts to the choice of an orientation for the tangent space at $\mathbf{0}$.

To extend this definition to the case of a general manifold we recall from example 3.2.3 that $H_j(M, M \setminus \{x\}; \mathbf{Z}) \simeq H_j(U(x), U(x) \setminus \{x\}; \mathbf{Z})$, where $U(x)$ is a small neighborhood of x (homeomorphic to $\mathbf{R}^m$) such that $M \setminus U(x)$ is a deformation retract of $M \setminus \{x\}$. The homological definition of orientation at $x \in M$ is therefore the choice of a generator for $H_m(M, M \setminus \{x\}; \mathbf{Z})$. Next we note that the inclusion $(M, M \setminus U(x)) \subset (M, M \setminus \{x\})$ induces isomorphisms

$$H_j(M, M \setminus U(x); \mathbf{Z}) \to H_j(M, M \setminus \{x\}; \mathbf{Z}). \tag{3.2.5}$$

In view of this isomorphism, which will be proven below, it is reasonable to define a *local orientation* on M as the choice of a generator for $H_m(M, M \setminus U(x); \mathbf{Z})$. To give a global definition we introduce a "coherence condition" as follows: Let $\mathcal{U} = \{U_i\}$ be a covering of M by small contractible open sets such each $H_m(M, M \setminus U_i; \mathbf{Z}) \simeq \mathbf{Z}$ and let $\mathbf{m}_i \in H_m(M, M \setminus U_i; \mathbf{Z})$ be a generator, i.e., a local orientation. The *coherence condition* is the requirement that the generators $\mathbf{m}_i$ and $\mathbf{m}_j$ map to same element of $H_m(M, M \setminus U_i \cap U_j; \mathbf{Z})$ in the diagram:

$$H_m(M, M \setminus U_i; \mathbf{Z}) \longrightarrow H_m(M, M \setminus (U_i \cap U_j); \mathbf{Z}) \longleftarrow H_m(M, M \setminus U_j; \mathbf{Z}).$$

The coherence condition implies that the orientation at $x \in U_i \cap U_j$ is unambiguously defined. If the coherence condition can be fulfilled for a choice of a set generators for all $H_m(M, M \setminus U_i; \mathbf{Z})$, then we say that we M is *orientable*. Once such a choice is fixed, we say M is *oriented*. Note that the homological definition makes no reference to the tangent bundle and is therefore meaningful even for topological manifolds.

It remains to prove (3.2.5) which we prove by introducing a useful algebraic fact. We have the commutative row exact diagram

$$\begin{array}{ccccccccc}
H_m(M \setminus U(x)) & \to & H_m(M) & \to & H_m(M, M \setminus U(x)) & \to & H_{m-1}(M \setminus U(x)) & \to & H_{m-1}(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_m(M \setminus \{x\}) & \to & H_m(M) & \to & H_m(M, M \setminus \{x\}) & \to & H_{m-1}(M \setminus \{x\}) & \to & H_{m-1}(M)
\end{array}$$

where all vertical arrows except possibly the middle one are isomorphisms. Applying the following useful algeberaic lemma we see that the middle vertical arrow is also an isomorphism:

Lemma 3.2.3 (Five Lemma) - *Assume that in the following row exact commutative diagram of modules, all vertical arrows except possibly the middle one are isomorphisms:*

$$\begin{array}{ccccccccc}
A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A'_1 & \to & A'_2 & \to & A'_3 & \to & A'_4 & \to & A'_5
\end{array}$$

Then the middle arrow is also an isomorphism.

Proof - The proof is a simple diagram chase. To show injectivity let $f_i : A_i \to A_{i+1}$, $f'_i : A'_i \to A'_{i+1}$ and $h_i : A_i \to A'_i$. Let $c \in A_3$ and assume $h_3(c) = 0$. Then $f_3(c) = 0$ since h_4 is an isomorphism and the diagram commutes. Therefore there is $c' \in A_2$ such that $f_2(c') = c$, and so $f_2'h_2(c') = 0$. Hence there is $c'' \in A_1'$ such that $f_1'(c'') = h_2(c')$. Set $c''' = h_1^{-1}(c'')$, then $f_1(c''') = c'$ and $c = 0$ by exactness. The proof of surjectivity is similar. ♣

The following theorem is very useful:

Theorem 3.2.2 *Let M be an orientable manifold, with orientation defined by the local orientations $\mathbf{m}_i \in H_m(M, M \setminus U_i; \mathbf{Z})$, and K a compact subset of M. Then there is a unique element $\mathbf{m}_K \in H_m(M, M \setminus K; \mathbf{Z})$ such $\mathbf{m}_K$ maps onto the generator $H_m(M, M \setminus \{x\}; \mathbf{Z})$ determined by the orientation for all $x \in K$. In particular if M is compact, a unique generator $\mathbf{m} \in H_m(M; \mathbf{Z})$ (called the* orientation *or* fundamental class *of M and generally denoted by $\lfloor M \rfloor$) defines the orientation of M.*

Proof - This proof is a good example of how local information can be patched together using the Mayer-Vietoris sequence, and it is instructive to work it out carefully. Let $\mathcal{U} = \{U_i\}$ be a covering of M as specified in the definition of local orientation. Assume $\{U_i\}_{i=1,\cdots,N}$ cover K, and notice that it suffices to show that there is a homology class $\mathbf{m} \in H_m(M, M \setminus \cup_{i=1}^N U_i; \mathbf{Z})$ whose image in $H_m(M, M \setminus \{x\}; \mathbf{Z})$ is the orientation at $x \in M$. The proof is by induction on the number of open sets covering K where the induction hypothesis is as follows: Let $A = U_1 \cup \cdots \cup U_k$, $B = U_{k+1}$. Then the induction hypothesis is that there are homology classes $\mathbf{m}_A \in H_m(M, M \setminus A; \mathbf{Z})$ and $\mathbf{m}_B \in H_m(M, M \setminus B; \mathbf{Z})$ whose images in $H_m(M, M \setminus (A \cap B); \mathbf{Z})$, are identical. By remark 3.2.1, the Mayer-Vietoris sequence is applicable to the case where, in the notation of theorem 3.2.1, $K = L = M$, A and B as defined above. Therefore we have the exact sequence

$$0 \rightarrow H_m(M, M \setminus A \cup B) \rightarrow H_m(M, M \setminus A) \oplus H_m(M, M \setminus B) \rightarrow H_m(M, M \setminus A \cap B) \tag{3.2.6}$$

For $k = 1$, by the coherence condition, $(\mathbf{m}_A, \mathbf{m}_B) \in H_m(M, M \setminus A) \oplus H_m(M, M \setminus B)$ maps to zero in $H_m(M, M \setminus (A \cap B))$ and so we can begin the induction. We *claim* that the images of $\mathbf{m}_{A \cup B}$ and $\mathbf{m}_C$ where $C = U_{k+2}$ in $H_m(M, M \setminus ((A \cup B) \cap C); \mathbf{Z})$ in the following diagram are identical:

$$H_m(M, M \setminus (A \cup B); \mathbf{Z}) \longrightarrow H_m(M, M \setminus (A \cup B) \cap C); \mathbf{Z}) \longleftarrow H_m(M, M \setminus C; \mathbf{Z}). \tag{3.2.7}$$

Once this done, we substitute $C = U_{k+2}$ for B and $A \cup B$ for A in (3.2.7) to obtain $\mathbf{m} \in H_m(M, M \setminus U_1 \cup \cdots \cup U_{k+2}; \mathbf{Z})$ as desired, i.e., we go from k to $k+1$. To show the claim we first note that

$$M \setminus ((A \cup B) \cap C) = (M \setminus (A \cap C)) \bigcap (M \setminus (B \cap C)),$$

and setting $D = A \cup B$ in the Mayer-Vietoris sequence, we obtain the exact sequence

$$0 \rightarrow H_m(M, M \setminus D \cap C) \rightarrow H_m(M, M \setminus A \cap C) \oplus H_m(M, M \setminus B \cap C) \rightarrow H_m(M, M \setminus A \cap B \cap C). \tag{3.2.8}$$

From the induction hypothesis we constructed elements $\mathbf{m}_{A \cap C} \in H_m(M, M \setminus (A \cap C); \mathbf{Z})$ and $\mathbf{m}_{B \cap C} \in H_m(M, M \setminus (B \cap C); \mathbf{Z})$ which are images of $\mathbf{m}_C \in H_m(M, M \setminus C; \mathbf{Z})$ under the map induced from the inclusions $(M, M \setminus C)$ in $(M, M \setminus (A \cap C))$ and $(M, M \setminus (B \cap C))$. Therefore the image of $(\mathbf{m}_{A \cap C}, \mathbf{m}_{B \cap C})$ in $H_m(M, M \setminus (A \cap B \cap C); \mathbf{Z})$ in the exact sequence (3.2.8) vanishes and we have unique element $\mathbf{m}_{(A \cup B) \cap C} \in H_m(M, M \setminus ((A \cup B) \cap C); \mathbf{Z})$ mapping to $(\mathbf{m}_{A \cap C}, \mathbf{m}_{B \cap C})$. Clearly this element is also the image of $\mathbf{m}_C$ in

$H_m(M, M\setminus((A\cup B)\cap C);\mathbf{Z})$ under the map induced by the inclusion of $(M, M\setminus C)$ in $(M, M\setminus((A\cup B)\cap C))$. We have the commutative row exact diagram

$$\begin{array}{ccccc} 0 & \to & H_m(M, M\setminus((A\cup B)\cap C)) & \longrightarrow & H_m(M, M\setminus(A\cap C))\oplus H_m(M, M\setminus(B\cap C)) \\ & & \nwarrow & & \nearrow \\ & & & H_m(M, M\setminus(A\cup B)) & \end{array}$$

In view of exactness and commutativity of the diagram, $\mathbf{m}_{(A\cup B)\cap C}$, which is the image of $\mathbf{m}_C$, is also the image of $\mathbf{m}_{A\cup B}\in H_m(M, M\setminus(A\cup B);\mathbf{Z})$ in $H_m(M, M\setminus((A\cup B)\cap C);\mathbf{Z})$. This proves the claim and the proof of the theorem is complete. ♣

The notion of orientability is defined relative to the case $R=\mathbf{Z}$. Notice that if $R=\mathbf{Z}/2$, then $H_m(M, M\setminus U_i;\mathbf{Z}/2)\simeq\mathbf{Z}/2$, and since $1=-1$ in $\mathbf{Z}/2$, there is trivially a unique choice for the local orientation classes. Therefore every manifold is $\mathbf{Z}/2$ orientable! This explains why $H_2(\mathbf{RP}(2);\mathbf{Z}/2)\simeq\mathbf{Z}/2$ (see example **??**) while $H_2(\mathbf{RP}(2);\mathbf{Z})=0$. $\mathbf{RP}(2)$ is not orientable. Sometimes it useful to set $R=\mathbf{Z}/2$, in which case we refer to $\mathbf{Z}/2$ *orientability.*

3.2.6 Some Basic Examples

Example 3.2.9 Let us compute the homology of $M\sharp N$ where M and N are compact oriented manifolds and θ satisfies the above condition so that $M\sharp N$ is an orientable manifold. By example 3.1.4, $H_m(M';R)=0$. The homology exact sequence for (M, M') is

$$\cdots\to H_j(M';R)\to H_j(M;R)\to H_j(M,M';R)\to H_{j-1}(M';R)\to\cdots$$

Since $H_j(M,M';R)=0$ for $j\neq n$ and $H_m(M;R)\to H_m(M,M';R)$ is an isomorphism by orientability, $H_j(M';R)\simeq H_j(M;R)$ for $j<m$. Applying the Mayer-Vietoris sequence we obtain

$$\cdots\to H_j(A;R)\to H_j(M';R)\oplus H_j(N';R)\to H_j(M\sharp N;R)\to H_j(A;R)\to\cdots$$

Note that *a priori* the map $H_{m-1}(A;R)\to H_{m-1}(M';R)\oplus H_{m-1}(N';R)$ depends on θ, however, under the orientability hypothesis $H_m(M\sharp N;R)\to H_{m-1}(A;R)$ is surjective and former map vanishes. It follows that

$$H_j(M\sharp N;R)\simeq H_j(M';R)\oplus H_j(N';R)\simeq H_j(M;R)\oplus H_j(N;R)\quad\text{for } j\neq 0,m.$$

Clearly $H_m(M\sharp N;R)\simeq R\simeq H_\circ(M\sharp N;R)$. In particular, for even dimensional compact orientable manifolds we have $\chi(M\sharp N)=\chi(M)+\chi(N)-2$. (See exercise 3.4.2 below). ♠

Example 3.2.10 Clearly the Euler characteristic of $\underbrace{\mathbf{CP}(2)\sharp\cdots\sharp\mathbf{CP}(2)}_{n\text{ copies}}$ is $n+2$. Let $N=S^1\times S^3$, then $\chi(M\sharp N)=\chi(M)-2$, and consequently there are compact orientable 4-manifolds of arbitrary Euler characteristic. Using corollary 3.1 we see that the same is true of manifolds of dimension $4k$. Applying the same corollary to $S^2\times M$ we see that there are compact orientable manifolds of dimension $4k+2$, $k\geq 1$, with arbitrary even Euler characteristic. It is a consequence of Poincaré duality (see section "Cohomology" below) that the Euler characteristic of an odd dimensional compact orientable manifold is zero, and compact orientable manifolds of dimension $4k+2$ have even Euler characteristic. ♠

Exercise 3.2.12 *With the above notation but without the assumptions of compactness or orientability, show that*

$$\chi(M\sharp N) = \begin{cases} \chi(M) + \chi(N) - 2, & if\ m=2k; \\ \chi(M) + \chi(N), & if\ m=2k\text{-}1. \end{cases}$$

(Use the technique of example 3.2.4, e.g. by applying formula (3.2.1) to the long exact sequence for the pair $(M\sharp N, N')$ which by excision is homologically indistinguishable from $(M', M' \cap N')$. Similarly, show that the alternating sum of the dimensions of $H_j(M', M' \cap N')$ is equal to $\chi(M') + 1 - (-1)^m$ and use example 3.2.3 to complete the proof.)

Exercise 3.2.13 *Let $M_1 = T^2$, $M_g = M_{g-1}\sharp M_1$. Show that the homology groups of M_g (called surface of genus g or sphere with g handles) are $\mathbf{Z}^{2g}$ in dimension 1, and $\mathbf{Z}$ in dimensions 0 and 2. In particular the Euler characteristic of M_g is $2 - 2g$.*

Example 3.2.11 Assume that the finite group G acts on the compact oriented manifold M in a *fixed-point free* manner, i.e., if $e \neq g \in G$ then x and $g.x$ are distinct for all $x \in M$. Assume also that the action of G is by simplicial maps for some fixed triangulation of M. Since $g^{|G|} = e$ ($|G|$ denotes the order of G), the map induced by g on $H_m(M;\mathbf{Z})$ is multiplication by ± 1. Assume the induced map is $+1$ for all $g \in G$, i.e., G acts a group of orientation preserving transformations of M. Since the action is fixed-point free, there is a subset $\{s_1, \cdots, s_k\}$ of oriented m-simplices of M such that the orientation class is

$$\lfloor M \rfloor = \sum_{g\in G} \sum_i g.(s_i).$$

Then the quotient manifold $N = G \setminus M$ is also orientable, and an orientation class for N is $\lfloor N \rfloor = \sum_i \bar{s}_i$ where $\bar{s}_i$ is the image of s_i in N. Consequently, the map $H_m(M;\mathbf{Z}) \rightarrow H_m(N;\mathbf{Z})$ induced by the projection $M \rightarrow N$ is multiplication by $|G|$. In particular, let $M = S^m$, and $G = \mathbf{Z}/2$ be the group generated by the antipodal map on S^m. The quotient space, *the real projective space*, will be denoted by $\mathbf{RP}(m)$. Then by exercise 3.4.3 $\mathbf{RP}(m)$ is orientable for m odd, and the induced map $H_m(S^m;\mathbf{Z}) \rightarrow H_m(\mathbf{RP}(m);\mathbf{Z})$ is multiplication by 2 in this case. ♠

Example 3.2.12 As a first step in the computation of the homology of $\mathbf{RP}(m)$, we obtain a cell decomposition for it. Clearly $\mathbf{RP}(1) \simeq S^1$ and consequently it has a cell decomposition with one cell in dimensions 0 and 1. Inductively assume that $\mathbf{RP}(m-1)$ has been realized as a cell complex with one cell in each of the dimensions $0, 1, \cdots, m-1$. Now $\mathbf{RP}(m-1)$ is the quotient of the equator of S^m under the anti-podal map. It follows that $\mathbf{RP}(m)$ can be obtained from $\mathbf{RP}(m-1)$ by attaching an m-cell (e.g., the northern hemisphere in S^m) with the attaching map on the boundary S^{m-1} being the obvious projection $S^{m-1} \rightarrow \mathbf{RP}(m-1)$. This gives the realization of $\mathbf{RP}(m)$ as a cell complex with one cell in each dimension $0, 1, \cdots, m$. ♠

Example 3.2.13 We use the cell decomposition described in example 3.4.2 to compute the homology of $\mathbf{RP}(3)$ and $\mathbf{RP}(4)$. Let U be a neighborhood of $\mathbf{RP}(2)$ in $\mathbf{RP}(3)$ such that $\mathbf{RP}(2)$ is a deformation retract of U, and let V be the interior of the disc B^3 which is the unique 3-cell of $\mathbf{RP}(3)$. Then $A = U \cap V$ has the homotopy type of S^2. Applying the Mayer-Vietoris sequence and example 3.1.5 we obtain

$$0 = H_1(A;\mathbf{Z}) \rightarrow \mathbf{Z}/2 \rightarrow H_1(\mathbf{RP}(3);\mathbf{Z}) \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \oplus \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow 0.$$

Clearly any homomorphism $\mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$ is either the zero map or is injective. Therefore exactness of the sequence requires $\mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$ to be injective, and consequently $H_1(\mathbf{RP}(3); \mathbf{Z}) \simeq \mathbf{Z}/2$. Furthermore, from the vanishing of $H_2(U; \mathbf{Z})$, $H_2(V; \mathbf{Z})$ and $H_1(A; \mathbf{Z})$ it follows that $H_2(\mathbf{RP}(3); \mathbf{Z}) = 0$. Since $\mathbf{RP}(3)$ is orientable (example 3.4.1), $H_3(\mathbf{RP}(3); \mathbf{Z}) \simeq \mathbf{Z}$. Clearly $H_\circ(\mathbf{RP}(3); \mathbf{Z}) \simeq \mathbf{Z}$. For $\mathbf{RP}(4)$, let U be a neighborhood of $\mathbf{RP}(3)$ in $\mathbf{RP}(4)$ such that $\mathbf{RP}(3)$ is a deformation retract of U, and V be the interior of the 4-cell B^4 that is attached to $\mathbf{RP}(3)$ to obtain $\mathbf{RP}(4)$. Then $A = U \cap V$ has the homotopy type of S^3. The Mayer-Vietoris sequence yields

$$0 \to H_4(\mathbf{RP}(4); \mathbf{Z}) \to H_3(A; \mathbf{Z}) \to H_3(U; \mathbf{Z}) \to H_3(\mathbf{RP}(4); \mathbf{Z}) \to 0.$$

Since the attaching map $\partial B^4 \to \mathbf{RP}(3)$ is the quotient mapping $S^3 \to \mathbf{RP}(3)$, it follows from example 4.2 that the homomorphism $H_3(A; \mathbf{Z}) \to H_3(U; \mathbf{Z})$ is multiplication by 2. Therefore $H_3(\mathbf{RP}(4); \mathbf{Z}) \simeq \mathbf{Z}/2$, $H_4(\mathbf{RP}(4); \mathbf{Z}) = 0$, and $\mathbf{RP}(4)$ is not orientable. It follows from proposition 3.1 that $H_j(\mathbf{RP}(4); \mathbf{Z}) \simeq H_j(\mathbf{RP}(3); \mathbf{Z})$ for $j \leq 2$. The same arguments are applicable to all real projective spaces as noted in exercise 3.4.5 below. ♠

Exercise 3.2.14 *(a) Show that the homology of* $\mathbf{RP}(2n+1)$ *is*

$$H_j(\mathbf{RP}(2n+1); \mathbf{Z}) \simeq \begin{cases} \mathbf{Z}, & \text{if } j = 0, 2n+1; \\ \mathbf{Z}/2, & \text{if } j = 2k-1 \text{ and } 1 \leq k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

(b) Show that the homology of $\mathbf{RP}(2n)$ *is*

$$H_j(\mathbf{RP}(2n); \mathbf{Z}) \simeq \begin{cases} \mathbf{Z}, & \text{if } j = 0; \\ \mathbf{Z}/2, & \text{if } j = 2k-1 \text{ and } 1 \leq k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.2.14 Let $N_1 = \mathbf{RP}(2)$, and $N_g = N_{g-1} \sharp N_1$, where $\sharp$ is the operation defined in example 3.2.10 Let us compute the homology of N_2. Let N_1' be the manifold obtained from N_1 by removing a small disc, and N_1'' be another copy of N_1'. We regard N_1' and N_1'' as open subsets corresponding to the first and second factors in the decomposition $N_2 = N_1 \sharp N_1$. Let U and V be neighborhoods of N_1' and N_1'' in N_2 such that $N_2 = U \cup V$, N_1' (resp. N_1'') is a deformation retract of (or even diffeomorphic to) U (resp. V), and $U \cap V$ has the homotopy type of a circle. Then from the Mayer-Vietoris sequence we obtain

$$0 \to H_2(N_2; \mathbf{Z}) \to H_1(A; \mathbf{Z}) \to H_1(U; \mathbf{Z}) \oplus H_1(V; \mathbf{Z}) \to H_1(N_2; \mathbf{Z}) \to 0, \tag{3.2.9}$$

where we are using the fact that the map $H_\circ(A; \mathbf{Z}) \to H_\circ(U; \mathbf{Z}) \oplus H_\circ(V; \mathbf{Z})$ is injective. It is clear that N_1' can be regarded as the quotient of $S = S^2 \setminus (B_1 \cup B_1')$ where antipodal points are identified, and B_1 and B_1' are two small discs around the north and south poles of S^2 which are mapped onto each other by the antipodal transformation. There is a homotopy $F : S \times I \to S$ such that $F(x, 0) = x$, $F(\cdot, 1)$ maps S onto the equator, and $F(\cdot, t)$ commutes with the antipodal map for all t. Since the quotient of S^1 under the antipodal map is homeomorphic to S^1, we see that N_1' and N_1'' have the homotopy type of the circle. Hence $H_1(U; \mathbf{Z}) \simeq \mathbf{Z} \simeq H_1(V; \mathbf{Z})$. We

Claim: The mapping

$$\mathbf{Z} \to H_1(A;\mathbf{Z}) \to H_1(U;\mathbf{Z}) \oplus H_1(U;\mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}$$

is $j \to (2j, 2j)$.

Proof of Claim - We consider the decomposition $\mathbf{RP}(2) = N_1' \cup B$ where B is a small 2-disc and $N_1' \cap B$ has the homotopy type of a circle. To prove the claim it suffices to show that the map induced by the inclusion $N_1' \cap B \hookrightarrow N_1'$ is multiplication by 2. From the Mayer Vietoris sequence we have

$$\begin{array}{ccccccc} H_1(N_1' \cap B;\mathbf{Z}) & \to & H_1(N_1';\mathbf{Z}) & \to & H_1(\mathbf{RP}(2);\mathbf{Z}) & \to & 0 \\ \| & & \| & & \| & & \\ \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z}/2 & & \end{array}$$

The claim follows immediately.

Having proven the lemma, we substitute the conclusion in (3.2.9) to obtain

$$H_1(N_2;\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}/2, \qquad H_2(N_2;\mathbf{Z}) = 0.$$

N_2 is called the *Klein Bottle*, and N_g is known as the *sphere with g cross caps*. The homology of N_g can be computed in the same and is given in exercises 3.2.15 3.2.16 below. ♠

Exercise 3.2.15 *(a) Let N' be the manifold obtained from N by removing a small disc. Show that*

$$H_1(N_2';\mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}.$$

(b) Show that $H_2(N_3;\mathbf{Z}) = 0$ and hence N_3 is not orientable. (Consider the long exact sequence for the pair (N_3, N_2'). Note that $H_2(N_3, N_2';\mathbf{Z}) \simeq H_2(N_1', \partial N_1';\mathbf{Z})$ by excision. Use the long exact sequence for the pair $(N_1, \partial N_1')$ and injectivity of the map $H_1(\partial N_1';\mathbf{Z}) \to H_1(N_1';\mathbf{Z})$, as for example in the proof of lemma 4.3, to show the vanishing of $H_2(N_3, N_2';\mathbf{Z})$ and $H_2(N_3;\mathbf{Z})$.)

(c) - Let U and V be small neighborhoods of N_1' and N_2' in N_3, diffeomorphic to N_1' and N_2' respectively, and such that $U \cap V$ has the homotopy type of a circle. Show that the map

$$\mathbf{Z} \simeq H_1(U \cap V;\mathbf{Z}) \to H_1(U;\mathbf{Z}) \oplus H_1(V;\mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$$

is $j \to (2j, 2j, 2j)$.

(d) - Show that

$$H_1(N_3;\mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/2.$$

Exercise 3.2.16 *Using the above methods show that*

$$H_\circ(N_g;\mathbf{Z}) \simeq \mathbf{Z}, \qquad H_1(N_g;\mathbf{Z}) \simeq \mathbf{Z}^{g-1} \oplus \mathbf{Z}/2, \qquad H_j(N_g;\mathbf{Z}) = 0, \quad \text{for} \quad j \geq 2.$$

3.2.7 Grassmann Manifolds

Let $E \simeq \mathbf{C}^n$, and $\mathbf{s} = [s_1, \cdots, s_r]$ be a sequence of positive integers such that $0 < s_1 < \cdots < s_r < n$. Recall from chapter 1 that the (complex) generalized flag manifold $\mathbf{F_s}$ is the set of all sequences of subspaces $0 \subset V_1 \subset \cdots \subset V_r \subset E$ such that $\dim V_i = s_i$. In particular, for $r = 1$ and $\mathbf{s} = [1]$, $\mathbf{F_s} = \mathbf{CP}(n-1)$. Recall also that generalized flag manifolds for the case $r = 1$ are called (complex) Grassmann manifolds, and are denoted by $\mathbf{G}_{s,n-s}$ if $\mathbf{s} = [s]$. For $r = n-1$, $s_i = i$ necessarily, and $\mathbf{F_s}$ is denoted by $\mathbf{F}_n$. A point in the flag manifold $\mathbf{F}_n$ (or $\mathbf{F_s}$) is a called a *flag*. As noted in chapter 1, generalized flag manifolds are homogeneous spaces for $U(n)$ (or $SU(n)$) and $GL(n;\mathbf{C})$ (or $SL(n;\mathbf{C})$).

Example 3.2.15 There is another useful realization of the complex Grassmann manifold. Let $v_1, \cdots, v_k$ be a basis for the k-dimensional subspace $V \subset E = \mathbf{C}^{n+k}$. Then $0 \neq v_1 \wedge \cdots \wedge v_k \in \bigwedge^k E$, and two bases for V define the same element, up to a non-zero scalar, in $\bigwedge^k E$. In this way we obtain an embedding of $\mathbf{G}_{s,n}$ in $\mathbf{CP}(\binom{n+k}{k} - 1)$. We refer to this mapping $\pi : \mathbf{G}_{s,n} \to \mathbf{CP}(\binom{n+k}{k} - 1)$ as the *Plücker embedding*. The image of π is precisely the submanifold of $\mathbf{CP}(\binom{n+k}{k} - 1)$ represented by decomposable vectors in $\bigwedge^k E$. Recall that a vector $v \in \bigwedge^k E$ is *decomposable* if it can be written in the form $v = v_1 \wedge \cdots \wedge v_k$. We fix a basis $e_1, \cdots, e_{k+n}$ for E, and represent the basis $v_1, \cdots, v_k$ for V as an $(n+k) \times k$ matrix $A(V)$. Then the coordinates of $\pi(V)$ in $\bigwedge^k E$ relative to the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ are the $k \times k$ minors of $A(V)$. These are called the *Plücker coordinates*. Specializing to the case $k = 2 = n$, and writing $A(V) = (a_{i\alpha})$ with $1 \leq i \leq 4,\ 1 \leq \alpha \leq 2$, we see that the Plücker coordinates of a point are $(a_{i1}a_{j2} - a_{i2}a_{j1})_{1 \leq i,j \leq 4}$. Set $\zeta_{ij} = a_{i1}a_{j2} - a_{i2}a_{j1}$, then $\zeta_{ij} = -\zeta_{ji}$ and

$$\zeta_{12}\zeta_{34} + \zeta_{23}\zeta_{14} + \zeta_{31}\zeta_{24} = 0. \tag{3.2.10}$$

Since $\mathbf{G}_{2,2}$ is a complex manifold of dimension four, this quadratic relation actually defines $\pi(\mathbf{G}_{2,2})$ as a submanifold of $\mathbf{CP}(5)$. ♠

Example 3.2.16 In this example and exercise (3.2.17) we use the Plücker coordinates to investigate the curvature tensor of a four dimensional Riemannian manifold. The result will be useful in the application of Gauss-Bonnet theorem to four dimensional manifolds later in this chapter, but will not used elsewhere. The parametrization of $\mathbf{G}_{2,2}$ by Plücker coordinates is valid for the real Grassmann manifold $\mathbf{G}^{\circ}_{2,2}(\mathbf{R})$ of oriented 2-planes in $\mathbf{R}^4$ as well. With the above notation, except that a_{ij}'s are real, we note that the sectional curvature of the plane spanned by v_1, v_2 is

$$R(v_1 \wedge v_2) = -\frac{1}{4}\sum_{ijkl} R_{ijkl}\zeta_{ij}\zeta_{kl},$$

the components of the curvature tensor R_{ijkl} are relative to the orthonormal basis $e_1, \cdots, e_4$. The equations defining the Grassmann manifold $\mathbf{G}^{\circ}_{2,2}(\mathbf{R}) \subset \mathbf{R}^6$ are (3.2.10) and the normalization condition

$$\sum_{i<j} \zeta_{ij}^2 \ - \ 1 \ = \ 0.$$

The critical points of the sectional curvature function $R(v_1 \wedge v_2)$ are the points where $\mathrm{grad}(R(v_1 \wedge v_2))$ is orthogonal to $\mathbf{G}^\circ_{2,2}(\mathbf{R})$. Here grad is computed relative to the inner product $(e_i \wedge e_j, e_k \wedge e_l) = \delta_{ik}\delta_{jl}$ for $i < j$ and $k < l$, on $\mathbf{R}^6$. Without loss of generality we may assume $e_1 \wedge e_2$ is critical. Then, it is a simple calculation to see that the condition on $\mathrm{grad}K(e_1 \wedge e_2)$ becomes $R_{1213} = R_{1214} = 0$. Interchanging the roles of e_1 and e_2 we obtain the additional relation $R_{1223} = R_{1224} = 0$. If furthermore the vector e_3 is such that the plane $e_1 \wedge e_3$ is critical for the restriction of $R(v_1 \wedge v_2)$ to all planes which intersect the planes spanned by $\{e_1, e_2\}$ and $\{e_3, e_4\}$ nontrivially, then a similar calculation implies $R_{1314} = R_{1323} = 0$. Therefore we can assume that the orthonormal basis $e_1, \cdots, e_4$ is such that

$$R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1314} = R_{1323} = 0.$$

Therefore by appropriate choice of basis we can make six out of the twenty coefficients of the curvature tensor zero. ♠

Exercise 3.2.17 *Show that for the choice of frame $e_1, \cdots, e_4$ described in example (3.2.16), the plane spanned by e_i, e_j, $i \neq j$, is critical for the sectional curvature function.*

Remark 3.2.4 To generalize example 3.2.15 to the case of the Grassmann manifold of k-planes in $\mathbf{C}^{k+n}$, one represents the Plücker coordinates as $k \times k$ minors of a $(k+n) \times k$ matrix just as above for the case $k = n = 2$. Then there are a number of quadratic equations between the Plücker coordinates similar to (3.2.10). The set of all such relations defines the Grassmann $\mathbf{G}_{k,n}$ as a subvariety of $\mathbf{CP}(\binom{n+k}{k} - 1)$. See [HP] for a treatment of equations defining Grassmann manifolds.

It is useful to note that we have an embedding

$$\mathbf{p} = (p_1, \cdots, p_r) : \mathbf{F_s} \longrightarrow \prod \mathbf{G}_{s_i, n-s_i}, \tag{3.2.11}$$

where for a flag $f : 0 \subset V_1 \subset \cdots \subset V_r \subset \mathbf{C}^n$, $p_i(f)$ is the point in $\mathbf{G}_{s_i,n-s_i}$ represented by the subspace V_i. Therefore a sequence $\{f_m\}$ converges to a flag f if $p_i(f)$ converge to $p_i(f)$ in $\mathbf{G}_{s_i,n-s_i}$. Note also that $p_i : \mathbf{F_s} \to \mathbf{G}_{s_i,n-s_i}$ is a fibre bundle with the fibre $p_i^{-1}(V_i)$ consisting of all flags of the form $f' : 0 \subset V'_1 \subset \cdots \subset V'_r \subset \mathbf{C}^n$ with $V'_i = V_i$.

The computation of the homology of generalized flag manifolds will be a good demonstration of the application of cell complexes, and furthermore flag manifolds have important implications for geometric problems. Since $\mathbf{F}_n$ and the Grassmann manifolds play a more important role than other flag manifolds, we concentrate on these two cases and indicate the changes necessary for the general case. In this way we simplify the notation. We begin by describing a cell structure for $\mathbf{G}_{k,n}$.

Let $e_1, \cdots, e_{n+k}$ be the standard basis for $E = \mathbf{C}^{n+k}$ and E_j be the subspace spanned by $e_1, \cdots, e_j$. We call $f_\circ : 0 \subset E_1 \subset \cdots \subset E_{n+k-1} \subset E$ the *standard flag*. To a k-dimensional subspace $V \subset E$ we assign the sequence $\mathbf{a} = \mathbf{a}_V = [a_1, \cdots, a_k]$ of non-negative integers $0 \leq a_1 \leq \cdots a_k \leq n$ where a_i is the smallest integer such that $\dim(V \cap E_{a_i+i}) = i$. We call $\mathbf{a}_V$ the *symbol sequence* of V. It is trivial that the mapping $V \to \mathbf{a}_V$ is onto the set of sequences $[a_1, \cdots, a_k]$ with the property $0 \leq a_1 \leq \cdots \leq a_k \leq n$. We denote the subset of $\mathbf{G}_{k,n}$ consisting of subspaces with a given symbol sequence $\mathbf{a} : 0 \leq a_1 \leq \cdots a_k \leq n$ by $\mathcal{V}_\mathbf{a}$. If V has the symbol sequence $\mathbf{a}$ then it has a basis of the form

$$v_i = e_{a_i+i} + \sum_{j<a_i+i} c_{ji} e_j. \tag{3.2.12}$$

The basis $v_1, \cdots, v_k$ is not uniquely determined by V, however if we impose the additional condition that $c_{ji} = 0$ if $j = a_l + l$ for some $l < i$, then the basis is uniquely determined by V. Conversely, any such set $\{v_1, \cdots, v_k\}$ determines a k-dimensional subspace with symbol sequence $[a_1, \cdots, a_k]$. It follows immediately that $\mathcal{V}_{\mathbf{a}}$ is parametrized by the coefficients c_{ji} satisfying the condition

$$c_{ji} = 0 \quad \text{if} \quad j = a_l + l \text{ for some } l < i. \tag{3.2.13}$$

Therefore $\mathcal{V}_{\mathbf{a}}$ is analytically homeomorphic to $\mathbf{C}^{d_{\mathbf{a}}}$ where $d_{\mathbf{a}} = \sum a_i$. For a convenient matrix description of $\mathcal{V}_{\mathbf{a}}$ see example 3.2.17 below.

Exercise 3.2.18 *Show that for fixed k and n, the set of symbol sequences $\mathbf{a}$ can be naturally identified with $\mathcal{S}_{k+n}/\mathcal{S}_k \times \mathcal{S}_n$, and therefore its cardinality is $\binom{n+k}{k}$.*

In order to realize $\mathbf{G}_{k,n}$ as a cell complex we must still show that the inclusion of the open cell $\imath : \breve{B}^{2d_{\mathbf{a}}} \simeq \mathcal{V}_{\mathbf{a}}$ in $\mathbf{G}_{k,n}$ extends to a continuous map of the closed cell $B^{2d_{\mathbf{a}}}$ into $\mathbf{G}_{k,n}$ as described in the definition of a cell complex. We explicitly describe this extension of the map $\imath$ to the boundary $\partial B^{2d_{\mathbf{a}}}$ by regarding the boundary points as limits of going to infinity along rays. A point in $V \in \mathcal{V}_{\mathbf{a}}$ is specified by the coordinates $\{c_{ji}\}_{j<a_i+i}$ subject to (3.2.13). Let $\lambda > 0$ be a real number and denote the point in $\mathcal{V}_{\mathbf{a}}$ with coordinates $\{\lambda c_{ji}\}_{j<a_i+i}$ subject to (3.2.13) by $V(\lambda)$. To determine $\lim_{\lambda\to\infty} V(\lambda)$ we use the Plücker embedding $\pi : \mathbf{G}_{k,n} \to \mathbf{CP}(N)$ where $N = \binom{n+k}{k} - 1$. Recall that the image of $V(\lambda)$ under the Plücker embedding is the vector

$$(e_{a_k+k} + \lambda \sum_{j<a_k+k} c_{jk} e_j) \wedge (e_{a_{k-1}+k-1} + \lambda \sum_{j<a_{k-1}+k-1} c_{j\ k-1} e_j) \wedge \cdots \wedge (e_{a_1+1} + \lambda \sum_{j<a_1+1} c_{j1} e_j).$$

Expanding in terms of powers of λ we obtain:

$$\lambda^k \xi_k + \lambda^{k-1} \xi_{k-1} + \cdots + \xi_\circ.$$

It is clear that the desired limit is the vector ξ_l where $l \leq k$ is the largest integer such that $\xi_l \neq 0$. Therefore the extension of $\imath$ to the boundary is $\imath(\lim_{\lambda\to\infty} V(\lambda)) = \xi_l$. To see what the symbol sequence of the limiting subspace represented by ξ_l is, let $v_1, \cdots, v_k$ be a basis for a k-dimensional subspace of $\mathbf{C}^{k+n}$ in the form specified by (3.2.12). Then we can write

$$v_k \wedge \cdots \wedge v_1 = e_{a_k+k} \wedge \cdots \wedge e_{a_1+1} + \sum_{l_k > \cdots > l_1} c_{l_k \cdots l_1} e_{l_k} \wedge \cdots \wedge e_{l_1},$$

and the sequence $a_k + k, \cdots, a_1 + 1$ is characterized by the property that if $c_{l_k \cdots l_1} \neq 0$, then $a_k + k \geq l_k, a_{k-1}+k-1 \geq l_{k-1}, \cdots, a_1+1 \geq l_1$. From this characterization it follows immediately that if $\mathbf{b} = [b_1, \cdots, b_k]$ is the symbol sequence of the limiting subspace represented by ξ_l, then $a_i \geq b_i$ for all i. It is easy to see the converse, i.e., if $\mathbf{b} = [b_1, \cdots, b_k]$ is the symbol sequence for a k-dimensional subspace V and $a_i \geq b_i$ then V lies in $\bar{\mathcal{V}}_{\mathbf{a}}$. Thus it is reasonable to define a partial order on the set of all sequences $\mathbf{a} : 0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$ as follows:

$$\mathbf{b} \preceq \mathbf{a} \quad \text{if} \quad b_i \leq a_i \quad \text{for all } i.$$

If $\mathbf{b} \preceq \mathbf{a}$ and $\mathbf{b} \neq \mathbf{a}$, then we write $\mathbf{b} \prec \mathbf{a}$. Summarizing, we have shown

Proposition 3.2.4 *The inclusion $\imath : \breve{B}^{2d_{\mathbf{a}}} \simeq \mathcal{V}_{\mathbf{a}} \to \mathbf{G}_{k,n}$ extends to a continuous map of the closed cell $B^{2d_{\mathbf{a}}}$ mapping $\imath(\partial B^{2d_{\mathbf{a}}})$ to $\cup_{\mathbf{b}\prec\mathbf{a}} \mathcal{V}_{\mathbf{b}}$. This gives $\mathbf{G}_{k,n}$ the structure of a cell complex with only even dimensional cells.*

Corollary 3.2.1 is now applicable and we obtain

Corollary 3.2.4 *The homology of the Grassmann manifold is given by*

$$H_j(\mathbf{G}_{k,n}; \mathbf{Z}) = \begin{cases} 0 & \text{if } j = 2i+1 \text{ or } j > 2kn \\ \mathbf{Z}^{p_k(l)} & \text{if } j = 2l \text{ and } l \leq kn, \end{cases}$$

where $p_k(l)$ (the partition function) is the number of ways l can be written as a sum of k non-negative integers $l = a_1 + \cdots + a_k$ with $a_i \leq a_{i+1} \leq n$. The Euler characteristic of $\mathbf{G}_{k,n}$ is $\chi(\mathbf{G}_{k,n}) = \binom{n+k}{k}$, and we have the numerical identity

$$\sum_{j=0}^{kn} p_k(j) = \binom{n+k}{k}.$$

We denote the closure of the cell $\mathcal{V}_{\mathbf{a}}$ by $\bar{\mathcal{V}}_{\mathbf{a}}$ and call it the *Schubert cycle* associated to $\mathbf{a}$. Schubert cycles form a basis for the homology of the Grassmann manifold as stated in corollary 5.1. In general, the Schubert cycles are not manifolds. From the above description of the closure of $\imath(B^{2d_{\mathbf{a}}})$ it is not difficult to see that $\imath(\partial B^{2d_{\mathbf{a}}}) = \cup_{\mathbf{b}\prec\mathbf{a}} \mathcal{V}_{\mathbf{b}}$ and therefore $\bar{\mathcal{V}}_{\mathbf{a}}$ is *stratified* by those submanifolds $\mathcal{V}_{\mathbf{b}}$ for which $b_i \leq a_i$ for all i, i.e.,

$$\bar{\mathcal{V}}_{\mathbf{a}} = \cup_{\mathbf{b}\preceq\mathbf{a}} \mathcal{V}_{\mathbf{b}}.$$

Exercise 3.2.19 *What are the Scubert cell in the closure of $calV_{\mathbf{a}} \subset \mathbf{G}_{3,3}$ where $\mathbf{a} = [1,1,3]$? Explicitly describe how every point in $\mathcal{V}_{[0,1,2]}$ is the limit of a ray in $\mathcal{V}_{\mathbf{a}}$.*

Example 3.2.17 We can write down the equations defining Schubert cycles. The principle is most easily demonstrated by an example which makes it clear what the defining equations are in general. Consider for example the Grassmann manifold of 3-planes in $\mathbf{C}^7$, and let $\mathbf{a} = [1,3,4]$. Then in view of the description of the Schubert cell corresponding to the symbol sequence $\mathbf{a}$, we consider all 7×3 matrices of the form

$$\begin{pmatrix} \star & \star & \star \\ 1 & 0 & 0 \\ 0 & \star & \star \\ 0 & \star & \star \\ 0 & 1 & 0 \\ 0 & 0 & \star \\ 0 & 0 & 1 \end{pmatrix},$$

where $\star$ means an arbitrary complex number. Let ζ_{ijk} denote the determinant of the 3×3 matrix consisting of rows $i<j<k$ of this matrix. Then $\zeta_{ijk}=0$ if $i,j,k\geq 3$ or $i=2$ and $j,k\geq 6$. This set of linear equations (together with the quadratic equations defining the Grassmann manifold as explained in 3.2.4) define the closed Schubert cycle corresponding to the symbol $\mathbf{a}$. The generalization of this to Scubert cycles in $\mathbf{G}_{k,n}$ is immediate. ♠

Understanding the structure of the Schubert cycles is an interesting problem, but we shall not pursue this matter any further here.

3.2.8 Flag Manifolds

Next we compute the homology of the flag manifold $\mathbf{F}_n$. We fix the standard flag $f_\circ : 0\subset E_1\subset\cdots\subset E_{n-1}\subset\mathbf{C}^n$, and let $f : 0\subset V_1\subset\cdots\subset V_{n-1}\subset\mathbf{C}^n$ be a flag. To f we assign $n-1$ sequences $\mathbf{k}^j=\mathbf{k}^j(f)=[k_1^j,k_2^j,\cdots,k_j^j]$, $j=1,\cdots,n-1$ where $k_i^j=a_i^j+i$, and $\mathbf{a}^j=[a_1^j,\cdots,a_j^j]$ is the symbol sequence of $p_j(f)$ relative to the standard flag as described above. We now make the elementary observation (whose proof is left to the reader)

Lemma 3.2.4 *Let $V\subset V'\subset\mathbf{C}^n$ be linear subspaces. If* $\dim(V\cap E_{i-1})<\dim(V\cap E_i)$ *then*

$$\dim(V'\cap E_{i-1})<\dim(V'\cap E_i).$$

From 3.5.1 it follows immediately that

$$\{k_1^j,\cdots,k_j^j\}\subset\{k_1^{j+1},\cdots,k_{j+1}^{j+1}\}.$$

This fact allows us to assign a permutation $\sigma=\sigma(f)\in\mathcal{S}_n$ to every flag f. In fact, let $\sigma(f)(1)=k_1^1$ and $\sigma(f)(j)$ be the unique integer in $\{k_1^j,\cdots,k_j^j\}$ which is not among $\{k_1^{j-1},\cdots k_{j-1}^{j-1}\}$, and having determined $\sigma(f)(j)$ for $j=1,\cdots,n-1$, let $\sigma(f)(n)$ be the remaining integer among $\{1,\cdots,n\}$. The mapping $f\to\sigma(f)$ is onto the symmetric group. In fact for a given $\sigma\in\mathcal{S}_n$, let V_i be the span of $e_{\sigma(1)},\cdots,e_{\sigma(i)}$ in the definition of f to obtain a flag with $\sigma(f)=\sigma$. We call $\sigma(f)=\sigma$ the *permutation sequence* of the flag f.

Let $B(\sigma)=\{f\in\mathbf{F}_n|\sigma(f)=\sigma\}$. The structure of $B(\sigma)$ can be easily described. It is easy to see that $B(\sigma)$ is precisely the set of the flags $f : 0\subset V_1\subset\cdots\subset V_{n-1}\subset\mathbf{C}^n$ such that V_k is the span of the vectors $v_1,\cdots,v_k$ where

$$v_k=e_{\sigma(k)}+\sum_{j<\sigma(k)}c_{jk}e_j, \tag{3.2.14}$$

and c_{jk}'s are complex numbers (c_{jk} is defined only for $j<\sigma(k)$). The coefficients c_{jk} are not uniquely determined by the flag f, however, if we impose the additional requirement that

$$c_{jk}=0 \quad \text{if there is } i<k \text{ with } \sigma(i)=j, \tag{3.2.15}$$

then we obtain a natural one to one correspondence between the coefficients $\{c_{jk}\}$ and flags f with permutation sequence σ. Therefore $B(\sigma)$ is parametrized by the Euclidean space $\mathbf{C}^d$, where the dimension d remains to be determined.

While the dimension of the cells $B(\sigma)$ can be easily determined, it is convenient to introduce the notion of length function on the permutation group $\mathcal{S}_n$ at this point. There are different ways of representing permutations. By $\langle i_1 \cdots i_n \rangle$ we mean the permutation

$$\langle i_1 \cdots i_n \rangle = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix},$$

i.e., the permutation that moves the object in box 1 to box i_1 etc. The *length function* $L : \mathcal{S}_n \rightarrow \mathbf{Z}_+$ (non-negative integers) is defined algorithmically as follows: Let $\langle i_1 \cdots i_n \rangle \in \mathcal{S}_n$. Consider (i_1, i_2); and let $K(2) = 0$ if $i_1 < i_2$ and $K(2) = 1$ if $i_1 > i_2$. If necessary re-order the pair to obtain (j_1, j_2) with $j_1 < j_2$. Next consider (j_1, j_2, i_3), and let $K(3)$ be the number of adjacent transpositions necessary to make this 3-tuple in increasing order. Thus $K(3) = 0, 1$ or 2. Let (l_1, l_2, l_3) be the re-arrangement of (j_1, j_2, i_3) in increasing order. Continue the process in the obvious manner, and define

$$L(\langle i_1 \cdots i_n \rangle) = K(2) + K(3) + \cdots + K(n).$$

Let us demonstrate the procedure by an example

Example 3.2.18 Let us compute $L(\langle 6325174 \rangle)$:

63	36	$K(2) = 1$
362	236	$K(3) = 2$
2365	2356	$K(4) = 1$
23561	12356	$K(5) = 4$
123567	123567	$K(6) = 0$
1235674	1234567	$K(7) = 3$

Thus $L(\langle 6325174 \rangle) = 11$. ♠

Lemma 3.2.5 *With the above notation* $\dim_{\mathbf{R}} B(\sigma) = 2L(\sigma)$.

Proof - We had noted earlier that $B(\sigma)$ is the set of flags $f : 0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbf{C}^n$ such that V_k is the span of vectors $v_1, \cdots, v_k$, where v_k is given by (3.2.14) and the coefficients c_{jk} are complex numbers subject to the requirement (3.2.15). Let us set $\sigma(1) = i_1$, then there are $i_1 - 1$ adjacent transpositions involving i_1 in the computation of $L(\sigma)$, i.e., number of arbitrary coefficients c_{j1}. Let $\sigma(2) = i_2$. If $i_2 > i_1$ then there are $i_2 - 2$ adjacent transpositions involving i_2 and not i_1 in the computation of $L(\sigma)$. If $i_2 < i_1$, then there are $i_2 - 1$ adjacent transpositions involving i_2 and not i_1 in the definition of $L(\sigma)$. In either case this number is equal to the number of arbitrary coefficients c_{j2}. Proceeding in the obvious manner we immediately obtain the desired result. ♣

Just as in the case of the Grassmann manifolds we now have to make sure that the inclusion of the open disc $\imath : \mathbf{C}^{L(\sigma)} \simeq \check{B}^{2L(\sigma)} \hookrightarrow \mathbf{F}_n$ extends to a continuous mapping of the closed disc $B^{2L(\sigma)}$ as required by the definition of a cell complex. We do this by a procedure similar to one used for Grassmann manifolds, however, looking at $p_i(f) \in \mathbf{G}_{i,n-i}$, for $f \in \mathbf{F}_n$, and taking limits as before will not work as shown in the following example:

Example 3.2.19 Consider the flag in $f : V_1 \subset V_2 \subset V_3 \subset \mathbf{C}^4$ where

$$V_1 = \mathbf{C}(e_4 + a_3 e_3), \quad V_2 = V_1 + \mathbf{C}(e_3 + a_1 e_1 + a_2 e_2).$$

Now

$$p_1(f)(\infty) = \lim_{\lambda \to \infty} p_1(f)(\lambda) = \mathbf{C}e_3.$$

In the description of coordinates for Grassmann manifold given above, we have

$$p_2(f)(\lambda) = \mathbf{C}(e_3 + \lambda(a_1 e_1 + a_2 e_2)) + \mathbf{C}(e_4 - \lambda a_3(a_1 e_1 + a_2 e_2)).$$

Therefore

$$p_2(f)(\infty) = \lim_{\lambda \to \infty} p_2(f) = \mathbf{C}(a_1 e_1 + a_2 e_2) + \mathbf{C}(e_4 + a_3 e_3).$$

In particular, $p_1(f)(\infty) \not\subset p_2(f)(\infty)$ and this is why we cannot just take limits by looking at various $p_i(f)$'s and take limits as for Grassmann manifolds to obtain the bounday map for the flag manifold. ♠

Let f be specified by coordinates c_{jk} as in (3.2.14) subject to (3.2.15). Let $0 < \lambda \in \mathbf{R}$ and

$$v_k(\lambda) = e_{\sigma(k)} + \sum_{j < \sigma(k)} \lambda c_{jk} e_j,$$

and V_k^λ be the span of $v_1(\lambda), \cdots, v_k(\lambda)$. We also define the flag f_λ as

$$f_\lambda : 0 \subset V_1^\lambda \subset \cdots \subset V_{n-1}^\lambda \subset \mathbf{C}^n.$$

Now

$$v_1(\lambda) \wedge \cdots \wedge v_k(\lambda) = \lambda^k \xi_k + \lambda^{k-1} \xi_{k-1} + \cdots + \xi_\circ.$$

Let $l \leq k$ be the largest integer such that $\xi_l \neq 0$. Regarding $v_1(\lambda) \wedge \cdots \wedge v_k(\lambda)$ as a point in $\mathbf{CP}(\binom{n}{k} - 1)$, we obtain

$$\lim_{\lambda \to \infty} v_1(\lambda) \wedge \cdots \wedge v_k(\lambda) = \xi_l.$$

Since the set of decomposable vector is a compact submanifold of $\mathbf{CP}(\binom{n}{k} - 1)$, ξ_l represents a k-dimensional subspace of $\mathbf{C}^n$ which we denote by V_k^∞. It is a simple matter to verify that for $k < n$

$$V_k^\infty \subset V_{k+1}^\infty.$$

We now set

$$\imath(\lim_{\lambda \to \infty} f_\lambda) = f_\infty : 0 \subset V_1^\infty \subset \cdots \subset V_{n-1}^\infty \subset \mathbf{C}^n.$$

Exercise 3.2.20 *Let f be as in example 3.2.19. Show that*

$$V_1^\infty = \mathbf{C}e_3, \quad V_2^\infty = \mathbf{C}e_3 + \mathbf{C}(a_1e_1 + a_2e_2).$$

This extends the inclusion $\imath$ to the boundary $B^{2L(\sigma)}$ in a continuous manner. At this point it is reasonable to extend the definition of the partial order $\mathbf{b} \preceq \mathbf{a}$ on the set of symbol sequences, to a partial order on the permutation group $\mathcal{S}_n$. Given a permutation $\sigma \in \mathcal{S}_n$, for each i let $s_1^i < \cdots < s_i^i$ be the sequence of numbers $\sigma(1), \cdots, \sigma(i)$ re-arranged in increasing order. For $\sigma, \tau \in \mathcal{S}_n$ we define the partial order (generally called *Bruhat order*) on $\mathcal{S}_n$ by

$$\tau \preceq \sigma \ \text{ if for every } i \text{ and } j \leq i \quad t_j^i \leq s_j^i.$$

Here t_j^i's bear the same relationship to τ as s_j^i's do to σ. If $\tau \preceq \sigma$ and $\tau \neq \sigma$ we set $\tau \prec \sigma$. It is clear that this definition can be equivalently reformulated in terms of symbol sequences as follows: For $\sigma \in \mathcal{S}_n$ let $\mathbf{a}^i(\sigma) = [s_1^i - 1, s_2^i - 2, \cdots, s_i^i - i]$. Then $\tau \preceq \sigma$ if and only if $\mathbf{a}^i(\tau) \preceq \mathbf{a}^i(\sigma)$ for all i. Geometrically, this definition has the equivalent interpretation that for flags $f, f' \in \mathbf{F}_n$

$$\sigma(f) \preceq \sigma(f') \iff \mathbf{a}_{p_i(f)} \preceq \mathbf{a}_{p_i(f')} \quad \text{for all } \ i.$$

It is a simple matter to verify that for $f \in \mathbf{F}_n$

$$\sigma(f_\infty) \preceq \sigma(f_\lambda) = \sigma(f).$$

Summarizing we have

Proposition 3.2.5 *The inclusion $\imath : \mathbf{C}^{L(\sigma)} \simeq \breve{B}^{2L(\sigma)} \hookrightarrow \mathbf{F}_n$ extends to a continuous map of $B^{2L(\sigma)}$ to $\mathbf{F}_n$ mapping $\partial B^{2L(\sigma)}$ to $\cup_{\tau\prec\sigma} B(\tau)$. Thus $\mathbf{F}_n$ has the structure of a cell complex.* (One can also show that $\imath(\partial B^{2L(\sigma)}) = \cup_{\tau\prec\sigma} B(\tau)$.)

The machinery of section 3 for the computation of homology groups of cell complexes is now applicable and we obtain

Corollary 3.2.5 *The homology of $\mathbf{F}_n$ is given by*

$$H_j(\mathbf{F}_n; \mathbf{Z}) = \begin{cases} 0 & \text{if } j = 2l+1 \text{ or } j > n(n-1) \\ \mathbf{Z}^{n(l)} & \text{if } j = 2l \leq n(n-1) \end{cases}$$

where $n(l)$ is the number of permutations σ with $L(\sigma) = l$. In particular, $\chi(\mathbf{F}_n) = n!$. (Note $\dim_{\mathbf{R}} \mathbf{F}_n = n(n-1)$.)

Exercise 3.2.21 *Show that relative to the Bruhat order, $e = id.$ is the smallest and the permutation ϵ which reverses the order of the integers $1 < 2 < \cdots < n$ (i.e., $\epsilon(i) = n - i + 1$) is the largest element of $\mathcal{S}_n$ and $L(\epsilon) = \frac{n(n-1)}{2}$. Prove also that if $\tau \prec \sigma$, then $L(\tau) < L(\sigma)$.*

Exercise 3.2.22 *Show that the elements of $\mathcal{S}_3$ can be organized in the form*

$$\begin{array}{ccc} & \langle 123\rangle & \\ \langle 213\rangle & & \langle 132\rangle \\ \langle 231\rangle & & \langle 312\rangle \\ & \langle 321\rangle & \end{array}$$

with $L(\sigma) = r - 1$ if σ is in the r^{th} row. Show also that for $\mathcal{S}_3$, $L(\tau) < L(\sigma)$ implies $\tau \prec \sigma$.

Exercise 3.2.23 *Show that the elements of $\mathcal{S}_4$ can be organized according to increasing value of the length function by the diagram*

$$\begin{array}{ccccccc} & & & \langle 1234\rangle & & & \\ & & \langle 1243\rangle & \langle 1324\rangle & \langle 2134\rangle & & \\ & \langle 1342\rangle & \langle 1423\rangle & \langle 2143\rangle & \langle 2314\rangle & \langle 3124\rangle & \\ \langle 1432\rangle & \langle 2341\rangle & \langle 2413\rangle & & \langle 3142\rangle & \langle 3214\rangle & \langle 4123\rangle \\ & \langle 2431\rangle & \langle 3241\rangle & \langle 3412\rangle & \langle 4132\rangle & \langle 4213\rangle & \\ & & \langle 3421\rangle & \langle 4231\rangle & \langle 4312\rangle & & \\ & & & \langle 4321\rangle & & & \end{array}$$

where if $\langle ijkl\rangle$ lies in the r^{th} row of the diagram, then $L(\langle ijkl\rangle) = r - 1$. Let $\langle ijkl\rangle$ and $\langle i'j'k'l'\rangle$ lie in rows r and $r+1$ respectively. Draw a line joining $\langle ijkl\rangle$ to $\langle i'j'k'l'\rangle$ if $\langle ijkl\rangle \prec \langle i'j'k'l'\rangle$. (For example, $\langle 1423\rangle \prec \langle 1432\rangle, \langle 2413\rangle$ and $\langle 4123\rangle$, but $\langle 1423\rangle$ is not comparable to the remaining elements in row 4).

Remark 3.2.5 The symmetry of the diagram in exercise 3.2.23 is valid for $\mathcal{S}_n$, and it is a consequence of the Poincaré duality which is treated in the chapter 7. The fact that $n(l)$ is increasing up to the middle and then decreases is also true for $\mathcal{S}_n$ and follows from the Hard Lefschetz theorem which will not be discussed in this volume. ♡

Exercise 3.2.24 *Let w_σ be the permutation matrix representing σ, and B be the group of $n \times n$ nonsingular upper triangular complex matrices. For $b \in B$ denote the columns of bw_σ by $b_1, \cdots, b_n$. Then the flag $f : 0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbf{C}^n$ where V_k is the span of $b_1, \cdots, b_k$ has permutation sequence σ. Furthermore every flag with permutation sequence σ has a unique representation $b^f B/B = bw_\sigma B/B$ for some $b \in B$. Let $N = \{b \in B|$*all eigenvalues of b are $1\}$*, and $B.f$ (and similarly $N.f$) denote the orbit of the flag f under the action of B. Thus if f is a flag with jump sequence σ, then $B.f = N.f$ is the set of all flags with permutation sequence σ. Deduce that we have the decomposition*

$$GL(n, \mathbf{C}) = \cup_\sigma B w_\sigma B = \cup_\sigma N w_\sigma B,$$

into disjoint union. It is customary to refer to this decomposition as the *Bruhat decomposition*.

Our discussion of the homology of $\mathbf{F}_n$ extends easily to the case of the generalized flag manifold $\mathbf{F}_\mathbf{s}$. The trick is to look at the fibre bundles $p_{s_i} : \mathbf{F}_\mathbf{s} \to \mathbf{G}_{s_i,n-s_i}$ where $\mathbf{s} = [s_1, \cdots, s_r]$. Given $f \in \mathbf{F}_\mathbf{s}$, consistent with our previous notation, we denote the sequence corresponding to $p_{s_i}(f)$ by $\mathbf{a}^{s_i} = [a_1^{s_i}, \cdots, a_{s_i}^{s_i}]$, and set $k_j^{s_i} = a_j^{s_i} + j$ for $j \leq s_i$. We have the inclusion $\{k_j^{s_l}\}_{j\leq s_l} \subset \{k_j^{s_{l'}}\}_{j\leq s_{l'}}$ for $l < l'$.

Exercise 3.2.25 *Let $c_1 < \cdots < c_{s_1}$ be the re-arrangement of the set of integers $\{k_j^{s_1}\}_{j\leq s_1}$ in increasing order, $c_{s_1+1} < \cdots < c_{s_2}$ be the rearrangement of the set of integers $\{k_j^{s_2}\}_{j\leq s_2} \setminus \{k_j^{s_1}\}_{j\leq s_1}$ in increasing order etc. Let $\mathbf{c}$ be the sequence*

$$\mathbf{c} = [c_1 < \cdots < c_{s_1}; c_{s_1+1} < \cdots < c_{s_2}; \cdots ; c_{s_r+1} < \cdots < c_n]$$

of distinct positive integers $\leq n$ and $\sigma_\mathbf{c}$ be the permutation $\langle c_1 c_2 \cdots c_n \rangle$. Let $B(\mathbf{c})$ be the set of flags $f \in \mathbf{F}_\mathbf{s}$ with sequence $\mathbf{c}$. Show that $B(\mathbf{c}) \simeq \mathbf{C}^{L(\sigma_\mathbf{c})} \simeq \breve{B}^{2L(\sigma_\mathbf{c})}$. (It is convenient to describe $B(\mathbf{c})$ as a matrix as in example 3.2.17. Let $l = s_r$ and consider $n \times l$ matrices of the following form: The first s_1 columns have 1's at positions (c_j, j), $(j \leq s_1)$, zeros below them and arbitrary complex numbers above them. Let $t_2 = s_2 - s_1$. The next t_2 columns have 1's in positions (c_{s_j}, j), $(s_1 + 1 \leq j \leq s_2)$, zeros below them and the entries above the 1's are arbitrary complex numbers except for the requirement that if $i = c_k$ for some $k \leq s_1$ then all entries in row $(i, \star)$ are zero. The process is continued in the obvious manner. The description for the cell $B(\mathbf{c})$ can be summarized as an $n \times l$ matrix with blocks consisting of $s_1, s_2 - s_1, \cdots, s_r - s_{r-1}$ columns just as in the case of Grassmann manifolds with the additional requirement that every time a 1 appears, every entry of the matrix in the same row and to the right of it is 0. For example if $n = 5$ and $\mathbf{c} = [2 < 5; 4; 1 < 3]$ then we get the 5×3 matrix

$$\begin{pmatrix} \star & \star & \star \\ 1 & 0 & 0 \\ 0 & \star & \star \\ 0 & \star & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice that the integers $c_{s_r+1} < \cdots < c_n$ do not appear in the description.)

Exercise 3.2.26 *Withe the notation and hypotheses of exercise 3.2.25 prove that $\mathbf{F}_\mathbf{s}$ is the disjoint union of cells $B(\mathbf{c})$, and the inclusion of the $\imath : B(\mathbf{c}) \simeq \breve{B}^{2L(\sigma_\mathbf{c})} \hookrightarrow \mathbf{F}_\mathbf{s}$ extends continuously to the boundary as required by the definition of cell complexes. Deduce that the homology of $\mathbf{F}_\mathbf{s}$ is*

$$H_j(\mathbf{F}_\mathbf{s}; \mathbf{Z}) = \begin{cases} 0, & \text{if } j = 2l+1 \text{ or } j > \dim_\mathbf{R} \mathbf{F}_\mathbf{s}; \\ \mathbf{Z}^{n_\mathbf{c}(l)}, & \text{if } j = 2l \leq \dim_\mathbf{R} \mathbf{F}_\mathbf{s}; \end{cases}$$

where $n_\mathbf{c}(l)$ is the number of permutations of the form $\sigma_\mathbf{c}$ such that $L(\sigma_\mathbf{c}) = l$. In particular,

$$\chi(\mathbf{F}_\mathbf{s}) = \frac{n!}{s_1! \cdots s_r!}$$

Exercise 3.2.27 *Let $t_1 = s_1$,and $t_j = s_j - s_{j-1}$ for $j \geq 2$. Show that $\{\sigma_\mathbf{c}\}$ form a complete set of coset represenatatives for $\mathcal{S}_n/\mathcal{S}_{t_1} \times \mathcal{S}_{t_2} \times \cdots \times \mathcal{S}_{t_r}$. Here $\mathcal{S}_{t_1}$ is the subgroup consisting of permutations of $\{1, \cdots, t_1\}$,*

$\mathcal{S}_{t_2}$ is the subgroup consisting of permutations of $\{s_1+1,\cdots,s_2\}$ etc. Therefore the set $\{\sigma_{\mathbf{c}}\}$ inherits a partial order from the Bruhat order on $\mathcal{S}_n$. Show that the extension of $\imath$ to $\partial B^{2L(\sigma_{\mathbf{c}})}$ mentioned in exercise 3.2.26, maps the latter space onto $\cup_{\sigma_{\mathbf{c}'}\prec\sigma_{\mathbf{c}}} B(\mathbf{c}')$.

Exercise 3.2.28 *Show that*

$$L(\sigma_{\mathbf{c}}) = \min_{\tau\in\mathcal{S}_{t_1}\times\cdots\times\mathcal{S}_{t_r}} L(\sigma_{\mathbf{c}}\tau).$$

What is the coset representative of maximal length?

3.3 Basic Notions of Cohomology

3.3.1 The Axioms

We begin by pointing out some algebraic facts. For R-modules A and T, let $\mathrm{Hom}(A,T)$ be the R-module of all homomorphisms of A into T. Then any homomorphism $A \to B$ induces a homomorphism $\mathrm{Hom}(B,T) \to \mathrm{Hom}(A,T)$ via the composition $A \to B \to T$. In particular, from a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \tag{3.3.1}$$

we obtain a *not necessarily exact* sequence $0 \to \mathrm{Hom}(C,T) \to \mathrm{Hom}(B,T) \to \mathrm{Hom}(A,T) \to 0$. We say that the short exact sequence (3.3.1) *splits* if $B \simeq A \oplus C$ and the maps i and p are the obvious inclusion of A in B and projection of B onto C. There is a simple and useful criterion for the splitting of a short exact sequence, viz., (3.3.1) splits if and only if there is a homomorphism $f : C \to B$ such that $pf = \mathrm{id}_C$. In fact we decompose $B = B' \oplus B''$ with $B' = \mathrm{Im} f$ and $B'' = \{b - fp(b) | b \in B\}$. It is trivial to see that $B' \simeq C$ and $B'' = \ker p$ and that the sequence (3.3.1) splits. The homomorphism f is called a *splitting homomorphism.* As an application we have

Lemma 3.3.1 *If (3.3.1) is exact, then for any R-module T*

$$0 \to \mathrm{Hom}(C,T) \to \mathrm{Hom}(B,T) \to \mathrm{Hom}(A,T)$$

is exact. If furthermore (3.3.1) splits the map $\mathrm{Hom}(B,T) \to \mathrm{Hom}(A,T)$ *is also surjective.*

The proof of the lemma is a straightforward application of the definitions.

Example 3.3.1 Assume C in (3.3.1) is free (i.e., direct sum of a number of copies of R). We show that the sequence (3.3.1) splits. Let $x_1, x_2, \cdots$ be a basis for C and $y_1, y_2, \cdots$ be elements of B such that $p(y_j) = x_j$. Then the assignment $f(x_j) = y_j$ is the desired splitting homomorphism. We also note that the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{i_m} \mathbf{Z} \to \mathbf{Z}/m \to 0,$$

where i_m is multiplication by m, does not split. The homomorphism $\mathrm{Hom}(\mathbf{Z},\mathbf{Z}) \to \mathrm{Hom}(\mathbf{Z},\mathbf{Z})$ induced by i_m is not surjective. ♠

Singular and simplicial cohomology theories can be constructed in the same manner from the singular and simplicial homology theories. To be more precise, for a pair of topological spaces or a simplicial pair (X,A) let $C_n(X,A;R)$ denote the R-module of relative n-chains and $C^n(X,A;R) = \mathrm{Hom}(C_n(X,A;R),R)$. The boundary operators $\partial_n : C_n(X,A;R) \to C_{n-1}(X,A;R)$ induces the *coboundary operators* (note shift of index) $\partial_n^\star : C^n(X,A;R) \to C^{n+1}(X,A;R)$ which satisfy

$$\partial_{n+1}^\star \partial_n^\star = 0. \tag{3.3.2}$$

$C^n(X,A;R)$ is the R-module of *relative n-cochains*, and we define the R-modules of *relative coboundaries* $B^n(X,A;R)$ and *relative cocycles* $Z^n(X,A;R)$ as

$$B^n(X,A;R) = \mathrm{Im}\partial_{n-1}^\star, \qquad Z^n(X,A;R) = \ker \partial_n^\star.$$

The cohomology R-modules or groups are then defined as

$$H^n(X, A; R) = Z^n(X, A; R)/B^n(X, A; R).$$

We shall not make a detailed verification of the validity of the axioms, however, there is one important point regarding the the construction of the connecting homomorphism which should be emphasized. Recall that the exact sequence

$$0 \to C_n(A; R) \to C_n(X; R) \to C_n(X, A; R) \to 0$$

splits since the R-module $C_n(X, A; R)$ is free. Consequently, the sequence

$$0 \to C^n(X, A; R) \to C^n(X; R) \to C^n(A; R) \to 0$$

is exact and splits. Now the construction of the connecting homomorphism for homology can be repeated verbatim to give the connecting homomorphism for cohomology.

Example 3.3.2 Let M be a differentiable manifold and ω a p-form on M. Then the mapping which assigns to each simplicial p-chain $c = \sum_i a_i c_i$, $a_i \in \mathbf{R}$ the scalar

$$\sum_i a_i \int_{c_i} \omega,$$

is a p-cochain (we may assume the simplicial chains are piece-wise smooth so that integration will be meaningful). It is a consequence of the Stokes' theorem that the coboundary operator $\partial_p^\star$ acting on p-forms becomes exterior differentiation. Differential forms and this example provide an important link between differential geometry and topology. We will return to the relationship between forms and cohomology later. ♠

We can define cohomology axiomatically just as in the case of homology except that the arrows are reversed. More precisely, by a cohomology theory with R coefficients we mean an assignment of R-modules $H^j(X, A; R)$ (or simply $H^j(X, A)$ if $R = \mathbf{Z}$ or the omission of R does not cause any confusion) to every pair (X, A) of topological spaces (or simplicial complexes) and R-module homomorphisms

$$H^j(f) = f^\star : H^j(Y, B; R) \to H^j(X, A; R)$$

to every continuous map $f : (X, A) \to (Y, B)$ such that the following conditions are satisfied:

1. **Contravariance -** If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$, then $(g.f)^\star = f^\star.g^\star$;

2. **Homotopy -** If $f, g : (X, A) \to (Y, B)$ are homotopic, then the induced maps $H^i(f)$ and $H^i(g)$ are identical;

3. **Long Exact Sequence -** There is the exact sequence

$$\cdots \to H^i(X, A) \to H^i(X) \to H^i(A) \xrightarrow{\delta^i} H^{i+1}(X, A) \to \cdots,$$

where the maps $H^i(X) \to H^i(A)$ and $H^i(X, A) \to H^i(X)$ are induced by the inclusions $A \hookrightarrow X$ and $(X, \emptyset) \hookrightarrow (X, A)$. The map δ^i is called the *connecting homomorphism*;

4. **Excision** - If $U \subseteq X$ is an open subset such that $\bar{U} \subseteq \breve{A}$, then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces isomorphisms $H^i(X, A; R) \simeq H^i(X \setminus U, A \setminus U; R)$.

5. **Naturality** - For a continuous map $f : (X, A) \to (Y, B)$ the following diagram commutes:

$$\begin{array}{ccccccccccc} \cdots & \to & H^i(Y,B) & \to & H^i(Y) & \to & H^i(B) & \to & H^{i+1}(X,A) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H^i(X,A) & \to & H^i(X) & \to & H^i(A) & \to & H^{i+1}(X,A) & \to & \cdots \end{array}$$

6. **Normalization** - For a single point $\{p\}$, $H^i(\{p\}; R) = 0$ for $i > 0$ and $H_\circ(\{p\}; R) = R$ and $H^i(\text{id.}) = \text{id}$.

The methods described earlier for the computation of homology are applicable to the computation of cohomology as well. In fact, the cohomology long exact sequence or cohomology version of the Mayer Vietoris sequence (which is obtained by reversing the arrows in the homology version) can be used for the computation of cohomology groups once a cell decomposition of a space is explicitly known. We also note if (X, A) is a simplicial pair with finitely many simplices, then the coboundary operator $\partial^\star$ is the transpose of the boundary operator ∂ which suggests a puely algebaic relationship between homology and cohomology theories. The following proposition makes this relationship precise:

Proposition 3.3.1 *If $H_{n-1}(X, A; R)$ is free and R is a principal ideal domain, then $H^n(X, A; R)$ is canonically isomorphic to the dual of $H_n(X, A; R)$. In particular if R is a field and (X, A) is a simplicial pair with finitely many simplices, then $H^n(X, A; R)$ is canonically isomorphic to the dual of $H_n(X, A; R)$.*

Remark 3.3.1 Let ${}^fH^n(X, A; R)$ and ${}^fH_n(X, A; R)$ denote the free parts of the n^{th} cohomology and homology groups of the pair (X, A) where R is a principal ideal domain and assume the homology and cohomology R-modules are finitely generated. Then ${}^fH^n(X, A; R)$ is isomorphic to the dual of ${}^fH_n(X, A; R)$. Let K denote the field of quotients of R, then $H^n(X, A; K) \simeq H^n(X, A; R) \otimes_R K$ and $H_n(X, A; K) \simeq H_n(X, A; R) \otimes_R K$. Consequently, they are dual vector spaces of dimension equal to the rank of the free R-module ${}^fH^n(X, A; R)$. A more elaborate version of this proposition, including the above statements, is proven in the section "Some Algebraic Considerations". The second statement of proposition 3.3.1 is a simple exercise in linear algebra. ♡

3.3.2 Algebraic Limits

To introduce other cohomology theories we need the algebraic notions of limits. Let $\{M_\alpha\}$, $\alpha \in I$ some index set, be a family of abelian groups, or R-modules etc., and assume that the index set I is a *directed set*, i.e., I is a partially ordered set (the partial ordering denoted by $\preceq$) with the additional property that for all $\alpha, \beta \in I$ there is $\gamma \in I$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Assume furthermore that we are given a family of homomorphisms $f_{\beta\alpha} : M_\alpha \to M_\beta$, for $\alpha \preceq \beta$, such that $f_{\alpha\alpha} = \text{id.}$ and $f_{\gamma\alpha} = f_{\gamma\beta} f_{\beta\alpha}$. Let $M' = \cup M_\alpha$, and define an equivalence relation $\sim$ on M' by $m \sim m'$ where $m \in M_\alpha$ and $m' \in M_\beta$, if there is γ, $\alpha, \beta \preceq \gamma$ such that $f_{\gamma\alpha}(m) = f_{\gamma\beta}(m')$. We define the *direct* or *inductive limit* as

$$M = \varinjlim \{M_\alpha, f_{\alpha\beta}\} = \varinjlim M_\alpha = M'/\sim .$$

Notice that we have mappings $f_\alpha : M_\alpha \to M$ which assign to each $m \in M_\alpha$ the equivalence class in M' to which it belongs. Let $\{N_\alpha\}$ be another family of abelian groups, R-modules etc. indexed by I, and homomorphisms $g_{\beta\alpha}$ satisfying the same conditions as $f_{\beta\alpha}$. Assume that we have homomorphisms $\phi_\alpha : M_\alpha \to N_\alpha$ such that $g_{\beta\alpha}\phi_\alpha = \phi_\beta f_{\beta\alpha}$. Then it is straightforward to show that we have an induced map $\varinjlim M_\alpha \to \varinjlim N_\alpha$.

Exercise 3.3.1 *(a) Let $\{M_i\}_{i\in \mathbf{N}}$ be a family of R-modules with $M_{i-1} \subseteq M_i$ and f_{ji} is the inclusion of M_i in M_j for $i < j$. Show that $\varinjlim M_i = \cup M_i$. (b) Let $\{M_i\}_{i\in N}$ be a family of R-modules and set $N_j = \oplus_{i\leq j} M_i$. Define homomorphisms $f_{kj} : N_j \to N_k$, for $j \leq k$, by $f_{kj}(m_1, \cdots, m_j) = (m_1, \cdots, m_j, 0, \cdots, 0)$. Show that $\varinjlim N_i = \oplus M_i$.* (Of course the same result holds if $\mathbf{N}$ is replaced by any totally ordered set.)

Exercise 3.3.2 *Let $0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$ be a short exact sequence for all α in some directed set I. Assume also that for all $\alpha \preceq \beta$ we have homomorphisms $f_{\beta\alpha} : A_\alpha \to A_\beta$, $g_{\beta\alpha} : B_\alpha \to B_\beta$ and $h_{\beta\alpha} : C_\alpha \to C_\beta$ such that the diagram*

$$\begin{array}{ccccccccc} 0 & \to & A_\alpha & \to & B_\alpha & \to & C_\alpha & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_\beta & \to & B_\beta & \to & C_\beta & \to & 0 \end{array}$$

commutes. Show that the direct limit sequence

$$0 \to \varinjlim A_\alpha \to \varinjlim B_\alpha \to \varinjlim C_\alpha \to 0$$

exists and is exact.

Exercise 3.3.3 *Let $M_i = \mathbf{Z}$ for $i = 1, 2, \cdots$ and for $i \leq j$, $f_{ji}(a) = p^{j-i}a$ where p is a prime. Show that*

$$\varinjlim M_i = \{\frac{n}{p^m} | m, n \in \mathbf{Z}\}.$$

Exercise 3.3.4 *Let $M_i = \mathbf{Z}/i$ for $i = 1, 2, \cdots$ and define partial order on $\mathbf{N}$ by $i \preceq j$ if $i|j$. Define $f_{ji} : M_i \to M_j$ to be multiplication by $\frac{j}{i}$ if $i|j$. Show that $\varinjlim M_i \simeq \mathbf{Q}/\mathbf{Z}$.*

Dual to the notion of direct limit is that of the *inverse* or *projective limit.* Here also we have a family of R-modules $\{M_\alpha\}_{\alpha\in I}$ where the index set I is directed. However we assume that for $\alpha \preceq \beta$ we have homomorphisms $f_{\alpha\beta} : M_\beta \to M_\alpha$ such that $f_{\alpha\alpha} = \mathrm{id}$ and $f_{\alpha\beta}f_{\beta\gamma} = f_{\alpha\gamma}$ if $\alpha \preceq \beta \preceq \gamma$. To define the inverse limit we set $M' = \prod M_\alpha$. Let $M \subseteq M'$ be the subset consisting of all sequences (x_α), $x_\alpha \in M_\alpha$, such that for $\alpha \preceq \beta$ $f_{\alpha\beta}(x_\beta) = x_\alpha$. Then M is the inverse limit of $\{M_\alpha, f_{\alpha\beta}\}$ and is denoted by $\varprojlim \{M_\alpha, f_{\alpha\beta}\}$ or more simply $\varprojlim M_\alpha$.

Exercise 3.3.5 *Show explicitly how the direct product can be realized as an inverse limit.*

Exercise 3.3.6 $M_i = \mathbf{Z}/p^i$ *where* p *is a prime and* $i = 1, 2, \cdots$. *Define* $f_{ij}(a) \equiv a \mod p^i$ *for* $i \leq j$ *and denote* $\varprojlim M_i$ *by* $\mathbf{Z}_p$ *(called the ring of* p*-adic integers). Show that every* p*-adic integer* x *has unique representation as the formal sum* $\sum_i a_i p^i$ *where* $a_i \in \{0, 1, \cdots, p-1\}$ *where addition and multiplication are defined in the obvious manner by treating* p *as an indeterminate.*

Exercise 3.3.7 $M_i = \mathbf{Z}/i$, $i \in \mathbf{N}$ *and define partial order on* $\mathbf{N}$ *by* $i \preceq j$ *if* $i|j$. *For* $i \preceq j$ *define* $f_{ij}(a) \equiv a \mod j$. *Show that* $\varprojlim M_i = \prod \mathbf{Z}_p$ *where product is taken over all primes* $p = 2, 3, 5, \cdots$.

Exercise 3.3.8 *Let* I *be a directed set and* $J \subset I$ *be a directed subset with the property that for all* $\alpha \in I$ *there is* $\beta \in J$ *with* $\alpha \preceq \beta$ *(we say* J *is* cofinal *in* I*). If* M_α *is a family of* R*-modules indexed by* I, *and* $f_{\beta\alpha} : M_\alpha \rightarrow M_\beta$, *then the direct limit of* $\{M_\alpha\}_{\alpha\in I}$ *and* $\{M_\beta\}_{\beta\in J}$ *are identical. Similar result is valid for inverse limits.*

A natural and important example of direct limits is when X is a non-compact manifold or simplicial complex, and $\{K_\alpha\}$ is a family of compact subsets with the property that for all K_α, K_β there is K_γ such that $K_\gamma \supseteq K_\alpha \cup K_\beta$ and $\cup K_\alpha = X$. We have homomorphisms

$$f_{\beta\alpha} : H^j(X, X \setminus K_\alpha; R) \rightarrow H^j(X, X \setminus K_\beta; R),$$

if $K_\alpha \subseteq K_\beta$. From the contravariance property of cohomology it follows that $f_{\gamma\beta} f_{\beta\alpha} = f_{\gamma\alpha}$ and $f_{\alpha\alpha} = \mathrm{id}$. Therefore $\varinjlim H^j(X, X \setminus K_\alpha; R)$ exists. It follows from exercise 3.3.8 that under the above hypothesis $\varinjlim H^j(X, X \setminus K_\alpha; R)$ is independent of the choice of the family $\{K_\alpha\}$. We set $H^j_c(X; R) = \varinjlim H^j(X, X \setminus K_\alpha; R)$ and call $H^j_c(X; R)$ the j^{th} *cohomology with compact support* of X.

Exercise 3.3.9 *Let* $X = \mathbf{R}^n$. *Show that* $H^j_c(X; \mathbf{Z})$ *vanishes except for* $j = n$ *in which case it is isomorphic to* $\mathbf{Z}$. (For example, let K_i, $i = 1, 2, \cdots$ be the closed ball of radius i etc.)

Remark 3.3.2 Let ω be a compactly supported smooth closed n-form on $\mathbf{R}^n$. One can show that there is a compactly supported $(n-1)$-form η such that $d\eta = \omega$ if and only if $\int_{\mathbf{R}^n} \omega = 0$. For $n = 1$ this is just a matter integrating, however, for $n > 1$, it is considerably more subtle. Let M be given a triangulation and ω be a closed 1-form on M. If a 1-cycle $c = \partial c'$ (i.e., c is a boundary) then $\int_c \omega = 0$. Assume $\int_c \omega = 0$ for every 1-cycle c. Fix a vertex $b \in M$ and for any $z \in M$ choose any path $\gamma(b, z)$ joining b to z. If z is not vertex we can go a subdivision to make z into a vertex, and then take the path to consist of 1-simplices. Then the condition $\int_c \omega = 0$ implies that $f(z) = \int_{\gamma(b,z)} \omega$ is well-defined and $df = \omega$. This may be regarded as a generalization of the fact that a periodic function of one variable is the derivative of a periodic function if its zeroth Fourier coefficient vanishes. ♡

Example 3.3.3 Let M be a compact manifold with boundary ∂M. If $\partial M = \emptyset$, then $H^j_c(M;R) = H^j(M;R)$. Assume $\partial M \neq \emptyset$, and $\breve{M} = M \setminus \partial M$ be the interior of M. We show that $H^j_c(\breve{M};R) \simeq H^j(M,\partial M;R)$. Let T_i's be small open tubular neighborhoods of ∂M such that ∂M is a deformation retract of each T_i, $T_{i+1} \subset T_i$ and $\cap T_i = \partial M$. Let $K_i = M \setminus T_i$ and consider the row exact commutative diagram

$$\begin{array}{ccccccccc}
H^{j-1}(M) & \rightarrow & H^{j-1}(M \setminus K_i) & \rightarrow & H^j(M, M \setminus K_i) & \rightarrow & H^j(M) & \rightarrow & H^j(M \setminus K_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{j-1}(M) & \rightarrow & H^{j-1}(\partial M) & \rightarrow & H^j(M, \partial M) & \rightarrow & H^j(M) & \rightarrow & H^j(\partial M)
\end{array}$$

It is clear that the vertical arrows except possibly for the middle one are isomorphisms, and therefore by the Five Lemma the middle arrow is also bijective. The required result follows immediately. Now assume M is orientable. By a similar argument, the homomorphism $H_m(M, M \setminus K_i) \rightarrow H_m(M, \partial M)$ is an isomorphism and there is a unique element $\lfloor(M,\partial M)\rfloor \in H_m(M,\partial M)$ which is mapped onto the orientation class $\mathbf{m}_{K_i} \in H_m(M, M\setminus K_i)$. The image of $\lfloor(M,\partial M)\rfloor \in H_m(M,\partial M)$ in $H_{m-1}(\partial M)$, under the connecting homomorphism, is the orientation class $\lfloor \partial M \rfloor$ of the boundary. This is easily proven by analyzing the effect of the connecting homomorphism on the homology classes representing orientation. ♠

Remark 3.3.3 It is useful to note that cohomology with compact support can be regarded as the quotient of cocycles over coboundaries if instead of all cochains we only consider those cochains γ which vanish on all chains contained in $X \setminus K$ for some sufficiently large compact subcomplex (or subset) $K = K(\gamma)$. This means that a p-cocycle γ is a coboundary if there is $(p-1)$-cocycle γ' with compact support such that $\partial^\star_{p-1}(\gamma') = \gamma$. The proof of this fact is straightforward and is a matter of checking through the definitions. ♡

3.3.3 Čech Cohomology

Let $\mathcal{U} = \{U_i\}$ be a covering of the topological space by open sets. Let $\mathcal{N}_n(\mathcal{U})$ be the set of $n+1$-tuple of indices $(i_\circ, i_1, \cdots, i_n)$ such that $U_{i_\circ} \cap \cdots \cap U_{i_n} \neq \emptyset$, and $\mathcal{N}(\mathcal{U}) = \cup_{n\geq 0}\mathcal{N}_n(\mathcal{U})$. $\mathcal{N}(\mathcal{U})$ is called the *nerve* of the covering. For $(i_\circ, \cdots, i_n) \in \mathcal{N}_n(\mathcal{U})$, let $\gamma(i_\circ, \cdots, i_n) \in R$ subject to the requirement

$$\gamma(i_{\sigma(\circ)}, \cdots, i_{\sigma(n)}) = \epsilon(\sigma)\gamma(i_\circ, \cdots, i_n), \tag{3.3.3}$$

where $\epsilon(\sigma)$ is the sign of the permutation $\sigma \in \mathcal{S}_{n+1}$. Therefore γ may be regarded as a mapping from the set indices $(i_\circ, i_1, \cdots, i_n)$ with property $U_{i_\circ} \cap \cdots U_{i_n} \neq \emptyset$ to R subject to the requirement (3.3.3). Every such γ is called an *n-Čech cochain*. Clearly the set of Čech cochains is an R-module under addition and scalar multiplication induced by those of R. We denote this R-module by $\check{C}^n(\mathcal{U}, M; R)$. Define the coboundary operator $\partial^\star_n : \check{C}^n(\mathcal{U}, M; R) \rightarrow \check{C}^{n+1}(\mathcal{U}, M; R)$ by the formula

$$\partial^\star_n \gamma(i_\circ, i_1, \cdots, i_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \gamma(i_\circ, \cdots, \hat{i_k}, \cdots, i_{n+1}), \tag{3.3.4}$$

where $\hat{i_k}$ means the index i_k is omitted. The verification of the fact that $\partial^\star_{n+1}\partial^\star_n = 0$ is entirely analogous to the familiar case of singular or simplicial (co)homology. The modules of *n-Čech cocycles* and *coboundaries* are naturally defined as the kernel of $\partial_n^\star$ and image of $\partial_{n-1}^\star$ and are denoted by $\check{Z}^n(\mathcal{U}, X; R)$ and $\check{B}^n(\mathcal{U}, X; R)$ respectively. We define

$$\check{H}^n(\mathcal{U}, X; R) = \check{Z}^n(\mathcal{U}, X; R)/\check{B}^n(\mathcal{U}, X; R).$$

Note $\check{H}^n(\mathcal{U}, X; R)$ depends on the choice of the covering $\mathcal{U}$ as shown by

Exercise 3.3.10 *By taking coverings of the circle consisting of two arcs and three arcs, show that the groups $\check{H}^j(\mathcal{U}, X; \mathbf{Z})$ may depend on the covering.*

Naturally we wish to get rid of dependence on the covering $\mathcal{U}$ in order to obtain a good cohomology theory. Let $\mathcal{V} = \{V_i\}$ be another covering of X. We say $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for every V_i there is $U_{r(i)}$ with $V_i \subseteq U_{r(i)}$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$ we write $\mathcal{U} \preceq \mathcal{V}$. Thus the set of coverings of a space X is a directed set, and it is easy to see given coverings $\mathcal{U}$ and $\mathcal{V}$ of X, there is a covering $\mathcal{W}$ which refines both $\mathcal{U}$ and $\mathcal{V}$. Now we claim that if $\mathcal{V}$ is a refinement $\mathcal{U}$, then there is an R-module homomorphism $r_{\mathcal{V}\mathcal{U}}^\star : \check{H}^n(\mathcal{U}, X; R) \rightarrow \check{H}^n(\mathcal{V}, X; R)$. In fact, given $c \in \check{C}^n(\mathcal{U}, M; R)$ let

$$r_{\mathcal{V}\mathcal{U}}(\gamma)(i_\circ, \cdots, i_n) = \gamma(r(i_\circ), \cdots, r(i_n)).$$

It is a simple matter to verify that $r_{\mathcal{V}\mathcal{U}}$ induces a homomorphism $r_{\mathcal{V}\mathcal{U}}^\star : \check{H}^n(\mathcal{U}, X; R) \rightarrow \check{H}^n(\mathcal{V}, X; R)$. Therefore we can take direct limits and define

$$\check{H}^n(X; R) = \varinjlim \check{H}^n(\mathcal{U}, X; R).$$

Dually one has the notion of Čech homology. Let $\check{C}_n(\mathcal{V}, X; R)$ be the free R-module with basis $\mathcal{N}_n(\mathcal{U})$. Denoting such a basis element by $(i_\circ, i_1, \cdots, i_n)$ and define the boundary operator by

$$\partial_n(i_\circ, i_1, \cdots, i_n) = \sum_k (-1)^k (i_\circ, \cdots, \hat{i_k}, \cdots, i_n).$$

We then proceed in the obvious manner and define the R-modules of n-Čech cycles $\check{Z}_n(\mathcal{U}, X; R)$ and boundaries $\check{B}_n(\mathcal{U}, X; R)$ as the kernel of ∂_n and image of ∂_{n+1}. We set

$$\check{H}_n(\mathcal{U}, X; R) = \check{Z}_n(\mathcal{U}, X; R)/\check{B}_n(\mathcal{U}, X; R).$$

As before if $\mathcal{V}$ is a refinement of $\mathcal{U}$ then we have a homomorphism $r_{\mathcal{V}\mathcal{U}} : \check{H}_n(\mathcal{U}, X; R) \rightarrow \check{H}_n(\mathcal{V}, X; R)$, we define the the n^{th} homology of X to be

$$\check{H}_n(X; R) = \varinjlim \check{H}_n(\mathcal{U}, X; R).$$

Rather than giving a formal proof of the equality of the Čech and singular or simplicial (co)homology groups for manifolds or simplicial complexes, we indicate why this is a reasonable expectation. Let for example $X = M$ be a manifold with a fixed simplicial decomposition. Consider the covering of the manifold by open sets U_i such that each U_i is a small open neighborhood of an m-simplex Δ_i and Δ_i is a deformation retract of U_i. Furthermore, we assume that the open sets U_i are such that each nonempty intersection $\Delta_{i_\circ}\cap\cdots\cap\Delta_{i_k}$ is a deformation retract of $U_{i_\circ}\cap\cdots\cap U_{i_k}$. In this manner we have a natural isomorphism between the simplicial and Čech modules which commutes with boundary operators. The required isomorphism then follows immediately. In particular, the Čech (co)homology of a simplex or a disc is identical to that of a point.

One would like to simplify the infinite limiting process in the definition of Čech (co)homology to a finite one amenable to calculation. As suggested by the above explanation for the equality of Čech and singular or simplicial theories, if the covering $\mathcal{U}$ is such that every non-empty intersection $U_{i_\circ}\cap\cdots\cap U_{i_k}$ has the (co)homology of a point, then the natural mapping

$$\check{H}^n(\mathcal{U}, X; R) \longrightarrow \check{H}^n(X; R) \tag{3.3.5}$$

is an isomorphism. The proof of this fact follows from general homological considerations similar to the proof of the idendependence of (co)Homology from the specific triangulation.

One reason for introducing Čech cohomology is that a number of geometric problems have a natural formulation in terms of Čech cohomology. Čech cohomology also easily generalizes to sheaf cohomology which has many applications in complex and algebraic geometry and will be discussed in this volume. Let us give the simplest example of a geometric question which translates into a problem in Čech cohomology.

Example 3.3.4 Let M be a manifold and $\mathcal{U} = \{U_\alpha\}$ be a covering of M by coordinate neighborhoods. Let $\varphi_{\alpha\beta} : \varphi_\beta(U_\alpha\cap U_\beta) \to \varphi_\alpha(U_\alpha\cap U_\beta)$ denote a transition function as defined in chapter 1. We want to investigate orientability of the manifold M. Define the Čech cochain $w_1 \in \check{C}(\mathcal{U}, M; \mathbf{Z}/2)$ by

$$w_1(\alpha,\beta) = \frac{\det D\varphi_{\alpha\beta}}{|\det D\varphi_{\alpha\beta}|}.$$

Note that $w_1(\alpha,\beta) = \pm 1$ and we have identified the multiplicative group $\{\pm 1\}$ with $\mathbf{Z}/2$. It is trivial to verify that $w_1(\alpha,\beta)$ is actually a cocycle. Orientability of M implies that the coordinate systems and consequently the transition functions can be chosen such that $w_1(\alpha,\beta) = 1$. We show conversely that if $w_1(\alpha,\beta)$ is a coboundary then the manifold M is orientable. Assume

$$w_1(\alpha,\beta) = \frac{w'(\beta)}{w'(\alpha)}, \qquad \text{(multiplicative notation)}$$

for some cochain $w' \in \check{C}^\circ(\mathcal{U}, M; \mathbf{Z}/2)$. By composing the homeomorphism φ_α with a linear transformation of $\mathbf{R}^m$ of determinant $w'(\alpha) = \pm 1$ we obtain new transition functions $\varphi'_{\alpha\beta}$ such that

$$\frac{\det D\varphi'_{\alpha\beta}}{|\det D\varphi'_{\alpha\beta}|} = 1,$$

which means that M is orientable. It is customary to refer to the cohomology class defined by the cocycle $w_1 \in \check{H}^1(M; \mathbf{Z}/2)$ as the *first Stiefel-Whitney class* of M. Therefore we have shown that the orientability of a manifold is equivalent to the vanishing of the first Stiefel-Whitney class. ♠

Exercise 3.3.11 *Generalize example 3.3.4 by assigning to a real vector bundle* $E \to M$ *a cohomology class* $w_1(E) \in \check{H}^1(M; \mathbf{Z}/2)$ *whose vanishing is equivalent to the orientability of the vector bundle. Show that a real line bundle is trivial if and only if it is orientable.*

Example 3.3.5 Using Čech cohomology one can assign to a complex vector bundle bundle $E \to M$ a cohomology class $c_1(E) \in \check{H}^2(M; \mathbf{Z})$ called the *first Chern class* of $E \to M$. First we consider the case of a complex line bundle $L \to M$. Let $\mathcal{U} = \{U_j\}$ be a covering of M and we assume that for all $(n+1)$-tuple $(i_\circ, \cdots, i_n) \in \mathcal{N}(\mathcal{U})$, $U_{i_\circ} \cap \cdots \cap U_{i_n}$ is contractible. Let $\rho_{jk} : U_j \cap U_k \to \mathbf{C}^\star$ be the transition functions for the complex line bundle $L \to M$. Then the functions

$$f_{jk} \;=\; \frac{1}{2\pi i} \log \rho_{jk}$$

are not single-valued and depend on the choice of the branch for logarithm. Any two values differ by an integer. We may assume the branch is chosen such that $f_{jk} + f_{kj} = 0$. The fact that ρ_{jk} are transition functions translates into the relation

$$c_{jkl} \stackrel{\text{def}}{=} f_{kl} - f_{jl} + f_{jk} \in \mathbf{Z}.$$

Clearly $\{c_{jkl}\}$ defines an integer valued Čech cocycle in dimension two which we denote by $c_1(L)$. For a vector bundle $E \to M$ of rank k we consider $\wedge^k E \to M$ which is a complex line bundle. We define $c_1(E) = c_1(\wedge^k E)$ which means that if ρ'_{jk}'s denote the transition functions for the vector bundle $E \to M$, then we repeat the above construction with $\rho_{jk} = \det(\rho'_{jk})$. ♠

Exercise 3.3.12 *Let* $\mathcal{L} \to \mathbf{CP}(1)$ *be the tautological line bundle. Using a a covering of* $\mathbf{CP}(1)$ *by four open sets* $U_\circ, U_1, U_2, U_3$ *(e.g., corresponding to small neighborhoods of the faces of the standard triangulation of the of the tetrahedron) explicitly write down values* c_{ijk} *for* $c_1(\mathcal{L})$ *and show that it is a generator for* $\check{H}^1(\mathbf{CP}(1); \mathbf{Z}) \simeq \mathbf{Z}$.

Exercise 3.3.13 *Let* $E \to M$ *and* $E' \to M$ *be vector bundles of ranks* k *and* k' *respectively. Show that* $c_1(E \oplus E') = c_1(E) + c_1(E')$*, and* $c_1(E \otimes E') = k'c_1(E) + kc_1(E')$.

Example 3.3.6 Just as vanishing of $w_1(E)$ implied orientability, $c_1(\mathcal{L}) = 0$, for a complex line bundle $\mathcal{L} \to M$, implies triviality of $\mathcal{L}$. The proof is a little more subtle than that for the first Stiefel-Whitney class. The argument involves an important idea which will be used elsewhere (see example 3.5.5 as well). We use the notation of example 3.3.5. The assumption $c_1(\mathcal{L}) = 0$ implies that there are integers h_{jk} defined for $(j,k) \in \mathcal{N}_1(\mathcal{U})$ such that $h_{jk} + h_{kj} = 0$ and $c_{jkl} = h_{kl} - h_{jl} + h_{jk}$. Modifying f_{jk} (by a coboundary) to $f'_{jk} = f_{jk} - h_{jk}$ we see that $f'_{kl} - f'_{jl} + f'_{jk} = 0$. This means the branches of logarithms for $\log \rho_{jk}$ can be chosen accordingly which implies that there are C^∞ complex valued functions ψ_{jk} such that $\rho_{jk} = e^{2\pi i \psi_{jk}}$. The key point is that the cocycle ψ_{jk} is actually a coboundary by the following clever idea: Let ϕ_j be a partition of unity and set $\psi_k = \sum \phi_l \psi_{lk}$ where the summation is over all indices $(l,k) \in \mathcal{N}(\mathcal{U})$. Now

$$\psi_k - \psi_j = \sum_k \phi_l \psi_{lk} - \sum_l \phi_l \psi_{lj} = \sum_l \phi_l(\psi_{lk} + \psi_{jl}) = \psi_{jk},$$

by the cocycle property $\psi_{lk} - \psi_{jk} + \psi_{jl} = 0$. Therefore modifying the transitions functions by multiplication by the coboundary $e^{2\pi i(\psi_j - \psi_k)}$ we see that the line bundle is trivial. ♠

3.3.4 Dual Structures Simple Complexes

To better understand the meaning of cohomology it is judicious to introduce a structure which is dual to that of a simplicial complex. We shall not duel on the technical details necessary for establishing the rigorous foundations of this theory since that will be lengthy and may obscure the fundamental underlying geometric ideas. To define this dual structure it is convenient to introduce a special class of cell complexes.

Recall that a subset $Y \subset \mathbf{R}^m$ is a polyhedron if there are points $y_1, \cdots, y_l$ such that Y is the convex closure of y_i's. The basic facts about convex sets was discussed in §5 of chapter 2. Let X be a regular cell complex of dimension m. We assume that every cell c of X has the structure of a convex polyhedron so that the notion of a p-face of a cell is well-defined. Every p-face is then a polyhedron as well. We call a regular cell complex where every cell has a given structure of a convex polyhedron a *polyhedral complex* if the intersection of two cells (or polyhedra) is either empty or consists of a single polyhedron. Thus X is made up of convex polyhedra much in the same way as simplicial complexes are constructed from simplices. A polyhedral complex is called *simple* if every p-face ($p < m$) is contained in $\binom{m+1-p}{m+1-q}$ q-faces.

Example 3.3.7 Let M be a compact surface endowed with a triangulation. By choosing points (vertices) in the interiors of the triangles and joining vertices corresponding to adjacent triangles we obtain a cell decomposition of M which we denote by $\check{M}$. $\check{M}$ is a simple complex. Figure XXXX shows the structure of the projective plane as a simplicial and a simple complex. Notice that if M had boundary, then the dual complex $\check{M}$ would no longer be simple. ♠

Example 3.3.7 gives a clear indication of what the dual structure to a simplicial decomposition of a manifold is like at least in dimension two. However, directly generalizing the description in example 3.3.7 to arbitrary dimensions is laborious and replete with technical problems. The most important relationship between the triangulation of M and its dual structure $\check{M}$ is that every k-simplex Δ^k of M intersects a unique $(m-k)$-face of $\check{M}$ and vice versa; each intersection is transverse and consists of a single point. The fact that the dual structure is that of a simple complex is a straightforward counting argument, since the q-faces containing a given p-face δ^p dual to the $(m-p)$-simplex Δ^{m-p}, are in one to one correspondence with $(m-q)$-simplices contained in Δ^{m-p}. Lemmas 3.3.2, **??** and formula (**??**) below contain the most essential properties of the dual simple complex for the purpose of cohomology.

The vertices of the dual complex are in one to one correspondence with m-simplices of M. An edge δ^1 of the dual complex is specified by an $(m-1)$-simplex Δ^{m-1} and the vertices of δ^1 correspond to the m-simplices containing Δ^{m-1}. Similarly, k-faces of the dual complex $\check{M}$ are in one to one correspondence with the $(m-k)$-simplices of M. Let δ^k correspond to the $(m-k)$-simplex Δ^{m-k}. Then the vertices of δ^k correspond m-simplices containing Δ^{m-k}; the edges correspond to $(m-1)$-simplices containing Δ^{m-k}; 2-faces of δ^k correspond to $(m-2)$-simplices containing Δ^{m-k}, etc. In other words, the incidence relations for the dual complex is that given a k-face δ^k of $\check{M}$ corresponding to the $(m-k)$-simplex Δ^{m-k} of M, the l-faces of δ^k, $l < k$, correspond precisely to those $(m-l)$-simplices of M containing Δ^{m-k}. To topologize the dual structure so defined we proceed as follows: Let δ^m be an m-face of the dual structure corresponding to the vertex Δ°. Let $\delta_1^\circ, \cdots, \delta_l^\circ$ be the vertices of δ^m, i.e., $\Delta_1^m, \cdots, \Delta_l^m$ are the m-simplices in the star of the vertex Δ° and δ_j° corresponds to Δ_j^m. Let $x_1, \cdots, x_l \in \mathbf{R}^k$ with the properties

1. Any subset $x_{i_1}, \cdots, x_{i_m}$ of distinct points is linearly independent;

2. **0** lies in the convex closure $\mathcal{C}(x_1, \cdots, x_l)$ of $x_1, \cdots, x_l$;

3. No x_j lies in the convex closure of the remaining x_i's.

4. If δ^k, $k > 0$, corresponds to a Δ^{m-k} containing Δ°, and $\Delta^m_{i_1}, \cdots, \Delta^m_{i_r}$ are all m-simplices containing Δ^{m-k}, then $\mathcal{C}(x_{i_1}, \cdots, x_{i_r})$ is a face of $\mathcal{C}(x_1, \cdots, x_r)$.

It is not obvious that we can choose x_j's with the above properties, however, some experimentation will convince the reader that this is in fact possible. We shall not present a proof.

We topologize δ^m, the m-face of the simple complex $\check{M}$ dual to Δ°, via any homeomorphism with $\mathcal{C}(x_1, \cdots, x_l)$ mapping the vertex corresponding the m-simplex Δ^m_j to x_j. Let δ^m and γ^m be m-faces corresponding to vertices Δ° and Γ°. Then $\delta^m \cap \gamma^m = \emptyset$ unless Δ° and Γ° are vertices of a 1-simplex. In such a case we denote the 1-simplex by $\Phi(\Delta^\circ, \Gamma^\circ)$. Enumerate the m-simplices in the star of Δ° and Γ° such that $\Delta^m_1, \cdots, \Delta^m_r$ are all the m-simplices containing $\Phi(\Delta^\circ, \Gamma^\circ)$. Fixing homeomorphisms

$$\phi_{\Delta^\circ} : \delta^m \to \mathcal{C}(x_1, \cdots, x_l), \quad \phi_{\Gamma^\circ} : \gamma^m \to \mathcal{C}(y_1, \cdots, y'_l),$$

satisfying the provisions described above, we attach δ^m and γ^m together by attaching $\mathcal{C}(x_1, \cdots, x_l)$ and $\mathcal{C}(y_1, \cdots, y'_l)$ along their faces $\mathcal{C}(x_1, \cdots, x_r)$ and $\mathcal{C}(y_1, \cdots, y_r)$ by mapping the vertex x_i to vertex y_i for $i = 1, \cdots, r$. Note that this is possible in view of our enumeration of the m-simplices containing Δ° and Γ°. This specifies the topology of $\check{M}$ unambiguously.

Exercise 3.3.14 *Regarding $\check{M}$ as a partially ordered set via inclusion of faces, show that the associated simplicial complex can be (naturally) identified with the first barycentric subdivision of M.*

Lemma 3.3.2 *Let M be a smooth manifold equipped with a triangulation, and let $\check{M}$ denote the dual structure of a simple complex. Then for any vertex δ° of $\check{M}$ and k edges $\delta^1_i \supset \delta^\circ$, $i = 1, \cdots, k$ where $k \leq m$, there is a unique k-face of $\check{M}$ containing δ^1_i's. The correspondence between sets of k edges containing a given vertex δ° and k-faces containing δ° is a bijection for $k \leq m$.*

Proof - Let Δ^m be the m-simplex corresponding to δ°, and $\Delta^{m-1}_1, \cdots, \Delta^{m-1}_k$ be the $(m-1)$-simplices dual to $\delta^1_1, \cdots, \delta^1_k$. Then $\Delta^{m-1}_j \subset \Delta^m$ since $\delta^1_j \supset \delta^\circ$. Clearly $\cap_j \Delta^{m-1}_j$ is an $(m-k)$-simplex and the corresponding k-face contains δ^1_j's. Since an $(m-k)$-simplex $\Delta^{m-k} \subset \Delta^m$ is uniquely representable as the intersection of k faces of Δ^m of codimension 1. ♣

Now let δ^{k+1} be a $(k+1)$-face of $\check{M}$, then its boundary $\partial\delta^{k+1}$ is a polyhedral complex homeomorphic to the sphere S^k. Let Δ^{m-k-1} be the $(m-k-1)$-simplex dual to δ^{k+1}. Then the p-faces in $\partial\delta^{k+1}$, $(p \leq k)$, are in bijective correspondence with the $(m-p)$-simplices containing Δ^{m-k-1}. Therefore given a p-face $\delta^p \subset \partial\delta^{k+1}$ is contained in $\delta^q \subset \partial\delta^{k+1}$, $(p \leq q \leq k)$, if and only if we have the following inclusion relation on the dual simplices

$$\Delta^{m-k-1} \subset \Delta^{m-q} \subset \Delta^{m-p}.$$

Therefore trivial counting argument implies:

Lemma 3.3.3 *Let M be a smooth manifold equipped with a triangulation, and let $\check{M}$ denote the dual structure of a simple complex. Then the boundary of every face of $\check{M}$ with the induced polyhedral structure is a simple complex.*

Lemma 3.3.2 provides a very convenient way of describing a k-polyhedron which allows one to incorporate the orientation in the notation. In fact, if a vertex δ° for a k-polyhedron δ^k is specified, then it is uniquely determined by a set of edges $\delta_1^1, \cdots, \delta_k^1$ incident upon the vertex δ°. We then represent δ^k as $\delta_1^1 \wedge \cdots \wedge \delta_k^1$. The ordered set of edges $\delta_1^1, \cdots, \delta_k^1$ determines an orientation for the polyhedron δ^k. For example, we can identify δ^k with a polyhedron in $\mathbf{R}^k$ by mapping δ_j^1 to the standard basis vector e_j and assign the orientation induced from the ordered basis $e_1, \cdots, e_k$. In view of the $\wedge$ notation and the anti-symmetry in the definition of wedge product, changing the order of the edges changes the representation of δ^k by the sign of the corresponding permutation, and thus the orientation is incorporated in this notation.

In order to define cohomology groups we need to define the coboundary operator. We consider the structure of M as a simplicial complex and the dual simple complex $\check{M}$. Recall from chapter 3, section 3 that $\check{E}_k = H_k(\check{M}(k), \check{M}(k-1); R)$ is the free R-module with basis the k-polyhedra of $\check{M}$ and we have the boundary homomorphism $\varrho_k : \check{E}_k \rightarrow \check{E}_{k-1}$, but this is not of interest. We need to define the coboundary operator $\varrho^\star : \check{E}_k \rightarrow \check{E}_{k+1}$. To do so we make use of the duality between M and $\check{M}$, and consequently between $\check{E}_k$ and $E_{m-k} = H_{m-k}(M(k), M(k-1); R)$. The coboundary operator is defined by the requirement

$$\varrho^\star(\delta^{k-1})(\Delta^{m-k}) = \delta^{k-1}(\varrho(\Delta^{m-k})). \tag{3.3.6}$$

It follows that $\varrho^\star(\delta^k)$ is the sum of $(k+1)$-dimensional polyhedra of $\check{M}$ containing δ^k on their boundary with the appropriate orientations. To make this more precise, let δ° be a vertex of δ^k and $\delta_1^1, \cdots, \delta_k^1$ be the unique set of edges incident upon δ° such that $\delta^k = \delta_1^1 \wedge \cdots \wedge \delta_k^1$. Then

$$\varrho^\star(\delta_1^1 \wedge \cdots \wedge \delta_k^1) = \sum_j \delta_j^1 \wedge \delta_1^1 \wedge \cdots \wedge \delta_k^1, \tag{3.3.7}$$

where the summation is over the edges δ_j^1 incident upon the vertex δ°. Note that if δ_j^1 is not distinct from $\delta_1^1, \cdots, \delta_k^1$, then corresponding term in (**??**) vanishes. The expression on the right hand side of (3.3.7) appears to depend on the choice of the vertex δ°, however, since it is straightforward to see that it is well-defined.

Now assume that M is orientable and compatible local orientations have been fixed which means that for every vertex $x \in M$, the m-simplices in a neighborhood U_x of x are given compatible orientations. Thus if a simplex lies in $U_x \cap U_y$, then the corresponding orientations are identical. Now define

$$\jmath : E_k = H_k(\check{M}(k), \check{M}(k-1)) \longrightarrow H_{m-k}(M(m-k), M(m-k-1)), \qquad \jmath(\delta^k) = \Delta^{m-k}, \tag{3.3.8}$$

where Δ^{m-k} is the unique oriented $(m-k)$-simplex of M such that

$$\mathrm{KI}(\delta^k, \Delta^{m-k}) = 1$$

We have the following basic lemma:

Lemma 3.3.4 *The following diagram is commutative:*

$$\begin{array}{ccc} E_k = H_k(\check{M}(k), \check{M}(k-1)) & \xrightarrow{\jmath} & H_{m-k}(M(m-k), M(m-k-1)) \\ \varrho^\star \downarrow & & \varrho \downarrow \\ E_k = H_{k+1}(\check{M}(k+1), \check{M}(k)) & \xrightarrow{\jmath} & H_{m-k-1}(M(m-k-1), M(m-k-2)) \end{array}$$

Proof - Let Δ^k be a k-simplex of M and δ^{m-k} the corresponding dual $(m-k)$-face of $\check{M}$. It follows from the description of the coboundary operator that the $(m-k+1)$-faces of $\check{M}$ that appear $\varrho^\star(\delta^{m-k})$ are precisely the faces dual to the simplices in the expression $\varrho(\Delta^k)$. Therefore to check that they appear with desired signs, it suffices to consider a k-simplex in $\mathbf{R}^m$ and compute the relevant quantities. This is a straightforward calculation. ♠

It follows from the lemma that the matrices representing the linear maps ϱ and $\varrho^\star$ are identical for proper choice of bases. We therefore have proven

Theorem 3.3.1 (Poincaré Duality) *Let M be a compact orientable manifold (without boundary). Then $\jmath$ induces isomorphisms*

$$\jmath : H^{m-k}(M; R) \simeq H_k(M; R).$$

Remark 3.3.4 The assumption of orientability for the isomorphism $H^j(M; \mathbf{Z})$ and $H_{m-j}(M; \mathbf{Z})$ is essential, since for example, for the real projective plane the second Betti number is zero while $H_\circ(\mathbf{RP}(2); \mathbf{Z}) \simeq \mathbf{Z}$. ♡

An immediate consequence of Poincaré duality is

Exercise 3.3.15 *The Euler characteristic of an odd dimensional compact orientable manifold is zero.*

Exercise 3.3.16 *Consider the triangulation of the torus and the dual simple structure given in Figure 2.3. Show that the Poincaré dual to the cycle represented as AB is the sum of the bold line segments as shown in the figure (with proper orientations).*

Exercise 3.3.17 *Show how the first Stiefel-Whitney class can be defined in the context of the simple complex dual to a triangulation of a manifold.*

Example 3.3.8 Let U be an orientable compact odd dimensional manifold with smooth boundary $\partial U = M$. Let U' be another copy of U with $\partial U' = M' = M$. Attach U and U' together along their boundaries via the identity map of $M \rightarrow M' = M$ to obtain a new manifold N. It is easy to see that N is a compact orientable manifold. Let V and V' be neighborhoods of U and U' in N such that M is a deformation retract of $V \cap V'$ and U (resp U') is a deformation retract of V (resp. V'). Applying the Mayer-Vietories sequence we obtain

$$\cdots \rightarrow H_k(M; \mathbf{R}) \rightarrow H_k(U; \mathbf{R}) \oplus H_k(U'; \mathbf{R}) \rightarrow H_k(N; \mathbf{R}) \rightarrow H_{k-1}(M; \mathbf{R}) \rightarrow \cdots$$

Now $\chi(N) = 0$ and since the alternating sum of the dimensions in an exact sequence of vector spaces is zero (example 3.5 of chapter 3) we obtain

$$\chi(M) = 2\chi(U).$$

In particular, $\mathbf{CP}(2n)$ is not the boundary of any compact manifold. If U were even dimensional, then $\chi(M) = 0$ and consequently $2\chi(U) = \chi(N)$. This example will be used later to obtain an extension of the Gauss-Bonnet theorem to compact hypersurfaces. ♠

With the notion of simple complex dual to a triangulation well established, the existence of a product structure on cohomology becomes an inevitability as shown below where generalizations of Poincaré duality are also discussed. Later in this chapter we see another application of the notion simple complex by showing that there are linear equations relating the numbers of simplices of various dimensions for a triangulation of a compact manifold (in addition to the fact that the alternating sum of the number of simplices in each dimension is equal to the Euler characteristic as implied by lemma 2.1 of chapter 3.)

3.3.5 Cup and Cap Products

The fundamental feature of cohomology theory which distinguishes it from homology is the existence of a graded product $\cup$, called *cup product* which endows $\sum H^p(X;R)$ with the structure of a graded ring. First we look at this concept in the context of simple complexes dual to a triangulation of a compact manifold M in which case the definition of cup product is very intuitive. In fact, let δ^p and δ^q denote p and q-faces of $\check{M}$, and assume $\delta^p\cap\delta^q$ is a vertex δ°. Then, with orientations assigned, we have representations $\delta^p=\delta^1_1\wedge\cdots\wedge\delta^1_p$ and $\delta^q=\delta^1_{p+1}\wedge\cdots\wedge\delta^1_{p+q}$ where δ^1_j's are edges incident upon the vertex δ°. We may define

$$\delta^p\cup\delta^q=\delta^1_1\wedge\cdots\wedge\delta^1_p\wedge\delta^1_{p+1}\wedge\cdots\wedge\delta^1_{p+q}. \tag{3.3.9}$$

This manner of defining cup product, albeit being intuitively appealing, has technical problems. For example, there is no canonical way of choosing vertices for the faces of the simple complex dual to a triangulation and therefore it is not clear how to make sense out of (**??**)[1]. A technically viable and geometrically appealing definition can be formulated by regarding the triangulated manifold M as a partially ordered set and looking at the associated simplicial complex which is the first barycentric subdivision of M. We denote the first barycentric subdivision of M by M again and let $\check{M}$ denote the dual simple complex. Recall that a vertex of $\check{M}$ is a maximal sequence $\delta^\circ:\Delta_{i_\circ}\subset\cdots\subset\Delta_{i_m}$ of simplices (of the original triangulation) of M, and a k-face δ^k of $\check{M}$ is obtained by deleting k simplices from such a maximal sequence. Given a k-face and an l-face δ^l of $\check{M}$, we want to define $\delta^k\cup\delta^l$. Let $\delta^k:\Delta_{i_\circ}\subset\cdots\subset\Delta_{i_{m+1-k}}$ and $\delta^l:\Delta'_{i_\circ}\subset\cdots\subset\Delta'_{i_{m+1-l}}$. If the sequences representing δ^k and δ^l have exactly $m+1-(l+k)$ simplices, say $\Delta_{j_\circ}\subset\cdots\subset\Delta_{j_{m+1-l-k}}$, in common, then we set

$$\delta^k\cup\delta^l:\Delta_{j_\circ}\subset\cdots\subset\Delta_{j_{m+1-l-k}}.$$

We also set $\delta^k\cup\delta^l=0$ if the number of simplices in common to δ^k and δ^l is not $m+1-(l+k)$. Thus the cup product represents intersections of faces of the dual complex $\check{M}$. Cup product is then extended to the group of p and q-cochains by linearity.

Now assume that the simplices of M and the faces of $\check{M}$ are oriented. We wish to incorporate orientations into cup product. Notice that if M is oriented, then orientations for simplices of M induce orientations for faces of $\check{M}$ since every face of the latter intersects a unique simplex of M transversally at one point, and therefore general considerations on intersections of submanifolds are applicable to relate orientations of simplices of M, faces of $\check{M}$ and orientation of M. Fixing local orientations for M, we can unambiguously assign a number

[1] It is worthwhile to develop a mehod to make sense out of (**??**). This will be done in the subsection on Discrete Maurer-Cartan Equations.

$\epsilon = \pm 1$ to a flag $\Delta_{i_\circ} \subset \cdots \subset \Delta_{i_m}$ by the following rule: The simplices Δ_{i_j} are regarded as the vertices of the first barycentric subdivision of M and therefore the m-simplex of the first barycentric subdivision with vertices ordered as $[\Delta_{i_\circ}, \cdots, \Delta_{i_m}]$ is either positively or negatively oriented with respect to the orientation of M which accordingly defines ϵ as $+1$ or -1. Naturally any re-arrangement of the simplices/vertices $\Delta_{i_\circ}, \cdots, \Delta_{i_m}$ multiplies the orientation of the simplex $[\Delta_{i_\circ}, \cdots, \Delta_{i_m}]$ (of the first barycentric subdivision) by the sign of the corresponding permutation. Now assume orientations $\epsilon(\delta^k)$ and $\epsilon(\delta^l)$ are assigned to the faces δ^k and δ^l of $\check{M}$. Let the complement of the common simplices $\{\Delta_{j_\circ}, \cdots, \Delta_{j_{m+1-l-k}}\}$ (of δ^k and δ^l) in $\{\Delta_{i_\circ}, \cdots, \Delta_{i_{m+1-k}}\}$ and $\{\Delta'_{i_\circ}, \cdots, \Delta'_{i_{m+1-l}}\}$ be $\{\Delta_{p_\circ}, \cdots, \Delta_{p_k}\}$ and $\{\Delta'_{q_\circ}, \cdots, \Delta'_{q_l}\}$ respectively. Since δ^k and δ^l are oriented, the (ordered) simplices/vertices $\Delta_{j_\circ}, \cdots, \Delta_{j_{m+1-l-k}}, \Delta_{p_\circ}, \cdots, \Delta_{p_k}$ and $\Delta_{j_\circ}, \cdots, \Delta_{j_{m+1-l-k}}, \Delta'_{q_\circ}, \cdots, \Delta'_{q_l}$ (which represent δ^k and δ^l) have orientations which we denote by ϵ_1 and ϵ_2 respectively. Similarly, since M is oriented, the (ordered) simplices/vertices $\Delta_{j_\circ}, \cdots, \Delta_{j_{m+1-l-k}}, \Delta_{p_\circ}, \cdots, \Delta_{p_k}, \Delta'_{q_\circ}, \cdots, \Delta'_{q_l}$ define an orientation ϵ_M. Then the orientation assigned to

$$\delta^k \cup \delta^l : \Delta_{j_\circ} \subset \cdots \subset \Delta_{j_{m+1-l-k}}$$

is $\epsilon_1\epsilon_2\epsilon_M$. The definition of cup product is extended to cochains by linearity. With this definition, the following properties of the cup product p and q-cochains α^p and β^q are easily verified:

1. Cup product is distributive relative to addition.
2. $\varrho^\star_{p+q}(\alpha^p \cup \beta^q) = \varrho^\star_p(\alpha^p) \cup \beta^q + (-1)^p \alpha^p \cup \varrho^\star_q(\beta^q)$.
3. $\alpha^p \cup \beta^q = (-1)^{pq} \beta^q \cup \alpha^p$.

From (2) we see that the product of two cocycles is a cocycle. Furthermore

$$\varrho^\star_p(\alpha^p) \cup \beta^q = \varrho^\star_{p+q}(\alpha^p \cup \beta^q) - (-1)^p \alpha^p \cup \varrho^\star_q(\beta^q)$$

implies that if β^q is a cocycle, then $\varrho^\star_p(\alpha^p) \cup \beta^q$ is a coboundary. Consequently, cup product is actually defined at the level of cohomology.

Exercise 3.3.18 *Using the barycentric subdivision, its interpretation as flags of simplices and the notion of dual complex, show that the Poincaré dual to the intersection of two closed orientable submanifolds of a compact orientable manifold is the cup product of their Poincaré duals.*

In addition to the cup product there is also a pairing between H^p and H_j, for $j \geq p$, called *cap product.* Given a p-cochain α^p and j-chain c^j we define $\alpha^p \cap c^j$ to be the unique $j-p$-chain such that for all $(j-p)$-cochains β^{j-p} we have (the superscripts denoting the dimension of the chain or cochain will be often omitted)

4. $(\beta \cup \alpha)(c) = \beta(\alpha \cap c)$.

The explicit description of cap product is via the Poincaré duality isomorphism $D = D_k : H^k(M; \mathbf{Z}) \rightarrow H_{m-k}(M; \mathbf{Z})$. In fact, we set

$$\alpha \cap c = D_{m+p-j}(\alpha \cup D_j^{-1}(c)). \tag{3.3.10}$$

Clearly, cap product defined via equation (3.3.10) satisfies property (4) and

5. $\partial_{j-p}(\alpha \cap c) = \varrho_p^\star(\alpha) \cap c + (-1)^p \alpha \cap \partial_j(c)$.

From (5) we see that cap product ie well-defined at the level of (co)homology. Notice that for $j = p$ the product $\alpha \cap c$ is simply the evaluation of the cocycle α on the chain c. In order for cap and cup products to become optimally useful we have to understand their behavior relative to mappings and the connecting homomorphism. These *naturality* properties of cap/cup product are given by the following:

6. Let $f : X \to Y$, $\alpha \in H^k(Y; R)$ and $c \in H_k(X; R)$, then
$$\alpha \cap f_\star(c) = f_\star(f^\star(\alpha) \cap c).$$

7. Let $A \subset X$ be an open subset (or A is a subcomplex of the simplicial complex X), $\alpha \in H^j(X; R)$, and $c \in H_k(X, A; R)$, then
$$\imath^\star(\alpha) \cap \delta_k c = \delta_{k-j}(\alpha \cap c),$$
where $\imath$ is the inclusion of A in X, and δ is the connecting homomorphism. In particular, if $X = M$ is a manifold with boundary $\partial M \neq \emptyset$, then
$$\imath^\star(\alpha) \cap \lfloor \partial M \rfloor = \delta(\alpha \cap \lfloor M \rfloor).$$
Here $\imath$ is the inclusion of ∂M in M, $\lfloor M \rfloor \in H_m(M, \partial M; R)$ is an orientation class for M, and $\alpha \in H^k(M; R)$.

While the definition of cup/cap in terms of dual complex is geometrically enlightening, it is important to extend the notion of singular (co)homology and pairs of spaces. For the standard n-simplex $\Delta(n)$ with vertices $e_\circ, \cdots, e_n$ let its *k-front face* $f_k(\Delta(n))$ be the simplex which is the convex closure of $e_\circ, \cdots, e_k$. Similarly its *k-back face* $b_k(\Delta(n))$ is the convex closure of $e_{n-k}, \cdots, e_n$. This notion extends in the obvious manner to front and back faces of singular chains by linearity. Let $C_\star(X) = \sum_j C_j(X)$ denote the singular complex of a topological space X, and as usual set
$$C_\star(X) \otimes C_\star(X) \; = \; \sum_n \sum_{j+k=n} C_j(X) \otimes C_k(X),$$
with boundary operator $\partial_n : \sum_{j+k=n} C_j(X) \otimes C_k(X) \to \sum_{j+k=n-1} C_j(X) \otimes C_k(X)$ defined by $\partial_n = \partial_j \otimes \mathrm{id.} + (-1)^j \mathrm{id.} \otimes \partial_k$. Then we have the mapping $\tau : C_\star(X) \to C_\star(X) \otimes C_\star(X)$ defined by
$$\tau(c) = \sum_{j+k=n} c_{|f_k(\Delta(n)} \otimes c_{|b_k(\Delta(n)}, \tag{3.3.11}$$
where $c_{|f_k(\Delta(n)}$ is the singular cochain given by the restriction of c to its front face $|f_k(\Delta(n)$. The map τ is often called the *Alexander-Whitney diagonal approximation.* The cup product, in the context of singular cohomology, can be defined by utilizing this map. In fact, if α and β are j and k singular cochains, then we define
$$(\alpha \cup \beta)(c) = (\alpha \otimes \beta)(\tau(c)) = \alpha(c_{|f_j(\Delta(n))})\beta(c_{|b_k(\Delta(n))}),$$

where we have used the ring structure of R to multiply cochains. It is clear that if $\alpha \in C^j(X, A; R)$ and $\beta \in C^k(X, B; R)$, then $\alpha \cup \beta \in C^{j+k}(X; R)$ and it vanishes on $C_{j+k}(A; R) + C_{j+k}(B; R)$. Now if, for example, $B \subseteq A$ then we have $\alpha \cup \beta \in C^{j+k}(X, A; R)$. (In general, if the inclusion $C_\star(A) + C_\star(B) \hookrightarrow C_\star(A \cup B)$ induces isomorphisms on homology groups, then one concludes that at the level of cohomology $\alpha \cup \beta \in H^{j+k}(X, A \cup B; R)$ for cocycles α and β. For our purposes, the special case $B \subseteq A$ will suffice.) This defines the notion of cup product at the level of relative cochains. The fact that cup product is defined on cohomology follows from identity (3) given above which remains valid for singular chains. For a singular cochain $\alpha \in C^j(X, A; R)$ and singular chain $c \in C_k(X, A; R)$ with $k \geq j$ we define the cap product $\alpha \cap c \in C_{k-j}(X, A; R)$ by the requirement $(\beta \cup \alpha)(c) = \beta(\alpha \cap c)$ for all $\beta \in C^{k-j}(X, A; R)$ (see property (4) above). Properties (1) through (7) remain valid for singular relative (co)cycles with minor modifications. For example, for $f : (X, A) \rightarrow (Y, B)$, $\alpha \in H^k(Y, B; R)$ and $c \in H_k(X, A; R)$, property (6) remains valid. We shall not dwell on the proofs of these statements and refer the reader to [Hat], [L] or [S] for detailed treatment in simplicial and singular categories.

3.3.6 Duality Theorems

Let M be an orientable manifold. Recall that for every compact subset $K \subseteq M$ there is $\mathbf{m}_K \in H_m(M, M \setminus K; \mathbf{Z})$ defining the orientation of M as specified in theorem 4.1 of chapter 2. Therefore for $\alpha \in H^j(M, M \setminus K; \mathbf{Z})$, $\alpha \cap \mathbf{m}_K \in H_{m-j}(M; \mathbf{Z})$. In view of naturality property (5), we pass to the limit and obtain a homomorphism

$$D : H^j_c(M; \mathbf{Z}) \longrightarrow H_{m-j}(M; \mathbf{Z}).$$

The proof of Poincaré duality given in the last section generalizes to yield the following theorem:

Theorem 3.3.2 (Poincaré Duality) - *Let M be an orientable manifold. Then the map D is an isomorphism.*

We first look at some consequences of the product structure and the duality theorem.

Example 3.3.9 Consider the complex projective space $\mathbf{CP}(n)$. We know that its (co)homology is $\mathbf{Z}$ in even dimensions $\leq 2n$ and zero otherwise. We want to understand the ring structure of $H^\star(\mathbf{CP}(n); \mathbf{Z})$. Let $\omega \in H^2(\mathbf{CP}(n); \mathbf{Z}) \simeq \mathbf{Z}$ be a generator, and denote the cup product of ω with itself k times by ω^k. We show by induction on n that ω^k is a generator of $H^{2k}(\mathbf{CP}(n); \mathbf{Z})$. The case $n = 1$ is trivial. Recall that the inclusion $\imath : \mathbf{CP}(n-1) \rightarrow \mathbf{CP}(n)$ induces an isomorphism on (co)homology in dimensions $\leq 2n - 1$. Since $\imath^\star$ is a ring homomorphism, $\imath^\star(\omega^k)$ is a generator of $H^{2k}(\mathbf{CP}(n-1); \mathbf{Z})$ for $k < n$, and consequently ω^k is a generator of $H^{2k}(\mathbf{CP}(n); \mathbf{Z})$ for $k < n$. By Poincaré duality for $\mathbf{CP}(n)$, $D(\omega^{n-1})$ is a generator of $H_2(\mathbf{CP}(n); \mathbf{Z})$. From property (4) of cap/cup products with $\beta = \omega$ and $\alpha = \omega^{n-1}$ we obtain

$$\omega^n(\lfloor \mathbf{CP}(n) \rfloor) = \omega(D(\omega^{n-1})) = \pm 1,$$

which proves that ω^n is a generator of $H^{2n}(\mathbf{CP}(n); \mathbf{Z})$ as desired. ♠

Let M be a compact oriented manifold of dimension $m = 2n$. Then cup product $\cup$ defines a pairing

$$H^n(M; \mathbf{Z}) \otimes H^n(M; \mathbf{Z}) \longrightarrow H^m(M; \mathbf{Z}) \longrightarrow \mathbf{Z}, \qquad (3.3.12)$$

by $(\alpha \cup \beta)(\lfloor M \rfloor)$, where $\lfloor M \rfloor$ is the orientation class of M. Restricting the pairing to the free parts of the abelian groups $H^n(M; \mathbf{Z})$ and $H^n(M; \mathbf{Z})$ we obtain a new pairing which we call the *intersection pairing*. Choosing a basis for the free part ${}^f H^n(M; \mathbf{Z})$ of the middle cohomology, we represent this pairing as an integral matrix which is called the *intersection matrix* of M. An integral matrix is called *unimodular* if its determinant is ± 1. An important consequence of the Poincaré duality is

Corollary 3.3.1 *The intersection matrix is unimodular. In particular, the intersection pairing is non-degenerate.*

Proof - It suffices to prove unimodularity. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a basis for $H^n_f(M; \mathbf{Z}) \simeq \mathbf{Z}^r$ and Q be the intersection matrix relative to this basis. Let $\{a_1, \cdots, a_r\}$ be the dual basis for the free part of the n^{th} homology of M, $H_n(M; \mathbf{Z}) \simeq \mathbf{Z}^r$. Set $\beta_j = D^{-1}(a_j)$. Then $\beta_j = \sum P_{kj}\alpha_k$ for some integral matrix $P = (P_{kj})$. Now

$$\delta_{ij} = \alpha_i(a_j) = \alpha_i \cup \beta_j([M]),$$

which is the $(i,j)^{\text{th}}$ entry of the matrix $Q'P$ ($'$ denotes transpose). Since P is integral $\det(P) = \det(Q) = \pm 1$ as desired. ♣

Corollary 3.3.2 *The middle Betti number and the Euler characteristic of a compact orientable manifold of dimension $m = 2n + 2$ are even.*

Proof - Since the intersection matrix is anti-symmetric in this case, its non-degeneracy implies even dimensionality of $H^{2n+1}(M; \mathbf{R})$, and Poincaré duality implies the second assertion. ♣

It is customary to write $H^\star(X; R) = \sum_k H^k(X; R)$ and regard $H^\star(X; R)$ as an algebra over the ring R.

Exercise 3.3.19 *Using the structure of $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ as a simple complex, show that $H^\star(T^2; \mathbf{Z})$ is isomorphic to the exterior algebra over $\mathbf{Z}^2$. In particular, the intersection matrix of T^2 is the standard 2×2 skew-symmetric matrix* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proposition 3.2.3 is also valid for cohomology. In particular, if the (co)homology groups of X and Y are free R-modules, then

$$H^n(X \times Y; R) = \sum_{i+j=n} H^i(X; R) \otimes H^j(Y; R). \tag{3.3.13}$$

In the case of cohomology, however, we have the additional structure of cup product which we now want to clarify for product spaces. For simplicity assume that the (co)homology groups of X and Y are free R-modules. Let p_X and p_Y denote the projections of $X \times Y$ onto the first and second factors respectively. Let $\alpha \in H^i(X; R)$ and $\beta \in H^j(Y; R)$, then $p_X^\star(\alpha)$ and $p_Y^\star(\beta)$ are identified with $\alpha \otimes 1$ and $1 \otimes \beta$ relative to the decomposition (3.3.13). By abuse of notation, whenever there is no risk of confusion, we simply write α instead of $p_X^\star(\alpha) = \alpha \otimes 1$ and similarly for β. Then the cup product for the product space $X \times Y$ is related to that of its factors by

$$(\alpha \otimes \beta) \cup (\alpha' \otimes \beta') = (-1)^{\deg(\beta)\deg(\alpha')}(\alpha \cup \alpha') \otimes (\beta \cup \beta'). \tag{3.3.14}$$

Exercise 3.3.20 *Let $T^m = \mathbf{R}^m/\mathbf{Z}^m$ be the m-torus with orientation induced from the standard one of $\mathbf{R}^m$. Show that the cohomology ring $H^\star(T^m;\mathbf{Z})$ is isomorphic to the exterior algebra $\bigwedge^\star \mathbf{Z}^m$ with cup product given by exterior multiplication.*

Exercise 3.3.21 *Let M and M' be compact orientable manifolds of the same dimension $m = 2n$ and let Q and Q' be the intersection matrices of M and M' for a choice of bases for ${}^fH^n(M;\mathbf{Z})$ and ${}^fH^n(M';\mathbf{Z})$. Show that $Q \oplus Q'$ is the intersection matrix for $M\sharp M'$. In particular, the intersection matrix for the orientable surface M_g of genus g is the standard $2g \times 2g$ skew symmetric matrix.*

Now consider the case of a compact orientable manifold M of dimension $m = 4n$. The intersection matrix is a unimodular symmetric matrix Q regarded as a symmetric bilinear form on ${}^fH^{2n}(M;\mathbf{Z})$. The index of Q (i.e., number of positive eigenvalues of Q minus number of negative eigenvalues of Q) is independent of the choice of basis and depends only on the homotopy class of M and the choice of the orientation class. We denote the index of Q by $\sigma(M)$ and call it the *signature* of M. It is convenient to set $\sigma(M) = 0$ if dimension of M is not divisible by 4.

Exercise 3.3.22 *Let $M = P \times R$ be the product of the compact orientable manifolds of dimensions $2p+2$ and $2r+2$. Show that $\sigma(M) = 0$. Generalize by proving*

$$\sigma(M) = \sigma(P)\sigma(R)$$

with no restriction on the dimensions.

Exercise 3.3.23 *Let M be a compact manifold with a fixed orientation, and $\bar{M}$ denote M with the opposite orientation. Then $\sigma(M) = -\sigma(\bar{M})$. Deduce that there are compact orientable manifolds of dimension 4 of arbitrary signature (let e.g., $M = \mathbf{CP}(2)$ and consider $M\sharp\cdots\sharp M\sharp\bar{M}\sharp\cdots\sharp\bar{M}$).*

The following duality theorem is a generalization of the Poincaré duality:

Theorem 3.3.3 (Lefschetz Duality) - *For a compact orientable manifold M of dimension $m+1$ with boundary $\partial M \neq \emptyset$, we have commutative diagram*

$$\begin{array}{ccccccccc}
\rightarrow & H^l(M,\partial M) & \rightarrow & H^l(M) & \rightarrow & H^l(\partial M) & \rightarrow & H^{l+1}(M,\partial M) & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\rightarrow & H_{m-l+1}(M) & \rightarrow & H_{m-l+1}(M,\partial M) & \rightarrow & H_{m-l}(\partial M) & \rightarrow & H_{m-l}(M) & \rightarrow
\end{array}$$

where all the columns are isomorphisms given by cap product. (The coefficient ring is assumed to be $\mathbf{Z}$.)

Exercise 3.3.24 *Show that the conclusion of example 3.3.8 is also a consequence of Lefschetz duality.*

Corollary 3.3.3 *Let M be a compact connected orientable manifold of dimension $m = 4n+1$ with boundary $\partial M \neq \emptyset$. Then the signature of ∂M, with orientation induced from that of M, vanishes.*

Proof - Consider the long exact sequence for cohomology of the pair $(M, \partial M)$:

$$H^{2n}(M; \mathbf{Q}) \xrightarrow{\imath^\star} H^{2n}(\partial M; \mathbf{Q}) \xrightarrow{\delta^{2n}} H^{2n+1}(M, \partial M; \mathbf{Q}),$$

where $\imath$ is the inclusion $\partial M \hookrightarrow M$. From Lefschetz duality the connecting homomorphism δ^{2n} has the same matrix representation as $\imath_\star : H_{2n}(\partial M; \mathbf{Q}) \rightarrow H_{2n}(M; \mathbf{Q})$ which is dual to $\imath^\star$. Therefore $\imath^\star$ and δ^{2n} have equal ranks and consequently

$$\dim H^{2n}(\partial M; \mathbf{Q}) = 2 \dim \mathrm{Im}(\imath^\star).$$

Let $\alpha, \beta \in H^{2n}(M; \mathbf{Q})$, then by property 7

$$\imath^\star(\alpha) \cup \imath^\star(\beta) \lfloor \partial M \rfloor = \imath^\star(\alpha \cup \beta) \lfloor \partial M \rfloor = \delta_1((\alpha \cup \beta) \cap \lfloor M \rfloor),$$

where $\delta_1 : H_1(M, \partial M; \mathbf{Q}) \rightarrow H_\circ(\partial M; \mathbf{Q})$ is the connecting homomorphism. We decompose $\partial M = N_1 \cup \cdots \cup N_r$ into its connected components. The map $H_\circ(\partial M; \mathbf{Q}) \rightarrow H_\circ(M; \mathbf{Q})$ is surjective and is given by $\zeta : (a_1, \cdots, a_r) \rightarrow \sum \imath_{j\star}(a_j)$ where $\imath_j : N_j \hookrightarrow M$. Now

$$\imath^\star(\alpha) \cup \imath^\star(\beta) \lfloor \partial M \rfloor \in \mathrm{Im}(\delta_1) = \ker(\zeta).$$

It follows that from the definition of the intersection pairing for ∂M (which has $r \geq 1$ connected components) that $\imath^\star(H^{2n}(M; \mathbf{Q})$ is self-orthogonal relative to $Q_{\partial M}$. Since $\dim H^{2n}(\partial M; \mathbf{Q}) = 2 \dim \mathrm{Im}(\imath^\star)$, and $Q_{\partial M}$ is symmetric, it has same number of positive and negative eigenvalues, and $\sigma(\partial M) = 0$ proving the corollary. ♣

Another useful extension of Poincaré duality is

Theorem 3.3.4 (Alexander Duality) - *Let M be a compact orientable manifold and $A \subset M$ a closed subset*[2] *so that $M \setminus A$ is an open submanifold. Then*

$$H^{m-k}(M, A; R) \simeq H_k(M \setminus A; R).$$

Example 3.3.10 Let X_n be the union of n copies of $\mathbf{CP}(1)$ linearly embedded in $\mathbf{CP}(2)$ and in general position which means thatbevery two copies of $\mathbf{CP}(1)$ intersect at exactly one point and no three copies are concurrent. Let $U_n = \mathbf{CP}(2) \setminus X_n$. In this example we compute the (co)homology of U_n. The long exact sequence for cohomology gives (coefficient group $\mathbf{Z}$ is omitted for simplicity of notation):

$$\cdots \rightarrow H^{l-1}(X_n) \rightarrow H^l(\mathbf{CP}(2), X_n) \rightarrow H^l(\mathbf{CP}(2)) \rightarrow H^l(X_n) \rightarrow H^{l+1}(\mathbf{CP}(2), X_n) \rightarrow \cdots$$

Substituting from exercise 3.3 of chapter 3 and Alexander duality ($H^l(\mathbf{CP}(2), X_n) \simeq H_{4-l}(U_n)$) we obtain:

$$0 \rightarrow \mathbf{Z}^{\frac{1}{2}(n-1)(n-2)} \rightarrow H_2(U_n) \rightarrow H^2(\mathbf{CP}(2)) \rightarrow H^2(X_n) \rightarrow H_1(U_n) \rightarrow H^3(\mathbf{CP}(2)).$$

[2] It is necessary to make some additional assumption about the structure of A or modify the notion of cohomology. It will suffice to assume that A is the union of a finite number of (possibly transversally intersecting) compact submanifolds, or a subcomplex for a triangulation of M (see [S]).

Since each linearly embedded $\mathbf{CP}(1)$ in $\mathbf{CP}(2)$ represents the same generator of $H_2(\mathbf{CP}(2)) \simeq \mathbf{Z}$, the homomorphism $\mathbf{Z} \simeq H^2(\mathbf{CP}(2)) \to H^2(X_n) \simeq \mathbf{Z}^n$ is given by $j \longrightarrow (j, j, \cdots, j)$ and is therefore injective. Hence

$$H_2(U_n; \mathbf{Z}) \simeq \mathbf{Z}^{\frac{1}{2}(n-1)(n-2)}.$$

Since $H^3(\mathbf{CP}(2); \mathbf{Z}) = 0$, we also obtain

$$H_1(U_n; \mathbf{Z}) \simeq H^3(\mathbf{CP}(2), X_n; \mathbf{Z}) \simeq \mathbf{Z}^{n-1}.$$

Clearly $H_\circ(U_n; \mathbf{Z}) \simeq \mathbf{Z}$ and $H_j(U_n; \mathbf{Z}) = 0$ for $j \geq 4$. Setting $l = 1$ in the cohomology long exact sequence we easily obtain $H_3(U_n; \mathbf{Z}) = 0$. ♠

It is not difficult to determine the ring structure of $H^\star(U_n; \mathbf{Z})$ and explicitly exhibit generators for the algebra. To do so it is convenient to make use of the following simple algebraic lemma:

Lemma 3.3.5 *Let V be a free R-module of rank n and $V^\star = \mathrm{Hom}(V, R)$. Let $c_1, \cdots, c_n \in V$ and $\gamma_1, \cdots, \gamma_n \in V^\star$ be such that $\gamma_i(c_j) = \delta_{ij}$. Then both $\{c_i\}$ and $\{\gamma_j\}$ are bases for the respective modules.*

Proof - Let $\{e_i\}$ and $\{\epsilon_i\}$ be dual bases for V and $V^\star$, and set $c_i = \sum c_{ij} e_j$ and $\gamma_i = \sum \gamma_{ji} \epsilon_j$. Then

$$\delta_{ik} = \gamma_k(c_i) = \sum_j c_{ij} \gamma_{jk},$$

proving that the matrices $C = (c_{ij})$ and $\Gamma = (\gamma_{ij})$ are inverses to each other. ♣

Example 3.3.11 We continue with the notation of example 3.3.10. It is convenient to use differential forms to exhibit a set of generators for $H^\star(U_n; \mathbf{Z})$ and understand its ring structure. We may assume that one of the copies of $\mathbf{CP}(1)$ is actually the line at infinity so that U_n is the complement of $n-1$ affine lines, in general position, in $\mathbf{C}^2$ defined by the equations

$$L_j(z) \equiv a_j z_1 + b_j z_2 + c_j = 0, \qquad j = 1, \cdots, n-1.$$

To make things look simpler we make an affine transformation so that $L_1(z) = z_1$ and $L_2(z) = z_2$. Now define

$$\omega_j \;=\; \frac{1}{2\pi i} \frac{dL_j}{L_j}, \qquad j = 1, \cdots, n-1.$$

It is clear that $d\omega_j = 0$. We look at the point of intersection of two affine lines, and so may consider the lines $L_1(z) = 0$ and $L_2(z) = 0$ which intersect near $\mathbf{0}$. Consider the small circles $\gamma_1(t) = (\epsilon_1 e^{2\pi i t}, \rho_1)$ and $\gamma_2(t) = (\rho_2, \epsilon_2 e^{2\pi i t})$ where $\epsilon_j > 0$ is small. Then, for ρ_j's appropriately chosen from an open set of values,

$$\int_{\gamma_k} \omega_j \;=\; \delta_{jk}, \quad \text{for } k = 1, 2 \;\text{ and } j = 1, \cdots, n-1. \tag{3.3.15}$$

It is clear how to define circles γ_k, for $k \geq 3$ such that equation (3.3.15) remains valid for $k = 1, \cdots, n-1$. The circles γ_k are clearly (singular or simplicial) cycles in U_n. From (3.3.15), lemma 3.3.5 and example 3.3.10 ($H_1(U_n; \mathbf{Z}) \simeq \mathbf{Z}^{n-1}$) it then follows that ω_j's and γ_k's form bases for $H^1(U_n; \mathbf{Z})$ and $H_1(U_n; \mathbf{Z})$ respectively. Now consider the 2-forms $\eta_{jk} = \omega_j \wedge \omega_k$ for $1 \leq j < k \leq n-1$ and the 2-torus $T_{12}(t_1, t_2) = (\epsilon e^{2\pi i t_1}, \epsilon e^{2\pi i t_2})$. Now $\int_{T_{12}} \eta_{jk} = \delta_{1j}\delta_{2k}$ for $1 \leq j < k \leq n-1$. By working in the vicinity of the intersections of the lines $L_p = 0$ and $L_q = 0$ we similarly define 2-tori T_{pq} for $1 \leq p < q \leq n-1$ such that

$$\int_{T_{pq}} \eta_{jk} = \delta_{jp}\delta_{kq}, \quad \text{for } 1 \leq j < k \leq n-1, \quad \text{and } 1 \leq p < q \leq n-1. \tag{3.3.16}$$

From (3.3.16), lemma 3.3.5 and example 3.3.10 ($H_2(U_n; \mathbf{Z}) \simeq \mathbf{Z}^{\frac{1}{2}(n-1)(n-2)}$) it then follows that η_{pq}'s and T_{jk}'s form bases for $H^2(U_n; \mathbf{Z})$ and $H_2(U_n; \mathbf{Z})$ respectively. Therefore we have determined the ring structure[3] of $H^\star(U_n; \mathbf{Z})$, viz., as an abstract algebra over $\mathbf{Z}$ it is generated by $\omega_1, \cdots, \omega_{n-1}$ subject to the relations $\omega_j\omega_k + \omega_k\omega_j = 0$ and $\omega_j\omega_k\omega_l = 0$. ♠

As an application of the notion of Kronecker index and the duality theorems, we introduce the concept of linking number. Let M be an oriented manifold of dimension m and P and Q compact connected oriented submanifolds of dimensions $p > 0$ and $q > 0$ respectively. Assume P and Q are in general position and $p + q = m - 1$ so that $P \cap Q = \emptyset$ and $\lfloor P \rfloor \in H_p(M \setminus Q; \mathbf{Z})$ and $\lfloor Q \rfloor \in H_q(M \setminus P; \mathbf{Z})$. We consider the special case that $M = S^m$ or $\mathbf{R}^m$ so that P and Q are boundaries in M, i.e., $\lfloor P \rfloor = 0$ and $\lfloor Q \rfloor = 0$. We want to construct an invariant which shows how linked P and Q are. Intuitively speaking, if P and Q are not linked then we can find a compact oriented manifold P_1 such that $P_1 \cap Q = \emptyset$ and $\partial P_1 = P$ (and/or similarly with the roles of P and Q reversed). Therefore it makes sense to define the *linking number* of P and Q as

$$\mathrm{Lk}(P, Q) = \mathrm{KI}(P_1, Q).$$

This definition appears to be nonsymmetrical with respect to P and Q. The fact $\mathrm{Lk}(P, Q) = \pm\mathrm{Lk}(Q, P)$ is shown below using an interpretation of the linking number as the degree of a certain mapping. Also note that to make sure that the definition of linking number is meaningful, we have to show that the choice of the bounding manifold P_1 is immaterial. Let P_2 be another manifold with $\partial P_2 = P$. Then set $R = P_1 \cup P_2$ and after possibly changing the orientation on P_2 so that the induced orientations on P from P_1 and P_2 cancel out, we see that $\partial R = 0$ so that R is a cycle. Since the ambient manifold is S^m or $\mathbf{R}^m$, $R = \partial S$ and by example **??**, $\mathrm{KI}(R, Q) = 0$, which shows $\mathrm{Lk}(P, Q)$ is well-defined.

Let $M = \mathbf{R}^3$ or S^3 and P and Q be circles. Clearly if P and Q can be separated then $\mathrm{Lk}(P, Q) = 0$. Figure XXXXX shows that the converse is not true.

Let us cast this definition into a more homological framework. Let Q' be a small tubular neighborhood of Q which we assume is disjoint from P and contains Q as a deformation retract. Regarding $\lfloor P \rfloor \in H_p(M \setminus Q'; \mathbf{Z})$, then $\delta_{p+1}^{-1}(\lfloor P \rfloor) \in H_{p+1}(M, M \setminus Q'; \mathbf{Z})$ where δ_{p+1} is the connecting homomorphism. Note that δ_{p+1} is an isomorphism since $H_{p+1}(M; \mathbf{Z}) = 0 = H_p(M; \mathbf{Z})$. By excision $H_{p+1}(M, M \setminus Q'; \mathbf{Z}) \simeq H_{p+1}(Q', \partial Q'; \mathbf{Z})$.

[3]We have not shown that cup product translates into wedge product when cocycles are given as closed differential forms, although this fact should appear plausible in view of the description of cup product in terms of the dual simple complex (see formula 3.3.9). The proof of this fact and more systematic discussion of differential forms and cohomology is postponed.

By Lefschetz duality we have an isomorphism $H_{p+1}(Q', \partial Q'; \mathbf{Z}) \rightarrow H^q(Q'; \mathbf{Z})$, and we denote the composed isomorphism $H_{p+1}(Q', \partial Q'; \mathbf{Z}) \rightarrow H^q(Q'; \mathbf{Z}) \rightarrow H^q(Q; \mathbf{Z})$ by β. Set

$$\mathrm{Lk}(P, Q) = (\beta\delta_{p+1}^{-1}(\lfloor P \rfloor))\lfloor Q \rfloor.$$

It is not difficult to see that the homological definition of the linking number is equivalent, up to sign, to the geometric definition given above. In fact, note that $\delta_{p+1}^{-1}(\lfloor P \rfloor)$ assigns to $\lfloor P \rfloor$ the homology class $\lfloor P_1 \rfloor \in H_{p+1}(M, M \setminus Q'; \mathbf{Z})$. One use of the homological definition is that it is immediate that the linking number is a homotopy invariant.

Example 3.3.12 Let U and V be distinct 1-dimensional linear subspaces of $\mathbf{C}^2$. Set $P = S^3 \cap U$ and $Q = S^3 \cap V$ where S^3 is the unit sphere in $\mathbf{C}^2$. Then P and Q are circles and we want to compute $\mathrm{Lk}(P, Q)$. Since $\mathrm{Lk}(P, Q)$ is invariant under diffeomorphisms of S^3 we may assume $U = \{(z, 0)\}$ and $V = \{(0, z)\}$. It is convenient to project S^3 stereographically onto $\mathbf{R}^3$, i.e., we consider the mapping

$$F(x_1, x_2, x_3, x_4) = \frac{2}{1 - x_4}(x_1, x_2, x_3),$$

which maps the north pole $(0, 0, 0, 1)$ to the point at infinity. Under this map P is mapped to the circle $C = \{(2\cos\theta, 2\sin\theta, 0)\}$ and Q is mapped to the line $L = \{(0, 0, t)\}$. Now C is the boundary of the disc of radius 2 centered at $\mathbf{0}$ in the (x_1, x_2)-plane which intersects L only at $\mathbf{0}$. The intersection being transverse, we obtain $\mathrm{Lk}(P, Q) = \pm 1$. ♠

Recall from the construction of the complex projective space that we have a fibration $p : S^3 \rightarrow \mathbf{CP}(1) = S^2$. This is the ubiquitous *Hopf fibration*. For distinct points a and b in S^2, $P_a = p^{-1}(a)$ and $P_b = p^{-1}(b)$ are circles in S^3, and therefore we may ask what the linking number of P_a and P_b is. It is clear that P_a and P_b are the intersections of two distinct linear subspaces of $\mathbf{C}^2$ with S^3, and we are in the situation of example 3.3.12. Therefore $\mathrm{Lk}(P_a, P_b) = \pm 1$. Because of the importance of Hopf fibration, we record this result by the following corollary:

Corollary 3.3.4 *Let $p : S^3 \rightarrow S^2$ be the Hopf fibration, then*

$$\mathrm{Lk}(p^{-1}(a), p^{-1}(b)) = \pm 1$$

for distinct points $a, b \in S^2$.

Example 3.3.13 We showed earlier that the Poincaré dual to $\lfloor \mathbf{CP}(k) \rfloor$ embedded linearly in $\mathbf{CP}(n)$ as the first $k + 1$ coordinates, is ω^{n-k}, where ω is a generator $H^2(\mathbf{CP}(n); \mathbf{Z})$ which is the Poincaré dual to a copy of $\mathbf{CP}(n - 1)$ linearly embedded in $\mathbf{CP}(n)$. Let $f(z)$ be a homogeneous polynomial of degree d in $z = (z_\circ, \cdots, z_n)$ and Z_f denote the locus of zeros of f regarded as a subset of $\mathbf{CP}(n)$. Then the Poincaré dual to $\lfloor Z_f \rfloor$ is $d'\omega$, and we want to determine d'. Assume Z_f and a copy of $\mathbf{CP}(1)$ linearly embedded, e.g. given by $z_2 = \cdots = z_n = 0$, are in general position, then

$$(\lceil Z_f \rceil \cup \lceil \mathbf{CP}(1) \rceil)\lfloor \mathbf{CP}(n) \rfloor = d'.$$

On the other hand, the intersection of Z_f and $\mathbf{CP}(1)$ is the roots of the equation $f(z) = 0$ when $z_2, \cdots, z_n$ are set equal to 0. Under some genericity condition on f, this equation has d distinct roots. Since the intersections all have positive orientation, $d = d'$. By a similar argument, if we take k homogeneous polynomials $f_1, \cdots, f_k$ of degrees $d_1, \cdots, d_k$, then under some general position assumption, the locus of their zeros $Z_{f_1,\cdots,f_k}$ has Poincaré dual represented by $d_1 \cdots d_k \omega^k$, or equivalently represented by $d_1 \cdots d_k \lceil \mathbf{CP}(n-k) \rceil \in H^k(\mathbf{CP}(n); \mathbf{Z})$. In particular, for $k = n$ we recover the classical theorem of Bezout's to the effect that that number of solutions of n homogeneous equations in $n+1$ unknowns is the product of their degrees provided the zero sets of these equations intersect transversally. Of course solutions are counted as points in the projective space which is the same thing as counting the number of solutions of n equations in n unknowns and taking the solutions at infinity into account. Bezout's theorem is also valid for nontransverse intersections where there are multiple roots, provided mutiplicities are taken into account. We shall not go into a discussion this matter here. ♠

Exercise 3.3.25 *What is the Poincaré dual of* $\lfloor N \rfloor$ *where* N *is the image of* $\mathbf{CP}(1)$ *in* $\mathbf{CP}(n)$ *under the embedding*

$$[z_\circ, z_1] \longrightarrow [z_\circ^n, z_\circ^{n-1} z_1, \cdots, z_\circ z_1^{n-1}, z_1^n]?$$

Exercise 3.3.26 *Let* f *be a homogeneous polynomial of degree* r *in* $n+1$ *variables, and assume that* $Z_f \subset \mathbf{CP}(n)$ *is a complex manifold. Let* L *be a copy of* $\mathbf{CP}(n-1)$ *linearly embedded in* $\mathbf{CP}(n)$ *and* $\zeta \in \mathbf{CP}(n) \setminus (Z_f \cup L)$. *Consider the mapping* $g_\zeta : Z_f \to L$ *which assigns to* $z \in Z_f$ *the unique intersection of the line (i.e. copy of* $\mathbf{CP}(1)$ *linearly embedded in* $\mathbf{CP}(n)$*) through* z *and* ζ *with* L. *Compute the degree of* g_ζ.

We close this section by briefly indicating the standard proofs of the duality theorems.

Proof of Alexander Duality - It is convenient to break up the proof into a sequence of special cases:

1. Let U be the closed unit disc in $\mathbf{R}^m$ and A a simplex contained in the interior $\breve{U}$ of U. Then we have isomorphisms:

$$H^k(U, U \setminus A; R) \simeq H_{m-k}(U; R) \simeq H_{m-k}(A),$$

given by taking cap product with a generator of $H_m(U, U \setminus A; R)$. In fact recall that $H^k(U, U \setminus A; R)$ vanishes unless $k = m$ in which case $H^k(U, U \setminus A; R) \simeq R$. It is clear that the evaluation on a generator $c \in H_m(U, U \setminus A; R)$, i.e., cap product with c, gives the desired isomorphism.

2. With notation and hypothesis of part (1), assume U is contained in a coordinate neighborhood of M. Then by excision, the inclusion $(U, U \setminus A) \to (M, M \setminus A)$ induces isomorphisms on (co)homology, and consequently $H^k(M, M \setminus A; R) \simeq H_{m-k}(A)$. In view of the naturality property of cap product (property 6) the isomorphism is given by taking cap product with a generator of $H_m(M, M \setminus A; R)$.

3. We show by induction on the number of simplices of A that the isomorphism is valid for a general complex A. We have already established the required result if A is a single simplex. Decompose $A = A' \cup A''$ as the union of two nonempty subcomplexes. By induction hypothesis the result is true for A', A'', $A' \cap A''$ ($A' \cap A''$ maybe empty). Substituting $U = M = V$, $U' = M \setminus A'$ and $V' = M \setminus A''$ in the Mayer-Vietoris sequence and its analogue for cohomology we obtain the commutative row exact diagram (the coefficient group R is omitted for simplicity of notation):

$$\begin{array}{ccccc}
H^k(M, M\setminus A')\oplus H^k(M, M\setminus A'') & \rightarrow & H^k(M, M\setminus A) & \rightarrow & H^{k+1}(M, M\setminus (A'\cap A'')) \\
\downarrow & & \downarrow & & \downarrow \\
H_{m-k}(A')\oplus H_{m-k}(A'') & \rightarrow & H_{m-k}(A) & \rightarrow & H_{m-k-1}(A'\cap A'')
\end{array}$$

Notice that the first vertical arrow is the isomorphism given by the induction hypothesis and is given by cap product with the generators of $H_m(M, M\setminus A')$ and $H_m(M, M\setminus A'')$, and similarly for the remaining vertical arrows. The commutativity of the diagram follows from the construction of the orientation class and the naturality property of cap products. We are now in a position to apply the Five Lemma (chapter 3, §5) by incorporating in the above diagram the isomorphisms $H^k(M, M\setminus (A'\cap A''))\rightarrow H_{m-k}(A'\cap A'')$ and $H^{k+1}(M, M\setminus A')\oplus H^{k+1}(M, M\setminus A'')\rightarrow H_{m-k-1}(A')\oplus H_{m-k-1}(A'')$, to obtain the desired isomorphism of the middle vertical arrow $H^k(M, M\setminus A)\rightarrow H_{m-k}(A)$. This completes the proof of the Alexander Duality. ♣

Proof of Lefschetz duality - In the proof of Alexander duality let A be such that $M\setminus A$ is a small neighborhood of ∂M homeomorphic to $\partial M\times[0,\epsilon)$ and $A\simeq M$. Then we obtain the isomorphism

$$H^k(M, \partial M)\xrightarrow{\simeq} H_{m-k}(M).$$

(Recall that in the statement of theorem 3.3.3, $\dim(M) = m+1$ rather than m.) The isomorphism $H^k(M)\simeq H_{m-k}(M, \partial M)$ can be proven by an (inductive) argument similar to the proof of Alexander duality. The naturality of cap product and the fact that the image of a generator of $H_m(M, \partial M)$ in $H_{m-1}(\partial M)$ is an orientation class for ∂M complete the proof of Lefschetz duality. ♣

3.4 Applications

3.4.1 Graph of a Function and Lefschetz Fixed Point Theorem

Let N be a closed oriented submanifold of the compact oriented manifold M. Denote the image of the orientation class $\lfloor N \rfloor$ in $H_n(M;\mathbf{Z})$ by $\lfloor N \rfloor$ and the Poincaré dual to $\lfloor N \rfloor$ by $\lceil N \rceil \in H^{m-n}(M;\mathbf{Z})$. Notice that $\lceil N \rceil$ is characterized by the property that for all n-cocycles α we have

$$(\alpha \cup \lceil N \rceil) \cap \lfloor M \rfloor = \alpha \cap \lfloor N \rfloor.$$

An important case is when N is the graph Γ_f of a mapping $f : M \to M'$ regarded as a submanifold of $M \times M'$:

$$\Gamma_f = \{(x, f(x)) | x \in M\}.$$

We now proceed to calculate the Poincaré dual to $\lfloor \Gamma_f \rfloor$ and use it to significantly improve the version of the Lefschetz fixed point theorem which was proved in chapter 3, §2. Let the coefficient ring be $\mathbf{R}$ and $\{\alpha_j^r\}$ and $\{\theta_j^s\}$ be basis for $H^r(M;\mathbf{R})$ and $H^s(M';\mathbf{R})$ respectively. Let $\{\beta_j^{m-r}\}$ and $\{\varphi_j^{m'-s}\}$ be the dual basis for $H^{m-r}(M;\mathbf{R})$ and $H^{m'-s}(M';\mathbf{R})$ respectively, i.e.,

$$(\alpha_j^r \cup \beta_k^s)\lfloor M \rfloor = \delta_{m-s}^r \delta_k^j, \quad (\theta_j^r \cup \varphi_k^s)\lfloor M' \rfloor = \delta_{m'-s}^r \delta_k^j.$$

We know that a basis for $H^n(M \times M';\mathbf{R})$ is $\{\alpha_j^r \otimes \varphi_k^s\}_{r+s=n}$. Since Γ_f is an m-dimensional submanifold of $M \times M'$, $\lceil \Gamma_f \rceil \in H^{m'}(M \times M';\mathbf{R})$ and we have

$$\lceil \Gamma_f \rceil = \sum_{r,j,k} c_{jk}^r \alpha_j^r \otimes \varphi_k^{m'-r},$$

where the coefficients c_{jk}^r are to be determined. To do so we evaluate $(\beta_j^{m-r} \otimes \theta_k^r)\lfloor \Gamma_f \rfloor$ in two different ways. First we use the embedding $\jmath$ of M onto its graph $\jmath(x) = (x, f(x))$ and compute on M:

$$(\beta_j^{m-r} \otimes \theta_k^r)\lfloor \Gamma_f \rfloor = \jmath^\star(\beta_j^{m-r} \otimes \theta_k^r)\lfloor M \rfloor = (-1)^{r(m-r)} f_{jk}^r, \tag{3.4.1}$$

where (f_{jk}^n) is the matrix of the map induced by f on the n^{th} cohomology relative to the bases $\{\theta_j^n\}$ and $\{\alpha_j^n\}$. (Recall that $\beta_j^{m-r} \otimes \theta_k^r$ really means $\beta_j^{m-r} \otimes 1 \cup 1 \otimes \theta_k^r$ so that $\jmath^\star(\beta_j^{m-r} \otimes \theta_k^r) = \beta_j^{m-r} \cup f^\star(\theta_k^r)$.) On the other hand, computing on $M \times M'$ and using the definition of the Poincaré dual we get

$$\begin{aligned}(\beta_j^{m-r} \otimes \theta_k^r)\lfloor \Gamma_f \rfloor &= (\beta_j^{m-r} \otimes \theta_k^r) \cup (\textstyle\sum c_{lp}^s \alpha_l^s \otimes \varphi_p^{m'-s})\lfloor M \times M' \rfloor \\ &= \textstyle\sum (-1)^{rs} c_{lp}^s (\beta_j^{m-r} \cup \alpha_l^s)\lfloor M \rfloor (\theta_k^r \cup \varphi_p^{m'-s})\lfloor M' \rfloor \\ &= (-1)^{mr} c_{jk}^r.\end{aligned} \tag{3.4.2}$$

From (3.4.1) and (3.4.2) we obtain

$$c_{kj}^r = (-1)^r f_{jk}^r. \tag{3.4.3}$$

For $M = M'$, we may take $\alpha_j^r = \theta_j^r$ and $\beta_j^r = \varphi_j^r$ and we obtain

$$\lceil \Gamma_f \rceil = \sum (-1)^r f_{jk}^r \alpha_j^r \otimes \beta_k^{m-r}. \tag{3.4.4}$$

In particular, we have the following expression for the Poincaré dual of the diagonal (or the identity map):

$$\lceil \Gamma_{\text{id.}} \rceil = \sum (-1)^{rm} \beta_j^{m-r} \otimes \alpha_j^r. \tag{3.4.5}$$

Formulae (3.4.4) and (3.4.5) imply one of the fundamental achievements of classical algebraic topology, namely, the Lefschetz Fixed Point theorem. To derive this result, recall that cup product is the algebraization of the notion of intersection where the orientation on the normal bundles is taken into account. The set theoretic intersection $\Gamma_f \cap \Gamma_{\text{id.}}$ is naturally identified with the fixed point set of f. Hence $(\lceil \Gamma_f \rceil \cup \lceil \Gamma_{\text{id.}} \rceil)\lfloor M \times M \rfloor$, for f transversal to the diagonal counts the number of fixed points of f with each fixed point counted with multiplicity ± 1 according as $\det(I - Df(x))$ is positive or negative. Fixed points x such that $\det(I - Df(x)) \neq 0$ are called *nondegenerate*. It is convenient to call the number of fixed points counted in this manner the *algebraic sum of the fixed points.*

Theorem 3.4.1 (Lefschetz Fixed Point Theorem) *Let M be a compact orientable manifold and $f : M \rightarrow M$ a continuously differentiable map. Assume all the fixed points of f are nondegenerate. Then the alternating sum of the traces of the induced maps on (co)homology with* **R** *coefficients is equal to the algebraic sum of the fixed points.*

Proof - From the expressions 3.4.4 and 3.4.5 we obtain

$$(\lceil \Gamma_f \rceil \cup \lceil \Gamma_{\text{id.}} \rceil)\lfloor M \times M \rfloor = \sum (-1)^r f_{jj}^r \tag{3.4.6}$$

which is the alternating sum of the traces of the induced maps on cohomology. ♣

Remark 3.4.1 Although the hypothesis of theorem 5.1 can be relaxed in a number of ways, the assumption of continuous differntiability cannot be replaced with continuity since the sign of an intersection requires differentiability (see also exercise 5.3 below). Instead of a mapping of M to itself one may consider a *correspondence* N which is an m-dimensional submanifold of $M \times M$. Then $(\lceil N \rceil \cup \Gamma_{\text{id.}})\lfloor M \times M \rfloor$ is the algebraic sum of fixed points of the correspondence N (assuming some nondegeneracy condition). This generalization has important consequences in algebraic geometry and number theory. ♡

Exercise 3.4.1 *Let $f : S^m \rightarrow S^m$ be a C^1 map of degree d. Assume that fixed points of f^n are isolated for every n and $|d| > 1$. Show that f has infinitely many periodic points. Construct a continuous mapping of $f : S^2 \rightarrow S^2$ of degree $d \geq 2$ with only two periodic points and deduce that the hypothesis of continuous differentiability in theorem 5.1 cannot be replaced with that of continuity.*

Substituting $f = \text{id.}$ in (3.4.6) we obtain

$$(\lceil \Gamma_{\text{id.}} \rceil \cup \lceil \Gamma_{\text{id.}} \rceil)\lfloor M \times M \rfloor = \chi(M). \tag{3.4.7}$$

To understand this formula, let $f : M \to M$ be a perturbation of the identity map of M such that f is homotopic to the identity map and Γ_f is transverse to the diagonal. For example, we may take $f = \phi_t$ where ϕ_t is the one parameter group of a vector field ξ with simple isolated zeros and $t > 0$ is fixed and small. In view of the discussion of the singularities of vector fields in chapter 1 we also recover the Poincaré-Hopf theorem for compact orientable manifolds from the Lefschetz Fixed Point theorem.

Corollary 3.4.1 (Poincaré-Hopf) - *Let M be a compact orientable manifold and ξ a vector field on M with isolated zeros. Then*

$$\mathrm{Ind}(\xi) = \chi(M).$$

The problem of the existence of a vector field with given singularities satisfying $\mathrm{Ind}(\xi) = \chi(M)$ requires techniques we have not yet introduced.

3.4.2 Degree of a Map

Let $f : M \to N$ be a continuously differentiable map of compact connected orientable manifolds of the same dimension m, and assume that Df is generically (i.e., in this case, on a connected open dense subset) of rank m. We assume the map f is generically n to 1, and we want to give a homological interpretation to the number n. In $M \times N$ we consider Γ_f and $M \times \{y\}$. It is clear geometrically the number n is the intersection number of Γ_f and and $M \times \{y\}$, for generic y, provided we take the intersections with proper sign of the normal bundle as explained in chapter 1. So we have to compute $(\lceil \Gamma_f \rceil \cup \lceil M \times \{y\} \rceil)\lfloor M \times N \rfloor$. Let $\theta^m \in H^m(N; \mathbf{Z})$ be a generator such that $\theta^m(\lfloor N \rfloor) = 1$, and let $\beta^\circ \in H^\circ(M; \mathbf{Z})$ be the Poincaré dual to $\lfloor M \rfloor$. Therefore the Poincaré dual of $\lfloor M \times \{y\} \rfloor$ is $\beta^\circ \otimes \theta^m$. Using (3.4.3) ($N$ replacing the manifold M', α's, β's etc. are bases for the free part of cohomology with $\mathbf{Z}$ coefficients) we obtain

$$\begin{aligned}(\lceil \Gamma_f \rceil \cup \lceil M \times \{y\} \rceil)\lfloor M \times N \rfloor &= (\textstyle\sum (-1)^r f^r_{kj} \alpha^r_j \otimes \varphi^{m-r}_k) \cup (\beta^\circ \otimes \theta^m)\lfloor M \times N \rfloor \\ &= (-1)^m f^m (\alpha^m \otimes \theta^m)\lfloor M \times N \rfloor \\ &= (-1)^m f^m.\end{aligned}$$

Note that since $H^m(M; \mathbf{Z}) \simeq \mathbf{Z} \simeq H^m(N; \mathbf{Z})$, the map induced by f is simply multiplication by the integer f^m. Therefore the integer f^m is the homological interpretation of the number n which is the cardinality of a generic fibre of f provided all the intersections have the same sign. In general there may be cancellations due to the sign of the intersections. If $Df(x)$ is orientation preserving for all $x \in M$, for example if f is a holomorphic map of complex manifolds, then f^m is actually equal to n since all the intersections have positive sign in this case. The integer f^m whose sign depends on the choice of orientations for M and N, is called the *degree* of f and is denoted by $\deg(f)$. Clearly $\deg(f)$ depends only on the homotopy class of f and the choice of the orientations. The definition of degree as the induced map on the top (co)homology requires only continuity of the map f. If M has more than one connected component, then $\deg(f)$ is the sum of degrees of f restricted to the connected components.

Exercise 3.4.2 *Show that the map $e^{i\theta} \to e^{in\theta}$ has degree n. Construct a map of degree n of S^m to itself. Construct a vector field ξ with an isolated zero at $\mathbf{0} \in \mathbf{R}^m$ and* $\mathrm{Ind}(\xi, 0) = n$.

As another application of the concept of degree we have the following:

Proposition 3.4.1 *Let M and N be compact orientable manifolds with $\partial M \neq \emptyset = \partial N$, $\dim \partial M = \dim N = m-1$ and $f : \partial M \to N$ a continuous map. If f extends to a map of $M \to N$, then* $\deg(f) = 0$.

Proof - If f extends then we have the commutative diagram

$$\begin{tikzcd} \longrightarrow H_m(M, \partial M; \mathbf{Z}) \arrow[r] & H_{m-1}(\partial M; \mathbf{Z}) \arrow[rr] \arrow[dr] & & H_{m-1}(M; \mathbf{Z}) \longrightarrow \arrow[dl] \\ & & H_{m-1}(N; \mathbf{Z}) & \end{tikzcd}$$

Since $\lfloor \partial M \rfloor$ is the image of a generator of $H_m(M, \partial M; \mathbf{Z})$ in $H_{m-1}(\partial M; \mathbf{Z})$ under the connecting homomorphism (see example 3.3.3), the image of $\lfloor \partial M \rfloor$ in $H_{m-1}(M; \mathbf{Z})$ vanishes. The commutativity and row exactness of the diagram then implies $f_\star(\lfloor \partial M \rfloor) = 0$ and $\deg(f) = 0$.

Example 3.4.1 Let ξ be a vector field defined in a neighborhood U of $\mathbf{0}$ in $\mathbf{R}^n$, and assume $\mathbf{0}$ is an isolated singularity of ξ. In chapter 1 we showed that one may define $\mathrm{Ind}(\xi, \mathbf{0})$ by looking at the map $g(x) = \frac{\xi_x}{\|\xi_x\|}$ which is defined for $x \in U \setminus \{0\}$. Let S_ϵ be a small sphere of radius ϵ centered at $\mathbf{0}$, then by restricting g to S_ϵ we obtain a map $g' : S_\epsilon \to S^{n-1}$. Endowing S_ϵ and S^{n-1} with compatible orientations, we show that

$$\mathrm{Ind}(\xi, \mathbf{0}) = \deg(g').$$

In §2 we interpreted $\mathrm{Ind}(\xi, \mathbf{0})$ as an intersection if the singularity of ξ at $\mathbf{0}$ were simple. The proof involved showing that the map g' is a diffeomorphism with $\mathrm{Ind}(\xi, \mathbf{0}) = \pm 1$ according as g' is orientation preserving or reversing. The proof follows in this case immediately. For the general case we perturb ξ in the interior D_ϵ of S_ϵ while keeping its values on S_ϵ fixed. We may then assume that the perturbed vector field η has r simple zeros $x_1, \cdots, x_r$ in D_ϵ. Let D_i be a small open ball lying in D_ϵ and centered at x_i. Assume that D_i's are disjoint. Then $M = D_\epsilon \setminus (\cup D_i)$ is a manifold with boundary $\partial M = S_\epsilon \cup \partial D_1 \cup \cdots \cup \partial D_r$, and define $h : M \to S^{n-1}$ by $h(x) = \frac{\eta_x}{\|\eta\|}$. Clearly the degree of g' is equal to the sum of the degrees of h restricted to ∂D_i's. Since $\mathrm{Ind}(\xi, \mathbf{0}) = \sum \mathrm{Ind}(\eta, x_i)$, we obtain the desired result from the case of a simple zero. ♠

Example 3.4.2 Let $U \subset \mathbf{R}^n$ be an open relatively compact subset with C^1 boundary ∂U, $x \notin \partial U$, and S^{n-1} the unit sphere. Consider the mapping $g = g_x : \partial U \to S^{n-1}$ defined by $g(y) = \frac{y-x}{\|y-x\|}$. If $x \notin U$ then the map g extends to the interior of U and by proposition 3.4.1 $\deg(g) = 0$. If $x \in U$, let B be a small ball centered at x with $\bar{B} \subset U$. Set $U' = U \setminus B$, and let g' be the extension of g to $\partial U'$. Then g' extends to U', and consequently $\deg(g') = 0$, and since $\deg(g'_{|\partial B}) = \pm 1$, the same is true of $\deg(g)$. For $z \in S^{n-1}$, $g^{-1}(z)$ is the intersection of the ray L through x in the direction of z with ∂U. From the above considerations it is clear that $\deg(g) = \sum \pm 1$ where summation is over all intersections $L_{x,y}$ and ∂U for a generic y and the sign of intersection is taken into account. It follows that $x \in U$ (resp. $x \notin U$) according as L_y intersects ∂U in an odd (resp. even) number of points. In practical problems of computer graphics one encounters situations where it is necessary to devise an algorithm for deciding whether a point is in the inside or the outside of a given bounded region. This can be accomplished by counting the number of intersections of a generic ray with the boundary of the region. ♠

Example 3.4.3 Another implication of the idea of looking at the intersection number of the ray L_y and ∂U is the differentiable version of the *Jordan-Brouwer separation* theorem. Let M be a compact hypersurface in

$\mathbf{R}^{m+1}$ which we assume to be at least of class C^1 and does not contain any straight line segment. (The latter assumption is not necessary, is made for convenience and can be easily removed since by a small perturbation of M the requirement will be fulfilled.) We want to show that M separates $\mathbf{R}^{m+1}$ into regions which we call the *interior* and *exterior* of M. As in example 3.4.2 for $x \in \mathbf{R}^{m+1} \setminus M$, define

$$g_x : M \longrightarrow S^m, \quad g_x(y) = \frac{y-x}{||y-x||}.$$

Then the degree of g_x is the algebraic intersection number of $L_{x,y}$ and M for a generic $y \in M$, where $L_{x,y}$ is the ray emanating from x and passing through y. Now set

$$U_1 = \{x \notin M | \deg(g_x) \text{ is odd}\}, \quad U_2 = \{x \notin M | \deg(g_x) \text{ is even}\}.$$

It is easy to see that $\deg(g_x)$ is a continuous function of $x \in \mathbf{R}^{m+1} \setminus M$. Therefore to show that the complement of M consists of two disjoint open sets it suffices to show that both U_1 and U_2 are nonempty. Since M is compact, a line intersects it only in finitely many points. By looking at perturbations of a tangent line to M, one easily concludes that an open set of lines in $\mathbf{R}^{m+1}$ have more than one point of intersection with M. Let L be a line intersecting M at $y_1, y_2, \cdots$ and $x \in L$ be such that all the points y_j lie on the same side of x in L. Starting at $x \in L$ and moving along L we may assume we arrive at the y_j's in the order $y_1, y_2, \cdots$ Let $x' \in L$ be a point lying between y_1 and y_2. Consider the rays L_{x,y_2} and L_{x',y_2} which start at x and x' and move in the same direction. Clearly L_{x',y_2} intersects M at $y_2, \cdots$ which proves that both U_1 and U_2 are nonempty. Therefore $\mathbf{R}^{m+1} \backslash M$ has at least two connected components. To prove that $\mathbf{R}^{m+1} \backslash M$ has exactly two connected components, it is convenient to add the point at infinity and replace $\mathbf{R}^{m+1}$ by S^{m+1}. The long exact sequence for cohomology yields

$$\rightarrow \mathbf{Z} \rightarrow H^{m+1}(S^{m+1}, M; \mathbf{Z}) \rightarrow \mathbf{Z} \rightarrow 0,$$

proving that free nrank of $H^{m+1}(S^{m+1}, M; \mathbf{Z})$ is at most 2. By Alexander duality $H^{m+1}(S^{m+1}, M; \mathbf{Z}) \simeq H_\circ(S^{m+1} \backslash M; \mathbf{Z}) \simeq \mathbf{Z}^2$. That, is $S^{m+1} \backslash M$ consists of exactly two connected components. The standard version of the Jordan-Brouwer separation theorem asserts that any continuous injective map of S^m into S^{m+1} disconnects S^{m+1}. Except for the fact that we have replaced continuity by the requirement of continuous differentiability, the above argument implies the standard version of the theorem. Without assuming some smoothness, the geometric concept of intersection becomes problematic. For a proof of the standard version of Jordan-Brouwer separation theorem see [S]. ♠

3.4.3 Order of a Zero, Linking Number and Degree

Let N be a compact orientable manifold with boundary $\partial N \neq \emptyset$, and $\dim N = n$. Let $h : N \rightarrow \mathbf{R}^n$ be a continuously differentiable map and assume that $Z_h = \{x | h(x) = \mathbf{0}\}$ is a finite subset of N with $Z_h \cap \partial N = \emptyset$. We now extend the notion of the index of a vector field at a singular point to that of the order zero of h. The proofs are almost identical with the case of a vector field. Just as in the case of a vector field we consider the mapping $g(x) = \frac{h(x)}{\|h(x)\|}$ defined for $x \in N \setminus Z_h$ and define the order of zero of h at $x_\circ$ by

$$\mathrm{Ord}(h, x_\circ) = \frac{1}{c_{n-1}} \int_{\partial D} g^\star(\omega),$$

where ω is the volume element on S^{n-1}, c_{n-1} is the volume of S^{n-1} and D is a small closed disc centered at $x_\circ$ with the property $D \cap Z_h = \{x_\circ\}$. It is straighforward to verify that $\mathrm{Ord}(h, x_\circ)$ is an integer independent of the choice of the small disc, but its sign depends on the choice of the orientation. We also have

$$\mathrm{Ord}(h, x_\circ) = \deg(g_{|\partial D}).$$

Naturally we say h has a *simple* zero at $x_\circ$ if $\mathrm{Ord}(h, x_\circ) = \pm 1$. The sign $\pm$ reflects the fact that h is orientation preserving or reversing near $x_\circ$. If $h(x) \neq \mathbf{0}$ for $x \in \partial N$, then applying Stokes' theorem as before we obtain

$$\deg(g_{|\partial N}) = \sum \mathrm{Ord}(h, z), \tag{3.4.8}$$

where summation is over all $z \in Z$ which we assume to be finite. We now use this formula to relate the concepts of the linking number and degree.

Let P_1 and Q be compact orientable manifolds with $\partial P_1 = P \neq \emptyset$, and denote the dimensions of P and Q by p and q respectively. Assume that we have C^1 embeddings $f : P_1 \to \mathbf{R}^m$ and $f' : Q \to \mathbf{R}^m$ where $m = p + q + 1$. Then we have a mapping $h : N = P_1 \times Q \to \mathbf{R}^m$

$$h(x, y) = f'(y) - f(x).$$

We assume that f and f' are transverse so that $h_{|\partial N}$ does not vanish, the function h has only finitely many zeros and they are all simple. For (x, y) with $h(x, y) \neq \mathbf{0}$, we set $g(x, y) = \frac{h(x,y)}{\|h(x,y)\|}$. It is trivial that

$$\sum \mathrm{Ord}(h, (x, y)) = (-1)^{p+1}\mathrm{KI}(f'(Q), f(P_1)) = (-1)^q \mathrm{Lk}(f(P), f'(Q)),$$

where summation is over all zeros of h. Therefore, in view of (3.4.8), the linking number $\mathrm{Lk}(f(P), f'(Q))$ is $(-1)^q$ times the degree of the mapping

$$g_{|P\times Q} : (x, y) \longrightarrow \frac{f'(y) - f(x)}{\|f'(y) - f(x)\|}.$$

It follows from this formula that changing the roles of P and Q has the effect of changing the degree of the mapping $g_{|P\times Q}$ by $(-1)^m$. Therefore we have the symmetry condition

$$\mathrm{Lk}(f(P), f'(Q)) = (-1)^{p+q+1}\mathrm{Lk}(f'(Q), f(P)).$$

As a further application of these ideas we have the following:

Exercise 3.4.3 *Let D_R denote the disc of radius R centered at $0 \in \mathbf{C}$. Consider the mapping $f_t : D_R \to \mathbf{C}$ defined by $f_t(z) = z^n + tp(z)$, where p is a polynomial of degree $n-1$ and $0 \leq t \leq 1$. For R sufficiently large, $f_t(z) \neq 0$ for all $z \in \partial D_R$ and $0 \leq t \leq 1$. Let $g_t(z) = \frac{f_t(z)}{|f_t(z)|}$ for $z \in \partial D_R$. Show that the degree of g_t is n, and deduce the fundamental theorem of algebra.*

More generally,

Exercise 3.4.4 *Let $U \subset \mathbf{C}^m$ be an open relatively compact subset with C^1 boundary ∂U and $f, g : U \to \mathbf{C}^m$ holomorphic maps having continuous extensions to the boundary. Assume that f does not vanish on the boundary, has only isolated zeros in U and $\|f(z\| > \|g(z)\|$ for $z \in \partial U$. Then f and $f+g$ have the same number of zeros in U.* (This is of course Rouché's theorem. Note that the condition of holomorphy is used only in making sure that the zeros appear with the same sign. A similar result holds for not necessarily holomorphic maps but there may be many cancellations due to the differences in signs.)

3.4.4 Maximal Tori

Let G be an analytic group and T be a maximal torus. Denote the Lie algebras of G and T by $\mathcal{G}$ and $\mathcal{T}$ respectively, and set $\mathcal{G} = \mathcal{T} \oplus \mathcal{M}$ where $\mathcal{M}$ is the the orthogonal complement of $\mathcal{T}$ in $\mathcal{G}$ relative to an Ad-invariant inner product. Let $\xi \in \mathcal{T}$ be a generic element, so that all the eigenvalues of $\mathrm{ad}(\xi)$ on $\mathcal{M}$ are purely imaginary and non-zero. From example **??** we see that the vector field ξ' induced by ξ on G/T has only simple singularities and their number is precisely the order of $N(T)/T$. Therefore

$$\chi(G/T) = \mathrm{Ind}(\xi') = |N(T)/T|.$$

($|X|$ denotes the cardinality of the set X.) Furthermore,

Proposition 3.4.2 *Let G be a compact analytic group and K a closed subgroup such that $\chi(G/K) \neq 0$. Then every element of G is conjugate to element of K. In particular, all maximal tori are conjugate.*

Proof - The first assertion of the proposition is equivalent to the existence of a fixed point for the transformation $L_g : uK \to guK$ of $M = G/K$. Since g can be connectd to the identity, L_g is homotopic to the identity map and therefore its Lefschetz number is the Euler characteristic which is nonzero by assumption. This implies the first assertion. The second assertion follows from the first, non-vanishing of $\chi(G/T)$ and the fact that a torus can be generated by a single element. ♣

Exercise 3.4.5 *Let $G = U(n)$* or *$SU(n)$ and T be the subgroup of diagonal matrices in G. Show that $N(T)/T$ is isomorphic to the symmetric group on n letters.* (G/T is of course the flag manifold $\mathbf{F}_n$)

For other compact classical groups we have:

1. $G = SO(2n+1)$, T maximal torus, $\chi(G/T) = 2^n n!$ for $n \geq 1$.
2. $G = Sp(n)$, T maximal torus, $\chi(G/T) = 2^n n!$ for $n \geq 1$. ($Sp(1) \simeq SU(2)$.)
3. $G = SO(2n)$, T maximal torus, $\chi(G/T) = 2^{n-1} n!$ for $n \geq 2$. ($SO(4) \simeq SO(3) \times SU(2)$.)

In case $G = SO(2n+1)$ or $Sp(n)$, the group $N(T)/T$ is isomorphic to the semi-direct product of $(\mathbf{Z}/2)^n.\mathcal{S}_n$ with the action of $\mathcal{S}_n$ on $(\mathbf{Z}/2)^n$ given by

$$\sigma(i_1, \cdots, i_n) = (i_{\sigma(1)}, \cdots, i_{\sigma(n)}), \qquad i_k \in \mathbf{Z}/2.$$

In case $G = SO(2n)$, $N(T)/T$ is the subgroup $\{(i_1, \cdots, i_n), \sigma) \in (\mathbf{Z}/2)^n.\mathcal{S}_n | \sum_k i_k \equiv 0\}$ of $(\mathbf{Z}/2)^n.\mathcal{S}_n$. The above can be established using linear algebra. The case of the exceptional groups is more complex. At any rate the answers are

$$\chi(E_6/T) = 2^7.3^4.5, \ \chi(E_7/T) = 2^{10}.3^4.5.7, \ \chi(E_8/T) = 2^{14}.3^5.5^2.7, \ \chi(F_4/T) = 2^7.3^2, \ \chi(G_2/T) = 12.$$

3.4.5 Zeta Functions

An important feature of the Lefschetx Fixed Point theorem is that it reduces counting fixed points to the computation of traces on (co)homology groups. Thus it linearizes the problem and makes it more tractable. The following calculation demonstrates the value of this fact: Let $f : M \rightarrow M$ be a smooth transformation of a compact orientable manifold of dimension m, and denote by $f^{\circ k}$ the composition of f with itself k times. Information about the periodic points of a diffeomorphism is of interest in Dynamical Systems. The set of periodic points of the transformation f is identical with the set of fixed points of the maps $f^{\circ k}$ as k ranges over the positive integers. Let N_k be the number of periodic points of period k (or fixed points of $f^{\circ k}$), then one defines a generating function for the number of periodic points (following Weil and Artin-Mazur) by

$$\zeta_f(t) = \exp(\sum_{k=0}^{\infty} N_k \frac{t^k}{k}).$$

We want to investigate some of the implications of the Lefschetz fixed point theorem for this generating function. Assume all fixed points of $f^{\circ k}$ occur with positive sign (e.g., f is a complex analytic mapping of complex manifolds), and are nondegenerate. Then the numbers N_k are the alternating sums of the traces of the linear maps induced by $f^{\circ k}$ on (co)homology. Denoting the induced map $H^i(f)$ on cohomology by F_i we obtain

$$\zeta_f(t) = \exp(\sum_{i=0}^{m}(-1)^i \sum_{k=1}^{\infty} \mathrm{Tr}(F_i^k)\frac{t^k}{k}).$$

Since for a scalar λ we have $\sum_{k=1}^{\infty} \lambda^k \frac{t^k}{k} = -\log(1 - t\lambda)$, we obtain ($d_i = \dim H^i(M; \mathbf{R})$)

$$\exp(\sum_{k=1}^{\infty} \mathrm{Tr}(F_i^k)\frac{t^k}{k}) = \prod_{j=1}^{d_i} \frac{1}{1 - t\lambda_{ij}} = \frac{1}{\det(I - tF_i)},$$

where λ_{ij}'s are the eigenvalues (not necessarily distinct) of F_i. Hence, setting, $Q_i(t) = \det(I - tF_i)$, we obtain

$$\zeta_f(t) = \frac{Q_1(t)Q_3(t)\cdots}{Q_\circ(t)Q_2(t)\cdots}.$$

This means that we have an expression for the generating function of the number of periodic points as a rational function. The essential point in this calculation was the reduction of the problem to one about traces of linear transformations via the Lefschetz fixed point theorem.

Exercise 3.4.6 *Let $T : \mathbf{R}^m \rightarrow \mathbf{R}^m$ be a nonsingular linear transformation represented by an integral matrix relative to the standard basis. Then T induces a transformation $\tau : \mathbf{R}^m/\mathbf{Z}^m \rightarrow \mathbf{R}^m/\mathbf{Z}^m$ of the torus. Assume that 1 is not an eigenvalue of T. Show that all the fixed points of τ are nondegenerate and occur with the same sign. Deduce that the number of fixed points of τ is $|\det(I - T)|$.*

Exercise 3.4.7 *With the notation and the hypotheses of exercise 3.4.6, assume furthermore that all fixed points of $\tau^{\circ k}$ have positive sign, and T has no eigenvalue of absolute value 1. Show that the generating function for the number of periodic points of τ (see remark* **??** *above) is*

$$\zeta_\tau(t) \;=\; \frac{\det(I-tT)\det(I-t\wedge^3 T)\cdots}{(1-t)\det(I-t\wedge^2 T)\det(I-t\wedge^4 T)\cdots}.$$

Deduce that T has infinitely many periodic points (x is a periodic point of τ if there is n such that $\tau^n(x)=x$).

The zeta function ζ_f has its roots in number theory where it was originally introduced in a special case by Artin and vastly generalized by Weil. This is not the proper context for the discussion of the Weil zeta function, nevertheless because of its significance and relation to the Lefschetz Fixed Point theorem we give a heuristic and brief exposition of it, emphasizing its topological aspect and its meaning in the combinatorics of Young subgroups, certain card shuffles or equivalently cohomology of complex flag manifolds. Let V be a nonsingular variety over the finite field $\mathcal{F}_q$ of q elements. We will not dwell on the exact meaning of *variety* and *nonsingular* in this context; some examples will suffice for us. In chapter 1 we introduced the complex flag manifolds $\mathbf{F}_{\mathbf{s}}=GL(n,\mathbf{C})/P_{\mathbf{s}}$ as the set of sequences of subspaces $V_{s_1}\subset V_{s_2}\subset\cdots\subset V_{s_r}$ of $\mathbf{C}^n$, where $0<s_1<s_2<\cdots<s_r<n$ and $\dim V_{s_j}=s_j$, and studied some of their homological properties in chapter 3. The analogous varieties over a finite field are the set of sequences of subspaces of the same form in the vector space of dimension n over a finite field. They are also realized as homogeneous spaces by replacing $GL(n,\mathbf{C})$ and $P_{\mathbf{s}}$ by matrices of the same form with entries from a finite field. We denote the corresponding flag manifolds over the field of q^m elements by $\mathbf{F}_{\mathbf{s}}(\mathcal{F}_{q^m})$. In particular, we have the notion of projective space $\mathcal{F}_{q^m}\mathbf{P}(n)\simeq(\mathcal{F}_{q^m}^{n+1}\setminus\mathbf{0})/\mathcal{F}_{q^m}^{\times}$ of dimension n over $\mathcal{F}_{q^m}$. Another class of examples is obtained by considering a set of homogeneous equations $F_1(z)=\cdots=F_r(z)=0$ in $n+1$ variables $z=(z_\circ,\cdots,z_n)$ and assuming that the degrees and the coefficients of F_j's are integers not divisible by a given prime p and such that their zero set is a submanifold $V(\mathbf{C})$ of $\mathbf{CP}(n)$. Then reducing the equations modulo p we obtain a set of homogeneous equations over $\mathcal{F}_p$ and denote their zero set in $\mathcal{F}_{p^m}\mathbf{P}(n)$ by $V(\mathcal{F}_{p^m})$. All these examples are special cases of varieties *liftable to characteristic zero.* The dimension of such a variety is the same as its dimension as a complex manifold.

The fundamental problem considered by Weil [W] was that of counting the number of points in $V(\mathcal{F}_{q^m})$ which we denote by $N_m=N_m(V,q)$. While in this form the problem does not appear to be tractable, Weil noticed that a generating function for N_m exhibits remarkable topological/geometric properties. The generating function, denoted by $Z(t)=Z_V(t)$ and called the *Weil zeta function*, is defined as

$$Z(t) \;=\; \exp\Big(\sum_{m=1}^{\infty}N_m\frac{t^m}{m}\Big).$$

Before stating the fundamental properties of $Z(t)$, known as the *Weil conjectures*, let us explicitly compute it in a simple special case.

Example 3.4.4 The projective space $\mathcal{F}_q\mathbf{P}(n)\simeq(\mathcal{F}_q^{n+1}\setminus\mathbf{0})/\mathcal{F}_q^{\times}$ and has $\frac{q^{n+1}-1}{q-1}=q^n+\cdots+q+1$ points. Therefore

$$\sum_{m=0}^{\infty}N_m\frac{t^m}{m}=\sum_{j=0}^{n}\sum_{m=0}^{\infty}q^{jm}\frac{t^m}{m}=-\sum_{j=0}^{n}\log(1-q^jt),$$

and consequently $Z(t) = \prod_{j=0}^{n} \frac{1}{1-q^j t}$. ♠

The basic properties of the Weil zeta function of a variety V of dimension n (proven in [D]) are

1. $Z(t)$ is a rational function of t with rational coefficients.
2. $Z(t)$ satisfies the (Poincaré duality type) functional equation

$$Z(\frac{1}{q^n t}) = \pm q^{n\chi/2} t^{\chi} Z(t),$$

   where the integer χ is the Euler characteristic of the complex manifold $V(\mathbf{C})$ if V is liftable to characteristic zero.

3. $Z(t)$ can be expanded (uniquely) as a product of the form

$$Z(t) = \frac{Q_1(t)Q_3(t)\cdots Q_{2n-1}(t)}{Q_\circ(t)Q_2(t)\cdots Q_{2n}(t)},$$

   where $Q_\circ(t) = 1-t$, $Q_{2n}(t) = 1-q^n t$, $Q_i(t) = \prod_j (1-\alpha_{ij}t)$ with α_{ij}'s algebraic integers and $|\alpha_{ij}| = q^{i/2}$. For a liftable variety V, the degree of $Q_i(t)$ is equal to the i^{th} Betti number of the complex manifold $V(\mathbf{C})$.

It is trivial to see that these properties hold for the projective space. Notice that if one determines the number of points of $V(\mathcal{F}_{q^m})$ for a certain finite number of m's, then $Z(t)$ is completely determined and, in principle, one knows N_m for all m. For instance, if V is given by a single homogeneous equation in three variables with integer coefficients such that its degree and coefficients are not divisible by a prime p, and $V(\mathbf{C})$ is topologically a surface of genus g, then we only need to know the numbers $N_1, \cdots, N_g$ to completely determine $Z(t)$.

The properties of $Z(t)$ are deeply connected with topological considerations. For example, the proof of the properties depends, among other things, on the development of a "good" cohomology theory with Poincaré duality and a version of the Lefschetz Fxed Point theorem. In fact, let $V = \cup_m V(\mathcal{F}_{q^m})$ and Φ_m denote the *Frobenius* defined by raising every coordinate of a point in V to power q^m. Then V is invariant under Φ_m and the number of fixed points of Φ_m is the integer N_m. The idea is to use a version of the Lefschetz Fixed Point theorem to evaluate the generating function $Z(t)$. That this implies the rationality of the generating function $Z(t)$ was clarified earlier, however, property (3) is quite deep.

Example 3.4.5 In this example we use the properties of $Z(t)$ to obtain some combinatorial information about Young subgroups of the symmetric group. To do so we count the number of points in $\mathbf{F_s}(\mathcal{F}_q)$. First look at $GL(n, \mathcal{F}_q)$. The first column of a nonsingular matrix can be any nonzero vector and there are $q^n - 1$ such vectors; the second column can be any vector linearly independent from the first one and therefore there are $q^n - q$ such vectors, etc. Proceeding inductively we see that the order of $GL(n, \mathcal{F}_q)$ is

$$|GL(n, \mathcal{F}_q)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}).$$

From chapter 3, section 6, recall the notation $t_1 = s_1$ and $t_i = s_i - s_{i-1}$ for $i > 1$. The order of the subgroup $P_{\mathbf{s}}(\mathcal{F}_q)$ is then easily computed to be

$$|P_{\mathbf{s}}(\mathcal{F}_q)| = (\prod_{i<j} q^{t_i t_j}) \prod_i |GL(t_i, \mathcal{F}_q)|,$$

and consequently the number of points of $\mathbf{F}_{\mathbf{s}}(\mathcal{F}_q)$ is

$$N_1 = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^2-1)(q-1)}{\prod_i (q^{t_i}-1)(q^{t_i-1}-1)\cdots(q^2-1)(q-1)},$$

which is an integer, and can be expressed as a polynomial $N_1 = 1 + a_1 q + a_2 q^2 + \cdots$ in q. The meaning of the coefficients a_j will be determined shortly. Replacing q by q^m we obtain N_m. Substituting in the definition of $Z(t)$ and evaluating just as in the case of the projective space we obtain

$$Z(t) \;=\; \prod_j \frac{1}{(1-q^j t)^{a_j}}. \qquad (3.4.9)$$

From property (3) of $Z(t)$ we see that a_j is the $(2j)^{\text{th}}$ Betti number of the complex flag manifold $\mathbf{F}_{\mathbf{s}}$[4]. Its combinatorial meaning, given earlier, is as follows: Consider the Young subgroup $\mathcal{S}_{\mathbf{s}} = \mathcal{S}_{t_1} \times \cdots \times \mathcal{S}_{t_r}$ of $\mathcal{S}_n$ and recall that every coset of $\mathcal{S}_{\mathbf{s}}$ has a unique representative $\sigma_{\mathbf{c}} \in \mathcal{S}_n$ of minimal length (see exercise **??**). Call this number the length of the coset $\sigma_{\mathbf{c}}\mathcal{S}_{\mathbf{s}}$. Then a_j is the number of cosets of length j. In particular, we have shown that the generating function for the number of cosets of length j is

$$\sum a_j \lambda^j = \frac{(\lambda^n-1)(\lambda^{n-1}-1)\cdots(\lambda^2-1)(\lambda-1)}{\prod_i (\lambda^{t_i}-1)(\lambda^{t_i-1}-1)\cdots(\lambda^2-1)(\lambda-1)}.$$

This expression is of course also the *Poincaré series*, defined as $\sum_k \dim(H_k(\mathbf{F}_{\mathbf{s}}; \mathbf{R}))\lambda^k$, for the complex flag manifold $\mathbf{F}_{\mathbf{s}}$. The fact that one can obtain information about complex manifolds from correspondings objects over finite fields is no accident. Remarkable examples are [HN] and [M].

Exercise 3.4.8 *Consider the flag manifold $\mathbf{F}_n$ so that $t_1 = \cdots = t_n = 1$. Show by induction on the number of factors (i.e., $n-1$) in*

$$N_1 = (q^{n-1} + q^{n-2} \cdots + q + 1)(q^{n-2} + q^{n-3} + \cdots + q + 1)\cdots(q+1)$$

that the $(2j)^{\text{th}}$ Betti numbers a_j of $\mathbf{F}_n$ satisfy the inequalities $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{\lfloor n(n-1)/4 \rfloor}$ where $\lfloor x \rfloor$ indicates largest integer not exceeding x. Generalize to $\mathbf{F}_{\mathbf{s}}$ by showing that the even dimensional Betti numbers are nondecreasing up to the middle dimension. (The nondecreasing property of Betti numbers up to the middle dimension is a special case of the Hard Lefschetz theorem which will not be discussed in this volume.)

[4] This is the only place where the Weil conjectures are used. It is more judicious to arrive at the meaning of the integers a_j by group theoretic considerations, and use the result as a demonstration of the Weil conjectures in the special case of flag manifolds. The Bruhat decomposition discussed briefly in chapter 3, §6, is valid in exactly the same form for $GL(n, \mathcal{F}_q)$. Now a_j is the coefficient of q^j in $N_1 = 1 + a_1 q + a_2 q^2 + \cdots$, and it is not difficult to see from the Bruhat decomposition that this means that there are a_j "cells" $\mathcal{F}_q^j$ in $\mathbf{F}_{\mathbf{s}}(\mathcal{F}_q)$. Equivalently there are a_j cells $\mathbf{C}^j$ in $\mathbf{F}_{\mathbf{s}}$ which is the desired interpretation of a_j.

3.5 Some Algebraic Considerations

3.5.1 Free Resolutions

Let R be a commutative ring with identity and E an R-module. By a *free resolution* of E we mean an exact sequence of the form

$$\cdots \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} \cdots \longrightarrow F_1 \longrightarrow F_\circ \longrightarrow E \longrightarrow 0, \tag{3.5.1}$$

where each F_i is a free R-module and it is convenient to set $F_{-1} = E$. That free resolutions exist is immeditate since for given any R-module there is a surjective map of a free R-module onto it. It was noted earlier that tensoring does not necessarily preserve exactness of a sequence. In particular, the sequence

$$\cdots \xrightarrow{\mathsf{f}_{n+1}} F_n \otimes C \xrightarrow{\mathsf{f}_n} \cdots \longrightarrow F_1 \otimes C \longrightarrow F_\circ \otimes C \longrightarrow 0, \tag{3.5.2}$$

where A is an R-module and $\mathsf{f}_n = f_n \otimes \mathrm{id.} : F_n \otimes C \to F_{n-1} \otimes C$, is not necessarily exact. In the spirit of homology theory we define

$$\mathrm{Tor}_n(E, C) = \mathrm{Tor}_n^R(E, C) = \frac{\ker \mathsf{f}_n}{\mathrm{Im}\mathsf{f}_{n+1}}.$$

Similarly, applying $\mathrm{Hom}(., C)$ to an exact sequence may violate its exactness. Let $\mathsf{f}^{n+1} : \mathrm{Hom}(F_n, C) \to \mathrm{Hom}(F_{n+1}, C)$ be the map induced by the composition $\mathsf{f}^{n+1}(h) = hf_{n+1} : F_{n+1} \to A$ where $h \in \mathrm{Hom}(F_n, C)$. Then the sequence

$$\cdots \xleftarrow{\mathsf{f}_{n+1}} \mathrm{Hom}(F_n, C) \xleftarrow{\mathsf{f}_n} \cdots \longleftarrow \mathrm{Hom}(F_1, C) \longleftarrow \mathrm{Hom}(F_\circ, C) \longleftarrow 0, \tag{3.5.3}$$

is not necesarily exact. In the spirit cohomology theory, we define

$$\mathrm{Ext}^n(E, A) = \mathrm{Ext}_R^n(E, A) = \frac{\ker \mathsf{f}^{n+1}}{\mathrm{Im}\mathsf{f}^n}.$$

Collectively the R-modules $\mathrm{Tor}_n(E, C)$ and $\mathrm{Ext}^n(E, C)$ maybe called, although not commonly, *extorsion* to be distinguished from extortion.

These definitions, *a priori*, depend on the choice of free resolution. That extorsion is in fact independent of the choice of free resolution is proven below. But first we look at some examples, consequences and exercises.

Example 3.5.1 Let $R = \mathbf{Z}$, $E = \mathbf{Z}/m$ and $C = \mathbf{Z}/n$. Then a free resolution for E is

$$0 \longrightarrow \mathbf{Z} \xrightarrow{f_1} \mathbf{Z} \xrightarrow{f_\circ} \mathbf{Z}/m \longrightarrow 0,$$

where f_1 is multiplication by m and $f_\circ$ is reduction modulo m. Tensoring with C yields the sequence

$$0 \longrightarrow \mathbf{Z}/n \xrightarrow{\mathsf{f}_1} \mathbf{Z}/n \longrightarrow 0,$$

where f_1 is multiplication by m. Therefore

$$\mathrm{Tor}_\circ(\mathbf{Z}/m, \mathbf{Z}/n) \simeq \mathbf{Z}/(m,n) \simeq \mathrm{Tor}_1(\mathbf{Z}/m, \mathbf{Z}/n),$$

where (m, n) is the greatest common divisor of m and n. Similarly, since $\mathrm{Hom}(\mathbf{Z}, \mathbf{Z}/n) \simeq \mathbf{Z}/n$, the Ext^j groups are computed from the sequence

$$0 \longleftarrow \mathbf{Z}/n \xleftarrow{\mathsf{f}_1} \mathbf{Z}/n \longleftarrow 0,$$

where f_1 is multiplication by m. Therefore, we have $\mathrm{Ext}^j(\mathbf{Z}/m, \mathbf{Z}/n) \simeq \mathrm{Tor}_j(\mathbf{Z}/m, \mathbf{Z}/n)$. Since submodules of (finitely generated) free modules over a principal ideal domain (PID) are free (and finitely generated), the above argument and structure theorem for finitely generated modules over PID's enable one to compute extorsion for finitely generated modules over PID's. The details are are straightforward. In particular, $\mathrm{Ext}^j(E, A) = 0 = \mathrm{Tor}_j(E, A)$ for $j \geq 2$ for modules over a PID. Of course we are anticipating the fact that extorsion is independent of choice of the free resolution.

Exercise 3.5.1 *Let R be any commutaive ring with identity, and E and C R-modules. Show that* $\mathrm{Tor}_\circ(E, C) \simeq E \otimes C$, *and* $\mathrm{Ext}^\circ(E, C) \simeq \mathrm{Hom}(E, C)$. *Show also that if E is a free R-module, then* $\mathrm{Tor}_j(E, C) = \mathrm{Ext}^j(E, C) = 0$ *for all R-modules C and $j \geq 1$.*

Example 3.5.2 In this and next examples we use the fact that extorsion is independent of the choice of free resolution to obtain some nonobvious results about (co)homology groups of a space. Let $H_n = H_n(X, A; R)$ be the n^{th} homology group of a pair (X, A) with coefficient ring R a principal ideal domain. Denoting the corresponding modules of chains, cycles and boundaries by C_n, Z_n and B_n respectively, we obtain the free resolution

$$0 \longrightarrow Z_n \xrightarrow{f_2} C_n \xrightarrow{f_1} Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0, \tag{3.5.4}$$

where f_2 is the inclusion and $f_1 = \partial_n$. Tensoring with C we obtain the sequence

$$0 \longrightarrow Z_n \otimes C \xrightarrow{\mathsf{f}_2} C_n \otimes C \xrightarrow{\mathsf{f}_1} Z_{n-1} \otimes C \longrightarrow 0,$$

From this sequence we obtain

$$\mathrm{Tor}_1(H_{n-1}, C) \simeq \frac{\ker(\mathsf{f}_1)}{Z_n \otimes C} \simeq \frac{\ker(\partial_n \otimes \mathrm{id.})/B_n \otimes C}{Z_n \otimes C/B_n \otimes C}.$$

Since $Z_n \otimes C/B_n \otimes C \simeq (Z_n/B_n) \otimes C \simeq H_n \otimes C$, and $H_n(X, A; C) \simeq \ker(\partial_n \otimes \mathrm{id.})/(B_n \otimes C)$ we obtain the exact sequence

$$0 \longrightarrow H_n \otimes C \longrightarrow H_n(X, A; C) \longrightarrow \mathrm{Tor}_1(H_{n-1}, C) \longrightarrow 0. \tag{3.5.5}$$

This exact sequence gives a method for effectively computing $H_n(X, A; C)$ for any abelian group C once we know $H_n(X, A; \mathbf{Z})$, especially when we note that the exact sequence (3.5.5) splits. To see that this sequence splits, note that from the free resolution (3.5.4) we obtain the exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ which splits since submodules of free modules over a PID are free. Therefore $C_n = Z_n \oplus \tilde{B}_{n-1}$, where $\tilde{B}_{n-1} \simeq B_{n-1}$. Consequently, under this isomorphism,

$$\mathrm{Tor}_1(H_{n-1}, C) \simeq \ker(\tilde{B}_{n-1} \otimes C \to Z_{n-1} \otimes C), \text{ and } H_n(X, A; C) \simeq H_n \otimes C \oplus \mathrm{Tor}_1(H_{n-1}, C). \tag{3.5.6}$$

The content of this example is known as the *universal coefficient theorem for homology.* ♠

For future use we note the following elementary lemma whose proof is omitted:

Lemma 3.5.1 *Let R be a PID, F a free R-module and $Z \subset F$ a submodule with the property that if $\alpha x \in Z$ for $0 \neq \alpha \in R$ and $x \in F$, then $x \in Z$. Then every homomorphism $Z \rightarrow C$ extends to a homomorphism $F \rightarrow C$.*

Example 3.5.3 If we apply $\mathrm{Hom}(., C)$ to the free resolution (3.5.4) we obtain the sequence

$$0 \longleftarrow \mathrm{Hom}(Z_n, C) \xleftarrow{\mathsf{f}^2} \mathrm{Hom}(C_n, C) \xleftarrow{\mathsf{f}^1} \mathrm{Hom}(Z_{n-1}, C) \longleftarrow 0. \tag{3.5.7}$$

Now $\mathrm{Ext}^1(H_{n-1}, C) = \ker \mathsf{f}^2/\mathrm{Im}\mathsf{f}^1$, $\ker \mathsf{f}^2 = \{h : C_n \rightarrow C | h \text{ vanishes on } Z_n\}$, and

$$\mathrm{Im}\mathsf{f}^1 = \{h \in \ker \mathsf{f}^2 | h : B_{n-1} = C_n/Z_n \rightarrow C \text{ extends to } Z_{n-1}\}. \tag{3.5.8}$$

The cohomology groups $H^n = H^n(X, A; R)$ are computed from the complex

$$\cdots \longleftarrow \mathrm{Hom}(C_{n+1}, C) \xleftarrow{\partial^\star_{n+1}} \mathrm{Hom}(C_n, C) \xleftarrow{\partial^\star_n} \mathrm{Hom}(C_{n-1}, C) \longleftarrow \cdots$$

Now $\ker \partial^\star_{n+1} (\supset \ker \mathsf{f}^2)$ consists of homomorphisms $h : C_n \rightarrow C$ vanishing on B_n, and

$$\mathrm{Im}\partial^\star_n = \{h \in \ker \mathsf{f}^2 | h : B_{n-1} = C_n/Z_n \rightarrow C \text{ extends to } C_{n-1}\}.$$

It follows from lemma 3.5.1 that every homomorphism $Z_{n-1} \rightarrow C$ extends to a homormorphism $C_{n-1} \rightarrow C$. Therefore $\mathrm{Im}\partial^\star_n = \mathrm{Im}\mathsf{f}^1$. With this observation, it follows easily from (3.5.7) that we have the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}, C) \longrightarrow H^n(X, A; C) \longrightarrow \mathrm{Hom}(H_n(X, A; R), C) \longrightarrow 0. \tag{3.5.9}$$

This exact sequence, like (3.5.5) and for the same reasons, splits but not canonically. It is often called the *universal coefficient theorem for cohomology.* It enables one to algebraically compute cohomology from homology and contains proposition **??** as a special case. ♠

Exercise 3.5.2 *Prove the statements in remark 3.3.1.*

Exercise 3.5.3 *Let X be a simplicial complex (or topological space). Show that $H^1(X; \mathbf{Z})$ is torsion free.*

Exercise 3.5.4 *Let E be a finite abelian group with the property that every element has order p, a prime. Show that $\mathrm{Ext}^1(E, C) \simeq \mathrm{Hom}(E, C/pC)$ for a (finitely generated) abelian group C.*

Exercise 3.5.5 *Let $\mathbf{Z}_{<p>}$ be the set of rational numbers whose denominators are powers of p, a prime. Compute $\mathrm{Tor}^j(E, \mathbf{Z}_{<p>}/\mathbf{Z})$ for a finitely generated abelian group E.*

Exercise 3.5.6 *Let E and C be finite abelian groups such that for all $x \in E$ and $y \in C$, the orders of x and y are relatively prime. Show that $\mathrm{Tor}^j(E, C) = 0$ for all $j \geq 0$.*

Exercise 3.5.7 *Let $R = \mathbf{Z}[\mathbf{Z}/m]$ be the integral group algebra of the cyclic group $\mathbf{Z}/m$ (i.e., all polynomials $\sum_{i=0}^{m-1} a_i T^i$ where multiplication is formal subject to the relation $T^m = 1$ and $a_i \in \mathbf{Z}$). Regard $\mathbf{Z}$ as an R-modules by defining $T.a = a$ for all $a \in \mathbf{Z}$. Let $F_j = R$, f_{2j} be multiplication by $1 + T + \cdots + T^{m-1}$ for $j \geq 1$, and f_{2j+1} be multiplication by $T - 1$. Show that the sequence*

$$\cdots \longrightarrow F_j \xrightarrow{f_j} F_{j-1} \longrightarrow \cdots \longrightarrow F_\circ \mathbf{Z} \longrightarrow 0,$$

where $f_{-1} : R \to \mathbf{Z}$ is given by $f_{-1}(\sum a_i T^i) = \sum a_i$, is a free resolution of $\mathbf{Z}$ as an R-module. Let C be an R-module and $C_1 = (T-1)(C)$, $C_2 = (1 + T + \cdots + T^{m-1})(C)$. Show that $\mathrm{Ext}^\circ_R(\mathbf{Z}, C) = \{c \in C | Tc = c\}$ *and*

$$\mathrm{Ext}^j_R(\mathbf{Z}, C) = \begin{cases} \{c \in C | Tc = c\}/C_2 & \text{if } j \text{ is even and positive} \\ \{c \in C | (1 + T + \cdots + T^{m-1})c = 0\}/C_1 & \text{if } j \text{ is odd} \end{cases}$$

Finally in this subsection we consider the issue of independence of extorsion from the choice of free resolution. The argument is simple and instructive. An important property of free modules (motivated by the splitting homomorphism) is that if F is a free R-module and $C \to C' \to 0$ is an exact sequence of R-modules, then for every homomorphism $F \to C'$ there is a homomorphism $F \to C$ such that the diagram

$$\begin{array}{ccccc} & & F & & \\ & \swarrow & \downarrow & & \\ C & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

commutes. That free modules have this property is trivial. This property is the defining property of *projective* modules, but for our purposes, free modules will suffice. We refer to this property as the *universal mapping property of projective modules.* Now assume we have two free resolutions

$$\begin{array}{cccccccccccccc} \cdots & \xrightarrow{f_{n+1}} & F_n & \xrightarrow{f_n} & \cdots & \longrightarrow & F_1 & \longrightarrow & F_\circ & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow \beta_n & & & & \downarrow \beta_1 & & \downarrow \beta_\circ & & \downarrow \text{id.} & & \\ \cdots & \xrightarrow{g_{n+1}} & G_n & \xrightarrow{g_n} & \cdots & \longrightarrow & G_1 & \longrightarrow & G_\circ & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

where the maps β_j remain to be specified such that the diagram commutes. The fact that $\beta_\circ$ exists follows from the universal mapping property of projective modules. Now from exactness and existence of $\beta_\circ$ it follows that $\mathrm{Im}(\beta_\circ f_1) \subset \mathrm{Im}(g_1)$. Therefore the universal mapping property is applicable to give the existence of β_1. Proceeding inductively in the obvious manner we obtain β_j's such the above diagram commutes. Reversing the roles of the top and bottom free resolutions we similarly obtain $\gamma_j : G_j \to F_j$ such the corresponding diagram commutes diagram. Let $\Phi_j = \gamma_j \beta_j$ to obtain the commutative row exact diagram

$$\begin{array}{cccccccccccccc} \cdots & \xrightarrow{f_{n+1}} & F_n & \xrightarrow{f_n} & \cdots & \longrightarrow & F_1 & \longrightarrow & F_\circ & \longrightarrow & E & \longrightarrow & 0 \\ & \swarrow & \downarrow \Phi_n & \swarrow & & \swarrow & \downarrow \Phi_1 & \swarrow & \downarrow \Phi_\circ & & \downarrow \text{id.} & & \\ \cdots & \xrightarrow{f_{n+1}} & F_n & \xrightarrow{f_n} & \cdots & \longrightarrow & F_1 & \longrightarrow & F_\circ & \longrightarrow & E & \longrightarrow & 0 \end{array} \tag{3.5.10}$$

where the maps $s_k : F_k \to F_{k+1}$ (*chain homotopy*) will be now determined. Here s_k's are constructed to enable us to compare the endomorphisms of $\mathrm{Tor}_k(E, C)$'s induced by Φ_k's and the identity map. Clearly

$\Phi_\circ(x) - x \in \ker f_\circ = \mathrm{Im} f_1$, and therefore by the universal mapping property there is $s_\circ$ such that

$$\Phi_\circ(x) - x = s_\circ f_1(x). \tag{3.5.11}$$

From commutatvity of the squares, it follows that $\Phi_1(x) - x - s_\circ f_1(x) \in \ker f_1$ for $x \in F_1$. Therefore, by the universal mapping property, there is $s_1 : F_1 \to F_2$ such that

$$\Phi_1(x) - x = s_\circ f_1(x) + f_2 s_1(x).$$

Proceeding inductively in the obvious manner we obtain s_j such that

$$\Phi_j(x) - x = s_{j-1} f_j(x) + f_{j+1} s_j(x) \quad \text{for } j \geq 1 \tag{3.5.12}$$

and (3.5.11) for $j = 0$. The relations (3.5.11) and (3.5.12) remain valid after tensoring with C and where the maps Φ_j, f_j etc. are replaced with $\Phi_j \otimes \mathrm{id.}$, f_j etc. It follows from (3.5.11) and (3.5.12) that the endomorphisms induced by Φ_j's on the $\mathrm{Tor}_j(E, C)$'s are the identity maps. Since $\Phi_j = \gamma_j \beta_j$ we immediately obtain independence of $\mathrm{Tor}_j(E, C)$'s from the choice of free resolution. Similar argument applies to the $\mathrm{Ext}^j(E, C)$, and we can simply say that extorsion is independent of the choice of free resolution. Therefore we have shown the first statement of

Theorem 3.5.1 *Extorsion is independent of the choice of free resolution. Given a short exact sequence $0 \to C' \to C \to C'' \to 0$ of R-modules, there are long exact sequences*

$$\cdots \to \mathrm{Tor}_n(E, C') \to \mathrm{Tor}_n(E, C) \to \mathrm{Tor}_n(E, C'') \to \mathrm{Tor}_{n-1}(E, C') \to \cdots \to \mathrm{Tor}_\circ(E, C'') \to 0$$
$$0 \to \mathrm{Ext}^\circ(E, C'') \to \cdots \to \mathrm{Ext}^{n-1}(E, C') \to \mathrm{Ext}^n(E, C'') \to \mathrm{Ext}^n(E, C) \to \mathrm{Ext}^n(E, C') \to \cdots$$

(The homomorphisms $\delta_n : \mathrm{Tor}_n(E, C'') \to \mathrm{Tor}_{n-1}(E, C')$ and $\delta^n : \mathrm{Ext}^n(E, C') \to \mathrm{Ext}^{n+1}(E, C'')$ are called *connecting homomorphisms.*)

Proof - It remains to prove the the second assertion. Tensoring the free resolution of E (3.5.1) with the short exact sequence we obtain the following row exact commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_n \otimes C' & \longrightarrow & F_n \otimes C & \longrightarrow & F_n \otimes C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{n-1} \otimes C' & \longrightarrow & F_{n-1} \otimes C & \longrightarrow & F_{n-1} \otimes C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

This is exactly the same situation as where we constructed the connecting homomorphism for homology. The same argument is applicable to both Tor and Ext modules. ♣

Exercise 3.5.8 *Assume the following diagram of free R-modules is row exact and commututative:*

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & B_{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Show that tensoring with an R-module C yields a similar row exact commutative diagram and a long exact sequence. In particular, consider the case where R is a PID, Z_n, C_n and B_n are the R-modules of cycles, chains and boundaries, with $Z_n \to Z_{n-1}$, and $B_n \to B_{n-1}$ the zero maps and $C_n \to C_{n-1}$ is the boundary homomorphism. By computing the maps in the long exact sequence, give another proof of the universal coefficient theorem.

Finally in this subsection we complete the algebraic aspect of the computation of (co)homology of product spaces. The fundamental algebraic fact which completes the picture is

Theorem 3.5.2 (Künneth Formula) *Assume R is a PID and the R-modules F_k and F'_k are free. Then the homology of the complex $\mathsf{F} \otimes \mathsf{F}'$ is given by the exact sequence*

$$0 \longrightarrow \sum_{p+q=n} H_p(\mathsf{F}) \otimes H_q(\mathsf{F}') \longrightarrow H_n(\mathsf{F} \otimes \mathsf{F}') \longrightarrow \sum_{j+k=n-1} \mathrm{Tor}_1(H_j(\mathsf{F}), H_k(\mathsf{F}')) \longrightarrow 0$$

Proof - Since the boundary operator ∂'' consists of two terms, trying to emulate the proof of the universal coefficient theorem encounters some difficulties. To circumvent this problem we introduce complexes Z' : $\cdots \to Z'_1 \to Z'_\circ \to 0$ and $\mathsf{D} : \cdots \to D_2 \to D_1 \to 0$ where $Z'_k \subset F'_k$ is the R-module of k-cycles and $D_k = B'_{k-1} \subset Z'_{k-1}$ is the R-module of $(k-1)$-boundaries. The maps $Z'_k \to Z'_{k-1}$ and $D_k \to D_{k-1}$ are the zero maps. We have the short exact sequence

$$0 \longrightarrow F_p \otimes Z'_q \longrightarrow F_p \otimes F'_q \longrightarrow F_p \otimes D_q \longrightarrow 0. \tag{3.5.13}$$

Summing over $p+q = n$ and inserting the boundary operators we obtain the following row exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sum_{p+q=n} F_p \otimes Z'_q & \longrightarrow & \sum_{p+q=n} F_p \otimes F'_q & \longrightarrow & \sum_{p+q=n} F_p \otimes D_q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sum_{p+q=n} F_{p-1} \otimes Z'_q & \longrightarrow & \sum_{p+q=n-1} F_p \otimes F'_q & \longrightarrow & \sum_{p+q=n} F_{p-1} \otimes D_q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

where the middle vertical arrow is ∂'' and the outer vertical arrows are $\partial \otimes \mathrm{id}$. By a familiar argument (see construction of the long exact sequence for homology) we have the long exact sequence

$$\to H_n(\mathsf{F} \otimes \mathsf{Z}') \to H_n(\mathsf{F} \otimes \mathsf{F}') \to H_n(\mathsf{F} \otimes \mathsf{D}) \stackrel{\delta_n}{\to} H_{n-1}(\mathsf{F} \otimes \mathsf{Z}') \to$$

from which we obtain the short exact sequence

$$0 \longrightarrow \mathrm{coker}\delta_{n+1} \longrightarrow H_n(\mathsf{F} \otimes \mathsf{F}') \longrightarrow \ker \delta_n \longrightarrow 0. \tag{3.5.14}$$

Now it is a straightforward exercise to follow through the diagram and see that $\delta_n = \sum_{p+q=n} (-1)^p \otimes \jmath_q$ where $\jmath_q : D_q \hookrightarrow Z'_{q-1}$ is the inclusion. Therefore $\mathrm{coker}\delta_n = \sum_{p+q=n} H_p(\mathsf{F}) \otimes H_q(\mathsf{F}')$. Furthermore using the free resolution of $H_{q-1}(\mathsf{F}')$ similar to one in example 3.5.2 it follows easily that

$$\ker \delta_n = \sum_{p+q=n} \ker((-1)^p \otimes \jmath_q) = \sum_{p+q=n} \mathrm{Tor}_1(H_p(\mathsf{F}), H_{q-1}(\mathsf{F}')).$$

Substituting in (3.5.14) we obtain the desired result. ♣

Remark 3.5.1 By an argument similar to one for the universal coefficient theorem one shows easily that the sequence splits. However, as noted earlier, the splitting in this or the universal coefficient theorem is not natural.

3.5.2 Double Complexes

A *double complex* is a family of R-modules $\mathsf{E} = \{E_{i,j}\}$ where $i, j \in \mathbf{Z}$. A *first quadrant* double complex E is one for which $E_{i,j} = 0$ unless $i \geq 0$ and $j \geq 0$. Similarly we have the notion of a *second quadrant* etc. double complex. The R-modules $E_{i,j}$ are related in different manners in different contexts. First we consider the case where we have a double complex of the form

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E_{\circ,2} & \longrightarrow & E_{1,2} & \longrightarrow & E_{2,2} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E_{\circ,1} & \longrightarrow & E_{1,1} & \longrightarrow & E_{2,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E_{\circ,\circ} & \longrightarrow & E_{1,\circ} & \longrightarrow & E_{2,\circ} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.5.15}$$

where we denote by $\partial_{ij} : E_{i,j} \to E_{i+1,j}$ and $\partial'_{i,j} : E_{i,j} \to E_{i,j+1}$ and assume

1. $\partial_{i+1,j}\partial_{i,j} = 0$;
2. $\partial'_{i,j+1}\partial'_{i,j} = 0$;
3. The squares in the diagram commute.

In the spirit of cohomology we define

$$_jH^i = \frac{\ker \partial_{i,j}}{\mathrm{Im}\partial_{i-1,j}}, \quad {}'_iH^j = \frac{\ker \partial'_{i,j}}{\mathrm{Im}\partial'_{i,j-1}}.$$

We use the notation $_jZ^i$, $'_iZ^j$, $_jB^i$ and $'_iB^j$ for the corresponding cocycle and coboundary modules. Similar considerations apply to homology rather than cohomology and we denote the corresponding modules by $_jH_i$ and $'_iH_j$ etc. The following proposition describes a situation which occurs not infrequently as will be demonstrated in two examples that follow.

Proposition 3.5.1 *With the above notation and hypotheses, assume furthermore*

$$_jH^i \ = \ 0, \quad {}'_jH^i = 0 \ \text{ unless } j = 0,$$

i.e., except possibly for the first row and first column, all rows and columns are exact. Then

$$_\circ H^i \ = \ {}'_\circ H^i.$$

Proof - The idea of the proof is quite simple and instructive. We demonstrate the idea by showing how we assign a cohomology class $[c'] \in {}'_\circ H^2$ to a cohomology class $[c] \in {}_\circ H^2$. We use the notation $[c]$ to denote the cohomology class associated to a cocycle c. The details and extension to arbitrary i is straightforward and therefore omitted. Let $c \in {}_\circ Z_2$, then $\partial_{2,1}(\partial'_{2,\circ}c) = 0$, and therefore by exactness of the second row there is $c_1 \in E_{1,1}$ such that $\partial_{1,1}(c_1) = \partial'_{2,0}(c)$. Similarly, $\partial_{1,2}(\partial'_{1,1}(c_1)) = 0$ and there is $c' \in E_{\circ,2}$ such that $\partial_{\circ,2}(c') = \partial'_{1,1}(c_1)$. c' is a cocycle since $\partial_{\circ,3}$ is injective and

$$\partial_{\circ,3}\partial'_{\circ,2}(c') = \partial'_{1,2}\partial_{\circ,2}(c') = \partial'_{1,2}\partial'_{1,1}(c_1) = 0.$$

In the assignment $c \rightarrow c'$ we made several choices. However, an examination of the process shows that different choices lead to c''s that differ by coboundaries. Similarly one shows that under this map coboundaries are mapped to coboundaries so that there is induced map on cohomology. To see that the correspondence $c \rightarrow c'$ induces isomorphism on cohomology, it suffices to note by symmetry we also have the correspondence $c' \rightarrow c$ which, at the level of cohomology, is the inverse to $c \rightarrow c'$. Further details are omitted. ♣

Example 3.5.4 Let $\cdots \rightarrow F_1 \rightarrow F_\circ \rightarrow E \rightarrow 0$ and $\cdots \rightarrow F'_1 \rightarrow F'_\circ \rightarrow C \rightarrow 0$ be free resolutions of R-modules E and C. Define the double complex E by $E_{i,j} = F_{i-1} \otimes F'_{j-1}$ where $F_{-1} = E$, $F'_{-1} = C$ and the maps are induced from the free resolutions for E and C in the obvious manner. In this situation we are looking at homology rather than cohomology, nevertheless proposition 3.5.1 remains valid. The rows and columns of this diagram, except possibly the first row and first column, are exact, since tensoring with a free module preserves exactness of a sequence. From the first row and first column one computes $\mathrm{Tor}_j(E, C)$ and $\mathrm{Tor}_j(C, E)$ and $\mathrm{Tor}_\circ(E, C) = E \otimes C = \mathrm{Tor}_\circ(C, E)$. Proposition 3.5.1 is applicable to show $\mathrm{Tor}_j(E, C) \ \simeq \ \mathrm{Tor}_j(C, E)$. ♠

Example 3.5.5 We had noted (without proof) that if $\mathcal{U} = \{U_i\}$ is a locally finite covering of a manifold M such that every nonempty $U_{i_\circ} \cap \cdots \cap U_{i_k}$ is contractible, then cohomology of M is isomorphic to the Čech groups computed relative to the covering $\mathcal{U}$. Recall that a (real) Čech cochain $\gamma \in \check{C}^n(\mathcal{U}, \mathbf{R})$ is the assignment of a real (or complex) number $\gamma(i_\circ, \cdots, i_n)$ to every $(n+1)$-tuple $(i_\circ, \cdots, i_n) \in \mathcal{N}(\mathcal{U})$. We assume that under a permutation of the indices, $\gamma(i_\circ, \cdots, i_n)$ transforms according to the sign of the permutation. It is convenient to define $\check{C}^{-1}(\mathcal{U}, \mathbf{R}) = \mathbf{R}$ and accordingly a 0-tuple (or empty set) of indices to correspond to the open set M. For every set of indices $(i_\circ, \cdots, i_n) \in \mathcal{N}(\mathcal{U})$, let $\mathcal{A}^p(i_\circ, \cdots, i_n)$ be the set of p-forms on the contractible open set $U_{i_\circ} \cap \cdots \cap U_{i_n} \neq \emptyset$ with the usual proviso that a permutation of indices multiplies the p-form by the sign of the permutation. For the empty set of indices let $\mathcal{A}^p(\emptyset)$ be the space of all p-forms on M. Now consider the double complex E with

$$E_{k,p} \ = \ \sum \mathcal{A}^p(i_\circ, \cdots, i_k),$$

where summation is over all $i_\circ < \cdots < i_k$ with $(i_\circ, \cdots, i_k) \in \mathcal{N}(\mathcal{U})$. The vertical maps are all exterior differentiation, and the horizontal maps are given by the same expression as Čech coboundary operator (3.3.4). The fact that the columns except possibly for the first are exact follows from the Poincaré lemma. The exactness of the rows, except possibly for the first, requires proof. Let $\{\phi_i\}$ be a partition of unity subordinate to the locally finite covering $\mathcal{U}$ and for a p-form valued Čech cocycle γ we define the $(p-1)$-form valued Čech cochain ψ by

$$\psi(i_\circ, \cdots, i_{p-1}) \;=\; (-1)^p \sum_j \phi_j \gamma_{i_\circ, \cdots, i_{p-1}, j},$$

where summation is over all indices j such that $(i_\circ, \cdots, i_{p-1}, j) \in \mathcal{N}(\mathcal{U})$. Then

$$\begin{aligned}
\partial_p^\star(\psi)(i_\circ, \cdots, i_p) &= \textstyle\sum_l (-1)^{l+p} \psi(k_\circ, \cdots, k_{l-1}, k_{l+1}, \cdots, k_p) \\
&= \textstyle\sum_j \phi_j \sum_l (-1)^{l+p} \gamma(k_\circ, \cdots, k_{l-1}, k_{l+1}, \cdots, k_p, j) \\
&= \textstyle\sum_j \phi_j \gamma(k_\circ, \cdots, k_p) \\
&= \gamma(i_\circ, \cdots, k_p),
\end{aligned}$$

where we have used the cocycle property in the next to last equality. This proves that the rows, except possibly the first one are also exact. Now proposition 3.5.1 is applicable. Let $H^p_{dR}(M)$ denote the cohomology groups attached to the first column, i.e.,

$$H^p_{dR}(M) \;=\; \frac{\{\omega\ p\text{−form on } M | d\omega = 0\}}{\{d\eta | \eta\ (p-1)\text{−form on } M\}}.$$

Then, the conclusion of proposition 3.5.1 becomes in this case

$$H^p_{dR}(M) \;\simeq\; \check{H}^p(M; \mathbf{R}),$$

where we have used the isomorphism $\check{H}^p(M; \mathbf{R}) \simeq \check{H}^p(\mathcal{U}, M; \mathbf{R})$, in view of the assumption on $\mathcal{U}$. All the conclusions remain valid if we use complex Čech cochains and complex valued forms on M. The vector space $H^p_{dR}(M)$ is called the p^{th} *de Rham* cohomology group of M. ♠

Exercise **??** and the example 3.5.7 implement the proof of proposition 3.5.1 in the special case where the double complex E is the one described in example 3.5.5 by assigning a 2-form to $c_1(L)$ for a complex line bundle $L \to M$. We shall see that a remarkable result emerges from this construction.

Example 3.5.6 Let $c_1(L) = (c_{jkl})$ be the first Chern class of the complex line bundle $L \to M$ with transition functions ρ_{jk} as described in example 3.3.5. Then proceeding as in example 3.5.5 to assign an element of $E_{1,1}$ to $c_1(L)$ we see that there are complex valued 1-form θ_j defined on U_j such that

$$\theta_k - \theta_j = \frac{1}{2\pi i} \frac{d\rho_{jk}}{\rho_{jk}}.$$

This means $(\theta_k - \theta_j) \in E_{1,1}$ in the notation of example 3.5.5. It follows that the 2-forms $d\theta_j$ defined on the U_j satisfy the compatibility condition $d\theta_j = d\theta_k$ on $U_j \cap U_k$ and therefore patch together to give a globally defined closed 2-form ω_L on M. Therefore $[\omega_L] \in H^2_{dR}(M)$ is the cohomology class assigned to $c_1(L)$ under the isomorphism described in example 3.5.5. ♠

Example 3.5.7 Let us specialize exercise **??** to the case where M is a complex manifold of dimension 1 and the line bundle $L \rightarrow M$ is holomorphic, i.e., ρ_{jk}'s are holomorphic functions. As in example **??** of chapter 1 we write $\rho_{jk} = \frac{f_j}{f_k}$ where

1. f_j is a meromorphic function on U_j with at most one pole or one zero;
2. Each f_j is holomorphic and nonvanishing on $U_j \cap U_k$ (if nonempty) for $k \neq j$

Denoting a local coordinate on U_j by z, we have the ususal decomposition $d = \partial + \bar{\partial}$. Set $h_j = |f_j|^{-2}$, then it is a simple calculation that, with the notation of exercise **??**,

$$\omega_L = \frac{1}{2\pi i}\partial\bar{\partial}\log h_j.$$

Let $C_j \subset U_j$ denote a small circle containing the pole or zero (if any) of f_j in its interior. By Stokes' theorem

$$\int_M \omega_L = \frac{1}{2\pi i}\sum_j \int_{C_j} \bar{\partial}\log|f_j|^{-2} = \frac{1}{2\pi i}\sum_j \int_{C_j} \frac{df_j}{f_j}.$$

Therefore from elementary complex analysis, $\int_M \omega_L = c_1(L)\lfloor M \rfloor$ is just the number of poles minus the number of zeros of the functions f_j. ♠