On Rainbow Cycles in Edge Colored Complete Graphs *†

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Abstract

In this paper we consider optimal edge colored complete graphs. We show that in any optimal edge coloring of the complete graph K_n, there is a Hamilton cycle with at most $\sqrt{8n}$ different colors. Also we prove that in every proper edge coloring of the complete graph K_n, there is a rainbow cycle with at least $n/2-1$ colors (A rainbow cycle is a cycle whose all edges have different colors). We prove that for sufficiently large n, in any optimal edge coloring of K_n, a random Hamilton cycle has approximately $(1-e^{-1})n$ different colors. Finally it is proved that if using an abelian group G, we properly edge color K_n, for odd n, then it has a rainbow Hamilton cycle.

Introduction

Graph colorings is one of the most important concepts in graph theory. In the present paper we study the existence of a Hamilton cycle with many colors, also the existence of a Hamilton cycle with few colors in any proper edge coloring

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of a complete graph. A *rainbow cycle* is a cycle whose all edges have different colors. Given an optimally edge colored complete graph with n vertices, we study the number of colors appearing on its cycles. We show that there exists a Hamilton cycle with at most $\sqrt{8n}$ colors and a Hamilton cycle with at least $n(2/3 - o(1))$ colors. A random Hamilton cycle is also shown to have $n(1 - 1/e + o(1))$ colors on average. There are examples of optimal edge colorings that have no Hamilton cycle with less than $\log_2 n$ colors. Furthermore, in some optimal edge colorings, there is no Hamilton cycle with $n-1$ or n colors. We conjecture that there is always a Hamilton cycle with at most $O(\log n)$ colors and a Hamilton cycle with at least $n-2$ colors.

In [2] it is shown that for every $\epsilon > 0$ and $n > n_0(\epsilon)$, any complete graph K_n whose edges are colored so that no vertex is incident with more than $(1 - \frac{1}{\sqrt{2}} - \epsilon)n$ edges of the same color, contains a Hamilton cycle in which adjacent edges have distinct colors. Moreover, for every k, $3 \leq k \leq n$, any such K_n contains a cycle of length k in which adjacent edges have distinct colors. Let the edges of the complete graph K_n be colored so that no color is used more than $k = k(n)$ times. This coloring it is called a *k-bounded coloring*. Clearly, if $k = 1$, every Hamilton cycle is a rainbow cycle. Hahn and Thomason showed that for the existence of a rainbow Hamilton cycle it is enough k grows as fast as $\sqrt[3]{n}$ and conjectured that the growth rate of k could in fact be linear, see [4]. Frieze and Reed in [3] proved that there is an absolute constant A such that if n is sufficiently large and k is at most $\lceil n/(A \ln\, n) \rceil$, then in any k-bounded coloring of K_n, there exists a rainbow Hamilton cycle. Also in [1] It has been shown that , if n is sufficiently large and k is at most $\lceil cn \rceil$, where $c < \frac{1}{64}$, then in any k-bounded coloring of K_n, there exists a rainbow Hamilton cycle. For any graph G, we denote the sets of the vertices and edges of G by $V(G)$ and $E(G)$, respectively. A *matching* in G is a set of edges with no shared end points. We denote the *complete graph* with n vertices by K_n. A *Hamilton cycle* of G is a cycle that contains every vertex of G. A *proper k-edge coloring* of a graph G is an assignment of k colors to the edges of G such that no two adjacent edges have the same color. The *edge chromatic number* $\chi'(G)$ of a graph G, is the minimum k for which G has a k-edge coloring. A proper edge coloring of G with $\chi'(G)$ colors is called an *optimal edge coloring.* For any $uv \in E(G)$, we denote the color of uv by $c(uv)$. It is known that $\chi'(K_n) = n - 1$ for even n and $\chi'(K_n) = n$ for odd n, see [7, p.274].

Given an abelian group G of order n, one can identify vertices of K_n with elements of G. We can then color K_n by setting $c(v_iv_j) = v_i + v_j$ for all $v_i, v_j \in G$ where "+" denotes the operation of the group. The coloring is well-defined since G is abelian, and is proper since every element of G has an inverse. Moreover, it is optimal for odd n. We call this coloring the edge coloring of K_n *with respect to* G. We study the existence of rainbow cycles of different lengths in optimally edge colored complete graphs. As an example, we show that if the coloring is with respect to the abelian group $\mathbb{Z}_p$ where p is an odd prime, then for any possible length, there exists a rainbow cycle of that length. We also prove that every edge coloring with respect to an abelian group of odd order has a rainbow cycle of length $n - 1$. Finally, it is shown that there is always a rainbow cycle with length at least $n/2 - 1$. Note that existence of long rainbow cylces implies existence of Hamilton cycles with many colors.

Results

We start this section by showing that in any optimal edge colorings of the complete graph, there exists a Hamilton cycles with few colors. We accomplish this goal by carefully analyzing a gready algorithm that tries to construct a Hamilton cycle with as few colors as possible.

Theorem 1. *In any optimal edge coloring of the complete graph K_n, there is a*

Hamilton cycle with at most $\sqrt{8n}$ different colors.

Proof. Our proof relies on the following observation: Let $P_1, \ldots, P_k$ be k vertex-disjoint paths that cover $V(K_n)$. For $1 \le j \le k$, let v_j be an endpoint of P_j. There are $\binom{k}{2}$ edges connecting the v_j's, and $\chi'(K_n) \le n$ colors are used. We can thus find a set S of at least $k(k-1)/(2n)$ edges, all of the same color and all connecting the v_j's. Evidently, adding S to $P_1, \ldots, P_k$ decreases the number of paths by at least $k(k-1)/(2n)$ and increases the number of distinct colors that appear on the paths by at most one. In addition, the paths are still vertex-disjoint and cover $V(K_n)$.

To begin, let P_i be the path of length zero formed by the ith vertex of K_n for $1 \le i \le n$. Clearly, $P_1, \ldots, P_n$ cover $V(K_n)$ and are vertex-disjoint. Let us set $x_0 = n$. By the above observation and induction on $i \ge 0$, there are x_i vertex-disjoint paths that use at most i colors and cover $V(K_n)$, and $x_{i+1} \le x_i - x_i(x_i-1)/(2n)$.

Clearly, the function $1/(x(x-1))$ is decreasing in the range $x > 1$. Hence if m is a nonnegative integer and $x_m > 1$, we have

$$m \le \sum_{i=0}^{m-1} \frac{2n(x_i - x_{i+1})}{x_i(x_i-1)} \le \int_{x_m}^{x_0} \frac{2n}{x(x-1)} dx \le \int_{x_m}^{x_0} \frac{2n}{(x-1)^2} dx = \frac{2n}{x_m - 1} - \frac{2n}{n-1},$$

which implies that $x_m \le 1 + 2n/(m+2)$ for all nonnegative integers m. Recall that there are x_m vertex-disjoint paths with at most m colors on the edges which cover $V(K_n)$. We can connect these paths to form a Hamilton cycle by adding x_m edges; the resulting Hamilton cycle has at most $m + x_m$ colors. For $m = \lceil \sqrt{2n} \rceil - 2$, we have $x_m \le \sqrt{2n} + 1$. Thus at most $(\lceil \sqrt{2n} \rceil - 2) + (\sqrt{2n} + 1) \le 2\sqrt{2n}$ colors appear on the Hamilton cycle. □

Assume K_n is optimally edge colored. If n edges are randomly selected from $E(K_n)$ with replacement, then for every color c, the probability that none of the n edges have color c is equal to $(1 - 1/\chi'(K_n))^n$, or approximately equal to $1/e$ for large n.

Therefore the number of distinct colors appearing on the n edges is approximately $(1 - 1/e)n$. The following theorem shows that constraining the n random edges to form a Hamilton cycle does not have a large impact on this average number of distinct colors.

Theorem 2. *Given an optimal edge coloring of the complete graph K_n, the expected number of different colors that appear on the edges of a random Hamilton cycle of K_n is approximately equal to $(1 - e^{-1})n$, for large enough n.*

Proof. Let c be an arbitrary color used in the given optimal edge coloring of K_n and let C be the set of edges whose colors are c. The edges in C are a matching of size $\lfloor n/2 \rfloor$. Clearly K_n has $(n-1)!/2$ Hamilton cycles.

Assume S is a subset of C with size k. We can count the number of Hamilton cycles that contain S by considering the following transformation: For each edge in S, contract its two endpoints into a single vertex. If H is a Hamilton cycle of K_n that contains S, its transform is a Hamilton cycle of the graph K_{n-k}. Furthermore, every Hamilton cycle of K_{n-k} is the transform of exactly 2^k Hamilton cycles of K_n that contain S, because the directions of the alignments of the edges of S in H have no impact on the transform of H. Consequently, there are $2^{k-1}(n-k-1)!$ Hamilton cycles in K_n that contain S. Thus the probability that a random Hamilton cycle contains S is $2^k(n-k-1)!/(n-1)!$.

The principle of inclusion and exclusion now implies that the probability of the event that a random Hamilton cycle avoids all edges in C is

$$p = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k a_k, \tag{1}$$

$$\text{where, } a_k = \binom{\lfloor \frac{n}{2} \rfloor}{k} \frac{2^k(n-k-1)!}{(n-1)!} = \frac{1}{k!} \prod_{j=0}^{k-1} \frac{2(\lfloor \frac{n}{2} \rfloor - j)}{n-j-1}.$$

It is not hard to see that $a_k \leq 1/k!$ for $k \geq 3$. For any real number x, we have $1 + x \leq e^x$; hence for $k \leq n^{1/3}$ we have

$$a_k = \frac{1}{k!}\prod_{j=0}^{k-1}(1+\frac{(n-j-1)-2(\lfloor\frac{n}{2}\rfloor-j)}{2(\lfloor\frac{n}{2}\rfloor-j)})^{-1}$$

$$\geq \frac{1}{k!}\prod_{j=0}^{k-1}(1+\frac{j}{2(\lfloor\frac{n}{2}\rfloor-k)})^{-1} \geq \frac{1}{k!}\prod_{j=0}^{k-1}\exp(-\frac{j}{2(\lfloor\frac{n}{2}\rfloor-k)}) =$$

$$\frac{1}{k!}\exp(-\frac{k(k-1)}{4(\lfloor\frac{n}{2}\rfloor-k)}) \geq \frac{1}{k!}\exp(\frac{n^{2/3}}{4\lfloor\frac{n}{2}\rfloor-n^{1/3}}) = \frac{1}{k!}(1+o(1)),$$

where $o(1)$ is a function in terms of n and k that becomes arbitrarily small as n gets large. Splitting the right hand side of Equation 1 into two sums as in

$$p \;=\; \sum_{k=0}^{\lfloor n^{1/3}\rfloor}(-1)^k a_k \;+\; \sum_{k=\lfloor n^{1/3}\rfloor+1}^{\lfloor\frac{n}{2}\rfloor}(-1)^k a_k,$$

we note that the Taylor expansion of e^x for $x = -1$ yields

$$\begin{aligned}\sum_{k=0}^{\lfloor n^{1/3}\rfloor}(-1)^k a_k &= \sum_{k=0}^{\lfloor n^{1/3}\rfloor}\frac{(-1)^k}{k!}(1+o(1)) = \sum_{k=0}^{\lfloor n^{1/3}\rfloor}\frac{(-1)^k}{k!} + \sum_{k=0}^{\lfloor n^{1/3}\rfloor}\frac{(-1)^k\,o(1)}{k!} \\ &= (e^{-1}+o(1))+o(1) = e^{-1}+o(1).\end{aligned}$$

On the other hand, we have

$$|\sum_{k=\lfloor n^{1/3}\rfloor+1}^{\lfloor\frac{n}{2}\rfloor}(-1)^k a_k\,| \leq \sum_{k=\lfloor n^{1/3}\rfloor+1}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k!} = o(1),$$

since the series $\sum_{k=0}^{\infty}\frac{1}{k!}$ is convergent. Thus we get $p = e^{-1} + o(1)$. We know $n - 1 \leq \chi'(K_n) \leq n$, and the color c appears on a random Hamilton cycle with probability $1-p$. Thus we expect that $\chi'(K_n)(1-p) = n(1-e^{-1})(1+o(1))$ different colors appear on a random Hamilton cycle on average. $\square$

Theorem 3. *In any optimal edge coloring of K_n, there is a Hamilton cycle with at least $n(2/3 - o(1))$ colors.*

Proof. Suppose n is even. Without loss of generality assume $\{1, 2, \cdots, n-1\}$ is the set of colors used. Let A be an $n \times n$ square matrix where $A_{ij} = c(v_i v_j)$ if $1 \le i, j \le n$ and $i \ne j$, and $A_{ii} = n$ if $1 \le i \le n$. Clearly, A is a Latin square. By a result in [6], we can select $n - O(\log^2 n)$ entries of A that have different values, and are located on distinct rows and columns. Of these entries, only one can be on the diagonal. Furthermore, if A_{ij} is selected then A_{ji} can not be selected since values appearing on the selected entries should be distinct. This means that we can select $n - O(\log^2 n) - 1$ edges of K_n such that the selected edges have different colors, and the degree of each vertex is at most two in the graph induced by these edges. That is, the selected edges consist of vertex-disjoint paths and cycles. Every cycle has at least 3 edges. Thus we can delete one edge of every cycle to get $(2/3)(n - O(\log^2 n) - 1)$ edges forming vertex-disjoint paths. We can connect these paths to get a Hamilton cycle with at least $(2/3)(n - O(\log^2 n) - 1) = (2/3 - o(1))n$ colors.

When n is odd, K_n is colored using n colors. Moreover, for any of these n colors, exactly one vertex is not an endpoint of an edge of that color. Thus, we can extend the optimal edge coloring of K_n to an optimal edge coloring of K_{n+1}. Since $n+1$ is even, K_{n+1} has a Hamilton cycle with at least $(2/3 - o(1))(n+1)$ colors. We can now trivially construct a Hamilton cycle for K_n that has $(2/3 - o(1))n$ colors. $\square$

We examined, to some extent, the number of colors on Hamilton cycles. We will now consider long, but not necessarily Hamilton, rainbow cycles.

Theorem 4. *In any proper edge coloring of K_n $(n \ge 3)$, there is a rainbow cycle with length at least $n/2 - 1$.*

Proof. Let C be the longest rainbow cycle, and assume C has length $t < n/2 - 1$. We say that an edge in K_n is *good* if its color does not appear in C. Let u and v be two adjacent vertices of C. Each of u and v is adjacent to at most $t-3$ vertices of C by good edges, and each vertex is adjacent to at least $(n-1)-t$ good edges. Thus each of u and v is adjacent to at least $(n-1-t)-(t-3) \geq 3$ vertices of $V(K_n)\backslash V(C)$ by good edges. Hence we can find two distinct vertices $w, p \in V(K_n)\backslash V(C)$ such that uw and vp are good edges, and $c(uw) \neq c(vp)$.

We can assume that wp is not a good edge, since otherwise uw, wp, and pw can replace uv in C to form a rainbow cycle of length $t+2$, a contradiction. If two good edges connect w to two adjacent vertices $x, y \in V(C)$, one can obtain a rainbow cycle of length $t+1$ by using xw and wy instead of xy, another contradiction. It follows that the number of good edges joining w to the vertices of C is at most $t/2$. Similarly, there are at most $t/2$ good edges joining p to the vertices of C.

Using the fact that wp is not a good edge, there are at least $2((n-t-1)-t/2-1)$ good edges that connect $\{w, p\}$ to $V(K_n)\backslash(V(C) \cup \{w, p\})$ and have colors different from $c(wu)$ and $c(vp)$. Since $2((n-t-1)-t/2-1) > n-(t+2)$, there is a vertex $q \in V(K_n)\backslash(V(C) \cup \{w, p\})$ such that wq and pq are good, $c(wq) \neq c(vp)$, and $c(uw) \neq c(pq)$. Replacing uv by uw, wq, qp, and pq, we obtain a rainbow cycle longer than C. It implies that $t \geq n/2-1$. $\square$

We will now prove, in a series of theorems, several results on the existence of rainbow cycles of different lengths in colorings obtained from finite abelian groups.

Theorem 5. *Let p be a prime number and $n = p^m$ $(n \neq 3)$. Consider the edge coloring of K_n with respect to $\mathbb{Z}_p^m = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. Then K_n has a rainbow cycle with length $n-1$.*

Proof. Let F be a finite field of order p^m. Assume that α is a generator for cyclic group $F^* = F\backslash\{0\}$. First note that the additive group of F is isomorphic to $\mathbb{Z}_p^m$. We prove that $1, \alpha, \ldots, \alpha^{p^m-1}$ is a rainbow Hamilton cycle. To prove this assume that $i \neq j$ and $\alpha^i + \alpha^{i+1} = \alpha^j + \alpha^{j+1}$. From this equality we conclude that $\alpha^i(\alpha + 1) = \alpha^j(\alpha + 1)$. Therefore $\alpha^i = \alpha^j$ and it follows that $i = j$, which is a contradiction. $\square$

Theorem 6. *If p is an odd prime number and K_p is edge colored with respect to $\mathbb{Z}_p$, then for any r, $3 \leq r \leq p$, there is a rainbow cycle of length r.*

Proof. Consider the edge coloring for K_p, with respect to $\mathbb{Z}_p$. First assume that r is odd and $3 \leq r \leq p$. Consider the cycle formed by the labels $1, \ldots, r$. Since for any i, j, $1 \leq i, j \leq r - 1$, $i \neq j$, we have $2i + 1 \neq r + 1$, $2i + 1 \neq 2j + 1$, and this implies that this cycle is a rainbow cycle of length r. Next suppose that r is even and $3 < r < p$. If $r = p - 1$, then by Theorem 5, we have a rainbow cycle of length $p - 1$. If $r \leq \frac{p-1}{2}$, consider the cycle formed by the labels $1, 2, \ldots, r-1, r+1$. Since for any i, $1 \leq i \leq r - 2$, we have $2i + 1 \neq 2r$, $2i + 1 \neq r + 2$, $2r \neq r + 2$, and $2i + 1 \neq 2j + 1$ for any $i \neq j$, the given cycle is rainbow of length r. If $\frac{p+1}{2} \leq r < p - 1$, consider the cycle labeled by the numbers $1, 2, \ldots, \frac{p-1}{2}, \frac{p+3}{2}, \ldots, r + 1$. Since for any i, $1 \leq i \leq r$, $i \neq \frac{p-1}{2}, \frac{p+1}{2}$, $2i + 1 \neq r + 2$, $2i + 1 \neq p + 1$, $r + 2 \neq p + 1$ and $2i + 1 \neq 2j + 1$, for any $i \neq j$, so the given cycle is rainbow of length r. $\square$

Theorem 7. *Let p be an odd prime number and $n = p^m$. Consider the edge coloring of K_n with respect to $\mathbb{Z}_p^m = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. Then the following hold:*

(i) K_n has a rainbow Hamilton cycle.

(ii) If $l \,|\, n - 1$ and $l > 2$, then the vertices $K_n \backslash \{(0, \ldots, 0)\}$ can be covered by $\frac{n-1}{l}$ rainbow cycle.

Proof. (i) We apply induction on m. By Theorem 6 the assertion is true for $m = 1$. Now suppose that the assertion is true for $\mathbb{Z}_p^{m-1}$, we show that the assertion is also true for $\mathbb{Z}_p^m$. Assume that $l = p^{m-1}$ and $\alpha_1, \ldots, \alpha_l$ are those elements of $\mathbb{Z}_p^l$ such that

$\alpha_i + \alpha_{i+1} \neq \alpha_j + \alpha_{j+1}$, for any $i, j, 1 \leq i, j \leq l$, $i \neq j$. We claim that the following Hamilton cycle is a rainbow Hamilton cycle.

$$(\alpha_1, 0), \ldots, (\alpha_l, 0), (\alpha_1, 1), \ldots, (\alpha_l, 1), \ldots, (\alpha_1, p-1), \ldots, (\alpha_l, p-1)$$

we note that if $r \neq s$ or $i \neq j$, then $(\alpha_i + \alpha_{i+1}, 2r) \neq (\alpha_j + \alpha_{j+1}, 2s)$. Assume that $(\alpha_l, t) + (\alpha_1, t+1) = (\alpha_i, q) + (\alpha_{i+1}, q) = (\alpha_i + \alpha_{i+1}, 2t+1)$ or $(\alpha_l, t) + (\alpha_1, t+1) = (\alpha_l, s) + (\alpha_1, s+1)$. In the second case $s = t$ and there is nothing to prove. In the first case $i = l$, a contradiction.

(ii) Since $l \,|\, n-1$, there is a cyclic subgroup H of F^*, such that $|H| = l$. Assume that H is generated by β. If $H\alpha^t$ is a right coset of H in F^*, then the elements of $\alpha^t, \beta\alpha^t, \ldots, \beta^{l-1}\alpha^t$ forms a rainbow cycle of length l, because if

$$\beta^i\alpha^t + \beta^{i+1}\alpha^t = \beta^j\alpha^t + \beta^{j+1}\alpha^t,$$

then we find $\beta^i(\beta+1) = \beta^j(\beta+1)$, that is $i = j$. □

Theorem 8. *If G is an abelian group of order n and n is odd, then the edge coloring of K_n with respect to G has a rainbow Hamilton cycle.*

Proof. By cyclic decomposition of finite abelian groups, we have $G \simeq \mathbb{Z}_{p_1^{m_1}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$, where p_i's are prime numbers (not necessarily distinct), see [5, p.109]. If $k = 1$, the assertion follows from Theorem 7. Otherwise by elementary group theory, there is an abelian group H of odd order such that $G = H \times \mathbb{Z}_{p^m}$. Now if $a_1, \ldots, a_h$ $(h = |H|)$ is a rainbow Hamilton cycle in K_h which has been colored with respect to H, then as we saw in the proof of Theorem 7 Part (i),

$$(a_1, 0), \ldots, (a_h, 0), \ldots, (a_1, p^m - 1), \ldots, (a_h, p^m - 1)$$

is a rainbow Hamilton cycle in K_n.

□

We continue by a conjecture on the existence of Hamilton cycles with few different colors in any optimal edge coloring of K_n.

Conjecture 1. *There is a positive constant* c *such that in any optimal edge coloring of* K_n*, there exists a Hamilton cycle with at most* $c \log_2 n$ *different colors.*

The following shows that if the above conjecture is correct, then $c \geq 1$.

Lemma 1. *If* $n = 2^m$ $(m \geq 2)$ *and* K_n *is edge colored with respect to* $\mathbb{Z}_2^m$*, then the edge coloring has no rainbow Hamilton path. Moreover there are no rainbow cycles of length* $n, n-2$ *and* $n-3$*, but there is a rainbow cycle of length* $n-1$*. Also every Hamilton cycle has at least* m *different colors.*

Proof. First note that the color of no edge is 0. Let $v_1, \ldots, v_n$ be a rainbow Hamilton path. If we add all colors appeared on the edges of this Hamitonian path, we obtain the number $1+2+\cdots+(2^m-1)$. On the other hand, for any i, the number of elements of $\mathbb{Z}_{2^m}$ in which the i-th components are 1 is even, so this number should be zero. Furthermore since $c(v_iv_j) = v_i + v_j$, we conclude that $v_1 + v_n + 2\sum_{i=2}^{n-1} v_i = 0$ and it yields that $v_1 = v_n$, a contradiction. Also since the number of colors appeared in the edges is $n-1$, there is no a rainbow Hamilton cycle.

Now let $v_1, v_2, \ldots, v_{n-2}$ be a rainbow cycle of length $n-2$. Suppose that just color $0 \neq a$ is not appeared on its edges. If we add all colors appeared on the edges of this cycle, we find $2\sum_{i=1}^{n-2} v_i = a$ and this implies that $a = 0$, which is impossible.

Next assume that $v_1, v_2, \ldots, v_{n-3}$, is a rainbow cycle of length $n-3$. Let a and b be the only colors which are not appeared on the edges of this cycle. If we add all colors appeared on the edges of this cycle, we find $2\sum_{i=1}^{n-3} v_i = a + b$ and this implies that $a = b$, a contradiction.

By Theorem 5, K_n with the given edge coloring has a rainbow cycle of length $n-1$. We claim that every Hamilton cycle in this edge coloring of K_n has at least m

different colors. Using the fact that for any two consecutive vertices v_i and v_j of this Hamilton cycle $c(v_iv_j) = v_i + v_j$, and $v_i + (v_i + v_j) = v_j$, if W is the vector space over $\mathbb{Z}_2$ generated by all colors appeared on the edges of a Hamilton cycle, then we have $\dim W \leq m - 1$. It follows that $|W| \leq 2^{m-1}$, but $\{v_1, v_2, \ldots, v_n\} \subseteq v_1 + W$ and this contradicts $n = 2^m$. Hence every Hamilton cycle has at least m different colors. $\square$

Conjecture 2. *In any optimal edge coloring of K_n, there is a Hamilton cycle with at least $n - 2$ different colors.*

By a reformulation of the above conjecture we may say that in any optimal edge coloring of K_n, there is a Hamilton path with at least $n - 2$ different colors.

We now go on to following questions.

Question 1. Is it true that in any optimal edge coloring of K_n, there is a rainbow cycle of size at least $n - 2$?

Question 2. Is it true that in any optimal edge coloring of K_n, there is a rainbow path of size at least $n - 2$?

Besides being interesting in its own right, the truth of these questions would support Conjecture 2.

In closing the paper we want to discuss some special cases in the above questions. We note that there is an optimal edge coloring for K_8 which has a rainbow path of length 6 but no rainbow path of length 7. To see this consider the following symmetric matrix $A = [a_{ij}]$,

$$\begin{array}{cccccccc}
* & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & * & 3 & 2 & 5 & 4 & 7 & 6 \\
2 & 3 & * & 1 & 6 & 7 & 4 & 5 \\
3 & 2 & 1 & * & 7 & 6 & 5 & 4 \\
4 & 5 & 6 & 7 & * & 1 & 2 & 3 \\
5 & 4 & 7 & 6 & 1 & * & 3 & 2 \\
6 & 7 & 4 & 5 & 2 & 3 & * & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & *
\end{array}$$

If $V(K_8) = \{v_1, \ldots, v_8\}$ and set $c(v_i v_j) = a_{ij}$, for any i, j, $1 \leq i, j \leq 8, i \neq j$, we obtain an optimal edge coloring as desired. Indeed the path $v_1, v_8, v_7, v_6, v_2, v_5, v_3$ is a rainbow path with length 6. Similarly by considering the matrix $B = [b_{ij}]$ as follows,

$$\begin{array}{cccccccc}
* & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & * & 3 & 2 & 7 & 6 & 5 & 4 \\
2 & 3 & * & 1 & 5 & 4 & 7 & 6 \\
3 & 2 & 1 & * & 6 & 7 & 4 & 5 \\
4 & 7 & 5 & 6 & * & 1 & 2 & 3 \\
5 & 6 & 4 & 7 & 1 & * & 3 & 2 \\
6 & 5 & 7 & 4 & 2 & 3 & * & 1 \\
7 & 4 & 6 & 5 & 3 & 2 & 1 & *
\end{array}$$

we obtain an optimal edge coloring for K_8 with no rainbow cycle of length 7 which contains a rainbow cycle with length 6. Indeed the cycle $v_1, v_8, v_7, v_2, v_6, v_3$ is a rainbow cycle of length 6.

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