

ADDING A LOT OF COHEN REALS BY ADDING A FEW I

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ABSTRACT. In this paper we produce models $V_1 \subseteq V_2$ of set theory such that adding κ -many Cohen reals to V_2 adds λ -many Cohen reals to V_1 , for some $\lambda > \kappa$. We deal mainly with the case when V_1 and V_2 have the same cardinals.

1. INTRODUCTION

A basic fact about Cohen reals is that adding λ -many Cohen reals cannot produce more than λ -many of Cohen reals¹. More precisely, if $\langle s_\alpha : \alpha < \lambda \rangle$ are λ -many Cohen reals over V , then in $V[\langle s_\alpha : \alpha < \lambda \rangle]$ there are no λ^+ -many Cohen reals over V . But if instead of dealing with one universe V we consider two, then the above may no longer be true.

The purpose of this paper is to produce models $V_1 \subseteq V_2$ such that adding κ -many Cohen reals to V_2 adds λ -many Cohen reals to V_1 , for some $\lambda > \kappa$. We deal mainly with the case when V_1 and V_2 have the same cardinals.

2. MODELS WITH THE SAME REALS

In this section we produce models $V_1 \subseteq V_2$ as above with the same reals. We first state a general result.

Theorem 2.1. *Let V_1 be an extension of V . Suppose that in V_1 :*

- (a) $\kappa < \lambda$ are infinite cardinals,
- (b) λ is regular,
- (c) there exists an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals cofinal in κ ; in particular $cf(\kappa) = \omega$,

¹By “ λ -many Cohen reals” we mean “a generic object $\langle s_\alpha : \alpha < \lambda \rangle$ for the poset $\mathbb{C}(\lambda)$ of finite partial functions from $\lambda \times \omega$ to 2 ”.

(d) there exists an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$,²

and

(e) there exists a club $C \subseteq \lambda$ which avoids points of countable V -cofinality.

Then adding κ -many Cohen reals over V_1 produces λ -many Cohen reals over V .

Proof. We consider two cases.

Case 1. $\lambda = \kappa^+$. Force to add κ -many Cohen reals over V_1 . Split them into two sequences of length κ denoted by $\langle r_\iota : \iota < \kappa \rangle$ and $\langle r'_\iota : \iota < \kappa \rangle$. Also let $\langle f_\alpha : \alpha < \kappa^+ \rangle \in V_1$ be an increasing (mod finite) sequence in $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$. Let $\alpha < \kappa^+$. We define a real s_α as follows:

Case 1. $\alpha \in C$. Then

$$\forall n < \omega, s_\alpha(n) = r_{f_\alpha(n)}(0).$$

Case 2. $\alpha \notin C$. Let α^* and α^{**} be two successive points of C so that $\alpha^* < \alpha < \alpha^{**}$. Let $\langle \alpha_\iota : \iota < \kappa \rangle$ be some fixed enumeration of the interval (α^*, α^{**}) . Then for some $\iota < \kappa$, $\alpha = \alpha_\iota$. Let $k(\iota) = \min\{k < \omega : r'_\iota(k) = 1\}$. Set

$$\forall n < \omega, s_\alpha(n) = r_{f_\alpha(k(\iota)+n)}(0).$$

The following lemma completes the proof.

Lemma 2.2. $\langle s_\alpha : \alpha < \kappa^+ \rangle$ is a sequence of κ^+ -many Cohen reals over V .

Notation 2.3. For each set I , let $\mathbb{C}(I)$ be the Cohen forcing notion for adding I -many Cohen reals. Thus $\mathbb{C}(I) = \{p : p \text{ is a finite partial function from } I \times \omega \text{ into } 2\}$, ordered by reverse inclusion.

²Note that condition (d) holds automatically for $\lambda = \kappa^+$; given any collection \mathcal{F} of κ -many elements of $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$ there exists f such that for each $g \in \mathcal{F}$, $f(n) > g(n)$ for all large n . Thus we can define, by induction on $\alpha < \kappa^+$, an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ in $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$.

Proof. First note that $\langle \langle r_\iota : \iota < \kappa \rangle, \langle r'_\iota : \iota < \kappa \rangle \rangle$ is $\mathbb{C}(\kappa) \times \mathbb{C}(\kappa)$ -generic over V_1 . By c.c.c of $\mathbb{C}(\kappa^+)$ it suffices to show that for any countable set $I \subseteq \kappa^+$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I \rangle$ is $\mathbb{C}(I)$ -generic over V . Thus it suffices to prove the following:

for every $(p, q) \in \mathbb{C}(\kappa) \times \mathbb{C}(\kappa)$ and every open dense subset $D \in$
 (*) V of $\mathbb{C}(I)$, there is $(\bar{p}, \bar{q}) \leq (p, q)$ such that $(\bar{p}, \bar{q}) \Vdash \langle \check{s}_\alpha : \alpha \in I \rangle$
 extends some element of D .

Let (p, q) and D be as above. For simplicity suppose that $p = q = \emptyset$. By (e) there are only finitely many $\alpha^* \in C$ such that $I \cap [\alpha^*, \alpha^{**}) \neq \emptyset$, where $\alpha^{**} = \min(C \setminus (\alpha^* + 1))$. For simplicity suppose that there are two $\alpha_1^* < \alpha_2^*$ in C with this property. Let $n^* < \omega$ be such that for all $n \geq n^*$, $f_{\alpha_1^*}(n) < f_{\alpha_2^*}(n)$. Let $p \in \mathbb{C}(\kappa)$ be such that

$$\text{dom}(p) = \{ \langle \beta, 0 \rangle : \exists n < n^* (\beta = f_{\alpha_1^*}(n) \text{ or } \beta = f_{\alpha_2^*}(n)) \}.$$

Then for $n < n^*$ and $j \in \{1, 2\}$,

$$(p, \emptyset) \Vdash \check{s}_{\alpha_j^*}(n) = \check{r}_{f_{\alpha_j^*}(n)}(0) = p(f_{\alpha_j^*}(n), 0)$$

Thus (p, \emptyset) decides $s_{\alpha_1^*} \upharpoonright n^*$ and $s_{\alpha_2^*} \upharpoonright n^*$. Let $b \in D$ be such that $\langle b(\alpha_1^*), b(\alpha_2^*) \rangle$ extends $\langle s_{\alpha_1^*} \upharpoonright n^*, s_{\alpha_2^*} \upharpoonright n^* \rangle$, where $b(\alpha)$ is defined by $b(\alpha) : \{n : (\alpha, n) \in \text{dom}(b)\} \rightarrow 2$ and $b(\alpha)(n) = b(\alpha, n)$. Let

$$p' = p \cup \bigcup_{j \in \{1, 2\}} \{ \langle f_{\alpha_j^*}(n), 0, b(\alpha_j^*, n) \rangle : n \geq n^*, (\alpha_j^*, n) \in \text{dom}(b) \}.$$

Then $p' \in \mathbb{C}(\kappa)$ ³ and

$$(p', \emptyset) \Vdash \langle \check{s}_{\alpha_1^*}, \check{s}_{\alpha_2^*} \rangle \text{ extends } \langle b(\alpha_1^*), b(\alpha_2^*) \rangle.$$

For $j \in \{1, 2\}$, let $\{\alpha_{j_0}, \dots, \alpha_{j_{k_j-1}}\}$ be an increasing enumeration of components of b in the interval $(\alpha_j^*, \alpha_j^{**})$ (i.e. those $\alpha \in (\alpha_j^*, \alpha_j^{**})$ such that $(\alpha, n) \in \text{dom}(b)$ for some n). For $j \in \{1, 2\}$ and $l < k_j$ let $\alpha_{jl} = \alpha_{\iota_{jl}}$ where $\iota_{jl} < \kappa$ is the index of α_{jl} in the enumeration of the interval $(\alpha_j^*, \alpha_j^{**})$ considered in Case 2 above. Let $m^* < \omega$ be such that for all $n \geq m^*$, $j \in \{1, 2\}$ and $l_j < l'_j < k_j$ we have

³This is because for $n \geq n^*$, $f_{\alpha_1^*}(n) \neq f_{\alpha_2^*}(n)$ and for $j \in \{1, 2\}$, $f_{\alpha_j^*}(n) \notin \{f_{\alpha_j^*}(m) : m < n\}$, thus there are no collisions.

$$f_{\alpha_1^*}(n) < f_{\alpha_{1\ell_1}}(n) < f_{\alpha_{1\ell_1'}}(n) < f_{\alpha_2^*}(n) < f_{\alpha_{2\ell_2}}(n) < f_{\alpha_{2\ell_2'}}(n).$$

Let

$$\bar{q} = \{\langle \iota_{jl}, n, 0 \rangle : j \in \{1, 2\}, l < k_j, n < m^* \} \cup \{\langle \iota_{jl}, m^*, 1 \rangle : j \in \{1, 2\}, l < k_j \}.$$

Then $\bar{q} \in \mathbb{C}(\kappa)$ and for $j \in \{1, 2\}$ and $n < m^*$, $(\emptyset, \bar{q}) \Vdash \text{“}r'_{\iota_{jl}}(n) = 0 \text{ and } r'_{\iota_{jl}}(m^*) = 1\text{”}$, thus $(\emptyset, \bar{q}) \Vdash k(j, l) = \min\{k < \omega : r'_{\iota_{jl}}(k) = 1\} = m^*$. Let

$$\bar{p} = p' \cup \bigcup_{j \in \{1, 2\}} \{\langle f_{\alpha_{jl}}(m^* + n), 0, b(\alpha_{jl}, n) \rangle : l < k_j, (\alpha_{jl}, n) \in \text{dom}(b)\}.$$

It is easily seen that $\bar{p} \in \mathbb{C}(\kappa)$ is well-defined and for $j \in \{1, 2\}$ and $l < k_j$,

$$(\bar{p}, \bar{q}) \Vdash \text{“}\mathfrak{s}_{\alpha_{jl}} \text{ extends } b(\alpha_{jl})\text{”}.$$

Thus

$$(\bar{p}, \bar{q}) \Vdash \text{“}\langle \mathfrak{s}_\alpha : \alpha \in I \rangle \text{ extends } b\text{”}.$$

(*) follows and we are done. \square

Case $\lambda > \kappa^+$. Force to add κ -many Cohen reals over V_1 . We now construct λ -many Cohen reals over V as in the above case using C and $\langle f_\alpha : \alpha < \lambda \rangle$. Case 2 of the definition of $\langle s_\alpha : \alpha < \lambda \rangle$ is now problematic since the cardinality of an interval (α^*, α^{**}) (using the above notation) may now be above κ and we have only κ -many Cohen reals to play with. Let us proceed as follows in order to overcome this.

Let us rearrange the Cohen reals as $\langle r_{n,\alpha} : n < \omega, \alpha < \kappa \rangle$ and $\langle r_\eta : \eta \in [\kappa]^{<\omega} \rangle$. We define by induction on levels a tree $T \subseteq [\lambda]^{<\omega}$, its projection $\pi(T) \subseteq [\kappa]^{<\omega}$ and for each $n < \omega$ and $\alpha \in \text{Lev}_n(T)$ a real s_α . The union of the levels of T will be λ so $\langle s_\alpha : \alpha < \lambda \rangle$ will be defined.

For $n = 0$, let $\text{Lev}_0(T) = \langle \rangle = \text{Lev}_0(\pi(T))$.

For $n = 1$, let $\text{Lev}_1(T) = C, \text{Lev}_1(\pi(T)) = \{0\}$, i.e. $\pi(\langle \alpha \rangle) = \langle 0 \rangle$ for every $\alpha \in C$. For $\alpha \in C$ we define a real s_α by

$$\forall m < \omega, s_\alpha(m) = r_{1, f_\alpha(m)}(0).$$

Suppose now that $n > 1$ and $T \upharpoonright n$ and $\pi(T) \upharpoonright n$ are defined. We define $Lev_n(T)$, $Lev_n(\pi(T))$ and reals s_α for $\alpha \in Lev_n(T)$. Let $\eta \in T \upharpoonright (n-1)$, $\alpha^*, \alpha^{**} \in Suc_T(\eta)$ and $\alpha^{**} = \min(Suc_T(\eta) \setminus (\alpha^* + 1))$. We then define $Suc_T(\eta \frown \langle \alpha^{**} \rangle)$ if it is not yet defined ⁴.

Case A. $|\alpha^{**} \setminus \alpha^*| \leq \kappa$.

Fix some enumeration $\langle \alpha_\iota : \iota < \rho \leq \kappa \rangle$ of $\alpha^{**} \setminus \alpha^*$. Let

- $Suc_T(\eta \frown \langle \alpha^{**} \rangle) = \alpha^{**} \setminus \alpha^*$,
- $Suc_T(\eta \frown \langle \alpha^{**} \rangle \frown \langle \alpha \rangle) = \langle \rangle$ for $\alpha \in \alpha^{**} \setminus \alpha^*$,
- $Suc_{\pi(T)}(\pi(\eta \frown \langle \alpha^{**} \rangle)) = \rho = |\alpha^{**} \setminus \alpha^*|$,
- $Suc_{\pi(T)}(\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle \iota \rangle) = \langle \rangle$ for $\iota < \rho$.

Now we define s_α for $\alpha \in \alpha^{**} \setminus \alpha^*$. Let ι be such that $\alpha = \alpha_\iota$. let $k = \min\{m < \omega : r_{\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle \iota \rangle}(m) = 1\}$, Finally let

$$\forall m < \omega, s_\alpha(m) = r_{n, f_\alpha(k+m)}(0).$$

Case B. $|\alpha^{**} \setminus \alpha^*| > \kappa$ and $cf(\alpha^{**}) < \kappa$.

Let $\rho = cf \alpha^{**}$ and let $\langle \alpha_\nu^{**} : \nu < \rho \rangle$ be a normal sequence cofinal in α^{**} with $\alpha_0^{**} > \alpha^*$.

Let

- $Suc_T(\eta \frown \langle \alpha^{**} \rangle) = \{\alpha_\nu^{**} : \nu < \rho\}$,
- $Suc_{\pi(T)}(\pi(\eta \frown \langle \alpha^{**} \rangle)) = \rho$.

Now we define $s_{\alpha_\nu^{**}}$ for $\nu < \rho$. Let $k = \min\{m < \omega : r_{\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle \nu \rangle}(m) = 1\}$ and let

$$\forall m < \omega, s_{\alpha_\nu^{**}}(m) = r_{n, f_{\alpha_\nu^{**}}(k+m)}(0).$$

Case C. $cf(\alpha^{**}) > \kappa$.

Let ρ and $\langle \alpha_\nu^{**} : \nu < \rho \rangle$ be as in Case B. Let

- $Suc_T(\eta \frown \langle \alpha^{**} \rangle) = \{\alpha_\nu^{**} : \nu < \rho\}$,
- $Suc_{\pi(T)}(\pi(\eta \frown \langle \alpha^{**} \rangle)) = \langle 0 \rangle$.

We define $s_{\alpha_\nu^{**}}$ for $\nu < \rho$. Let $k = \min\{m < \omega : r_{\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle 0 \rangle}(m) = 1\}$ and let

$$\forall m < \omega, s_{\alpha_\nu^{**}}(m) = r_{n, f_{\alpha_\nu^{**}}(k+m)}(0).$$

⁴Then $Lev_n(T)$ will be the union of such $Suc_T(\eta \frown \langle \alpha^{**} \rangle)$'s.

By the definition, T is a well-founded tree and $\bigcup_{n < \omega} Lev_n(T) = \lambda$. The following lemma completes our proof.

Lemma 2.4. $\langle s_\alpha : \alpha < \lambda \rangle$ is a sequence of λ -many Cohen reals over V .

Proof. First note that $\langle \langle r_{n,\alpha} : n < \omega, \alpha < \kappa \rangle, \langle r_\eta : \eta \in [\kappa]^{<\omega} \rangle \rangle$ is $\mathbb{C}(\omega \times \kappa) \times \mathbb{C}([\kappa]^{<\omega})$ -generic over V_1 . By c.c.c of $\mathbb{C}(\lambda)$ it suffices to show that for any countable set $I \subseteq \lambda$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I \rangle$ is $\mathbb{C}(I)$ -generic over V . Thus it suffices to prove the following:

For every $(p, q) \in \mathbb{C}(\omega \times \kappa) \times \mathbb{C}([\kappa]^{<\omega})$ and every open dense subset

(*) $D \in V$ of $\mathbb{C}(I)$, there is $(\bar{p}, \bar{q}) \leq (p, q)$ such that $(\bar{p}, \bar{q}) \Vdash \langle \check{s}_\alpha : \alpha \in I \rangle$ extends some element of D .

Let (p, q) and D be as above. for simplicity suppose that $p = q = \emptyset$. For each $n < \omega$ let $I_n = I \cap Lev_n(T)$. Then $I_0 = \emptyset$ and $I_1 = I \cap C$ is finite. For simplicity let $I_1 = \{\alpha_1^*, \alpha_2^*\}$ where $\alpha_1^* < \alpha_2^*$. Pick $n^* < \omega$ such that for all $n \geq n^*$, $f_{\alpha_1^*}(n) < f_{\alpha_2^*}(n)$. Let $p_0 \in \mathbb{C}(\omega \times \kappa)$ be such that

$$\text{dom}(p_0) = \{\langle 1, \beta, 0 \rangle : \exists n < n^* (\beta = f_{\alpha_1^*}(n) \text{ or } \beta = f_{\alpha_2^*}(n))\}.$$

Then for $n < n^*$ and $j \in \{1, 2\}$

$$(p_0, \emptyset) \Vdash \check{s}_{\alpha_j^*}(n) = \check{r}_{1, f_{\alpha_j^*}(n)}(0) = p_0(1, f_{\alpha_j^*}(n), 0).$$

thus (p_0, \emptyset) decides $s_{\alpha_1^*} \upharpoonright n^*$ and $s_{\alpha_2^*} \upharpoonright n^*$. Let $b \in D$ be such that $\langle b(\alpha_1^*), b(\alpha_2^*) \rangle$ extends $\langle s_{\alpha_1^*} \upharpoonright n^*, s_{\alpha_2^*} \upharpoonright n^* \rangle$. Let

$$p_1 = p_0 \cup \bigcup_{j \in \{1, 2\}} \{\langle 1, f_{\alpha_j^*}(n), 0, b(\alpha_j^*, n) \rangle : n \geq n^*, (\alpha_j^*, n) \in \text{dom}(b)\}.$$

Then $p_1 \in \mathbb{C}(\omega \times \kappa)$ is well-defined and letting $q_1 = \emptyset$, we have

$$(p_1, q_1) \Vdash \langle \check{s}_{\alpha_1^*}, \check{s}_{\alpha_2^*} \rangle \text{ extends } \langle b(\alpha_1^*), b(\alpha_2^*) \rangle.$$

For each $n < \omega$ let J_n be the set of all components of b which are in I_n , i.e. $J_n = \{\alpha \in I_n : \exists n, (\alpha, n) \in \text{dom}(b)\}$. We note that $J_0 = \emptyset$ and $J_1 = I_1 = \{\alpha_1^*, \alpha_2^*\}$. Also note that for

all but finitely many $n < \omega$, $J_n = \emptyset$. Thus let us suppose $t < \omega$ is such that for all $n > t$, $J_n = \emptyset$. Let us consider J_2 . For each $\alpha \in J_2$ there are three cases to be considered ⁵:

Case 1. There are $\alpha^* < \alpha^{**}$ in $Lev_1(T) = C$, $\alpha^{**} = \min(C \setminus (\alpha^* + 1))$ such that $|\alpha^{**} \setminus \alpha^*| \leq \kappa$ and $\alpha \in Suc_T(\langle \alpha^{**} \rangle) = \alpha^{**} \setminus \alpha^*$. Let i_α be the index of α in the enumeration of $\alpha^{**} \setminus \alpha^*$ considered in Case A above, and let $k_\alpha = \min\{m < \omega : r_{\pi(\langle \alpha^{**} \rangle) \frown \langle i_\alpha \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{2, f_\alpha(k_\alpha + m)}(0).$$

Case 2. There are $\alpha^* < \alpha^{**}$ as above such that $|\alpha^{**} \setminus \alpha^*| > \kappa$ and $\rho = cf \alpha^{**} < \kappa$. Let $\langle \alpha_{\nu^*}^{**} : \nu < \rho \rangle$ be as in Case B. Then $\alpha = \alpha_{\nu_\alpha}^{**}$ for some $\nu_\alpha < \rho$ and if $k_\alpha = \min\{m < \omega : r_{\pi(\langle \alpha^{**} \rangle) \frown \langle \nu_\alpha \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{2, f_\alpha(k_\alpha + m)}(0).$$

Case 3. There are $\alpha^* < \alpha^{**}$ as above such that $\rho = cf \alpha^{**} > \kappa$. Let $\langle \alpha_{\nu^*}^{**} : \nu < \rho \rangle$ be as in Case C. Then $\alpha = \alpha_{\nu_\alpha}^{**}$ for some $\nu_\alpha < \rho$ and if $k_\alpha = \min\{m < \omega : r_{\pi(\langle \alpha^{**} \rangle) \frown \langle 0 \rangle}(m) = 1\}$, then

$$\forall m < \omega, s_\alpha(m) = r_{2, f_\alpha(k_\alpha + m)}(0).$$

Let $m^* < \omega$ be such that for all $n \geq m^*$ and $\alpha < \alpha'$ in $J_1 \cup J_2$, $f_\alpha(n) < f_{\alpha'}(n)$. Let

$$\begin{aligned} q_2 = & \{ \langle \eta, n, 0 \rangle : n < m^*, \exists \alpha \in J_2 (\eta = \pi(\langle \alpha^{**} \rangle) \frown \langle i_\alpha \rangle \text{ or} \\ & \eta = \pi(\langle \alpha^{**} \rangle) \frown \langle \nu_\alpha \rangle \text{ or} \\ & \eta = \pi(\langle \alpha^{**} \rangle) \frown \langle 0 \rangle) \} \\ & \cup \{ \langle \eta, m^*, 1 \rangle : \exists \alpha \in J_2 (\eta = \pi(\langle \alpha^{**} \rangle) \frown \langle i_\alpha \rangle \text{ or} \\ & \eta = \pi(\langle \alpha^{**} \rangle) \frown \langle \nu_\alpha \rangle \text{ or} \\ & \eta = \pi(\langle \alpha^{**} \rangle) \frown \langle 0 \rangle) \} \end{aligned}$$

Then $q_2 \in \mathbb{C}([\kappa]^{<\omega})$ is well-defined and for each $\alpha \in J_2$, $(\phi, q_2) \Vdash k_\alpha = m^*$. Let

$$p_2 = p_1 \cup \{ \langle 2, f_\alpha(m^* + m), 0, b(\alpha, m) \rangle : \alpha \in J_2, (\alpha, m) \in \text{dom}(b) \}.$$

⁵Note that all the action in Cases 1-3 below is happening in the generic extension; in particular we did not yet determine the value of k_α .

Then $p_2 \in \mathbb{C}(\omega \times \kappa)$ is well-defined, $(p_2, q_2) \leq (p_1, q_1)$ and for $\alpha \in J_2$ and $m < \omega$ with $(\alpha, m) \in \text{dom}(b)$,

$$(p_2, q_2) \Vdash \check{s}_\alpha(m) = \check{r}_{2, f_\alpha(k_\alpha + m)}(0) = p_2(2, f_\alpha(k_\alpha + m), 0) = b(\alpha, m) = b(\alpha)(m),$$

thus $(p_2, q_2) \Vdash \text{“}\check{s}_\alpha \text{ extend } b(\alpha)\text{”}$ and hence

$$(p_2, q_2) \Vdash \text{“}\langle \check{s}_\alpha : \alpha \in J_1 \cup J_2 \rangle \text{ extends } \langle b(\alpha) : \alpha \in J_1 \cup J_2 \rangle\text{”}.$$

By induction suppose that we have defined $(p_1, q_1) \geq (p_2, q_2) \geq \dots \geq (p_j, q_j)$ for $j < t$, where for $1 \leq i \leq j$,

$$(p_i, q_i) \Vdash \text{“}\langle \check{s}_\alpha : \alpha \in J_1 \cup \dots \cup J_i \rangle \text{ extends } \langle b(\alpha) : \alpha \in J_1 \cup \dots \cup J_i \rangle\text{”}.$$

We define $(p_{j+1}, q_{j+1}) \leq (p_j, q_j)$ such that for each $\alpha \in J_{j+1}$, $(p_{j+1}, q_{j+1}) \Vdash \text{“}\check{s}_\alpha \text{ extends } b(\alpha)\text{”}$.

Let $\alpha \in J_{j+1}$. Then we can find $\eta \in T \upharpoonright j$ and $\alpha^* < \alpha^{**}$ such that $\alpha^*, \alpha^{**} \in \text{Suc}_T(\eta)$, $\alpha^{**} = \min(\text{Suc}_T(\eta) \setminus (\alpha^* + 1))$ and $\alpha \in \text{Suc}_T(\eta \frown \langle \alpha^{**} \rangle)$. As before there are three cases to be considered ⁶:

Case 1. $|\alpha^{**} \setminus \alpha^*| \leq \kappa$. Then let i_α be the index of α in the enumeration of $\alpha^{**} \setminus \alpha^*$ considered in Case A and let $k_\alpha = \min\{m < \omega : r_{\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle i_\alpha \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{j+1, f_\alpha(k_\alpha + m)}(0).$$

Case 2. $|\alpha^{**} \setminus \alpha^*| > \kappa$ and $\rho = cf \alpha^{**} < \kappa$. Let $\langle \alpha_\nu^{**} : \nu < \rho \rangle$ be as in Case B and let $\nu_\alpha < \rho$ be such that $\alpha = \alpha_{\nu_\alpha}^{**}$. Let $k_\alpha = \min\{m < \omega : r_{\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle \nu_\alpha \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{j+1, f_\alpha(k_\alpha + m)}(0).$$

Case 3. $\rho = cf \alpha^{**} > \kappa$. Let $\langle \alpha_\nu^{**} : \nu < \rho \rangle$ be as in Case C. Let $\nu_\alpha < \rho$ be such that $\alpha = \alpha_{\nu_\alpha}^{**}$ and let $k_\alpha = \min\{m < \omega : r_{\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle 0 \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{j+1, f_\alpha(k_\alpha + m)}(0).$$

Let $m^* < \omega$ be such that for all $n \geq m^*$ and $\alpha < \alpha'$ in $J_1 \cup \dots \cup J_{j+1}$, $f_\alpha(n) < f_{\alpha'}(n)$. Let

$$q_{j+1} = q_j \cup \{ \langle \check{\eta}, n, 0 \rangle : n < m^*, \exists \alpha \in J_{j+1} \text{ (for some unique } \eta \in T \upharpoonright j, \text{)} \}$$

⁶Again note that all the action in Cases 1-3 below is happening in the generic extension.

$$\begin{aligned}
 & \alpha^{**} \in \text{Suc}_T(\eta), \text{ we have } \alpha \in \text{Suc}_T(\eta \frown \langle \alpha^{**} \rangle) \\
 & \quad \text{and } (\bar{\eta} = \pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle i_\alpha \rangle \\
 & \quad \text{or } \bar{\eta} = \pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle \nu_\alpha \rangle \\
 & \quad \text{or } \bar{\eta} = (\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle 0 \rangle)) \} \\
 & \cup \{ \langle \bar{\eta}, m^*, 1 \rangle : \exists \alpha \in J_{j+1} \text{ (for some unique } \eta \in T \upharpoonright j, \\
 & \quad \alpha^{**} \in \text{Suc}_T(\eta), \text{ we have } \alpha \in \text{Suc}_T(\eta \frown \langle \alpha^{**} \rangle) \\
 & \quad \text{and } (\bar{\eta} = \pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle i_\alpha \rangle \\
 & \quad \text{or } \bar{\eta} = \pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle \nu_\alpha \rangle \\
 & \quad \text{or } \bar{\eta} = (\pi(\eta \frown \langle \alpha^{**} \rangle) \frown \langle 0 \rangle)) \}.
 \end{aligned}$$

It is easily seen that $q_{j+1} \in \mathbb{C}([\kappa]^{<\omega})$ and for each $\alpha \in J_{j+1}$, $(\phi, q_{j+1}) \Vdash -k_\alpha = m^*$. Let

$$p_{j+1} = p_j \cup \{ \langle j+1, f_\alpha(m^* + m), 0, b(\alpha, m) \rangle : \alpha \in J_{j+1}, (\alpha, m) \in \text{dom}(b) \}.$$

Then $p_{j+1} \in \mathbb{C}(\omega \times \kappa)$ is well-defined and $(p_{j+1}, q_{j+1}) \leq (p_j, q_j)$ and for $\alpha \in J_{j+1}$ we have

$$\begin{aligned}
 (p_{j+1}, q_{j+1}) \Vdash -\mathfrak{s}_\alpha(m) &= \mathfrak{r}_{j+1, f_\alpha(k_\alpha + m)}(0) = p_{j+1}(j+1, f_\alpha(k_\alpha + m), 0) = b(\alpha, m) = \\
 & b(\alpha)(m).
 \end{aligned}$$

Thus $(p_{j+1}, q_{j+1}) \Vdash -\mathfrak{s}_\alpha$ extends $b(\alpha)$. Finally let $(\bar{p}, \bar{q}) = (p_t, q_t)$. Then for each component α of b ,

$$(\bar{p}, \bar{q}) \Vdash -\mathfrak{s}_\alpha \text{ extends } b(\alpha).$$

Hence

$$(\bar{p}, \bar{q}) \Vdash -\langle \mathfrak{s}_\alpha : \alpha \in I \rangle \text{ extends } b.$$

(*) follows and we are done. □

Theorem 2.1 follows. □

We now give several applications of the above theorem.

Theorem 2.5. *Suppose that V satisfies GCH, $\kappa = \bigcup_{n < \omega} \kappa_n$ and $\bigcup_{n < \omega} o(\kappa_n) = \kappa$ (where $o(\kappa_n)$ is the Mitchell order of κ_n). Then there exists a cardinal preserving generic extension V_1 of V satisfying GCH and having the same reals as V does, so that adding κ -many Cohen reals over V_1 produces κ^+ -many Cohen reals over V .*

Proof. Rearranging the sequence $\langle \kappa_n : n < \omega \rangle$ we may assume that $o(\kappa_{n+1}) > \kappa_n$ for each $n < \omega$. Let $0 < n < \omega$. By [Mag 1], there exists a forcing notion \mathbb{P}_n such that:

- Each condition in \mathbb{P}_n is of the form (g, G) , where g is an increasing function from a finite subset of κ_n^+ into κ_{n+1} and G is a function from $\kappa_n^+ \setminus \text{dom}(g)$ into $\mathcal{P}(\kappa_{n+1})$ such that for each $\alpha \in \text{dom}(G)$, $G(\alpha)$ belongs to a suitable normal measure ⁷. We may also assume that conditions have no parts below or at κ_n , and sets of measure one are like this as well.
- Forcing with \mathbb{P}_n preserves cardinals and the *GCH*, and adds no new subsets to κ_n .
- If G_n is \mathbb{P}_n -generic over V , then in $V[G_n]$ there is a normal function $g_n^* : \kappa_n^+ \rightarrow \kappa_{n+1}$ such that $\text{ran}(g_n^*)$ is a club subset of κ_{n+1} consisting of measurable cardinals of V such that $V[G_n] = V[g_n^*]$.

Let $\mathbb{P}^* = \prod_{n < \omega} \mathbb{P}_n$, and let

$$\mathbb{P} = \{ \langle \langle g_n, G_n \rangle : n < \omega \rangle \in \mathbb{P}^* : g_n = \emptyset, \text{ for all but finitely many } n \}.$$

Then \mathbb{P} satisfies the $\kappa^+ - c.c.$ ⁸ and using simple modification of arguments from [Mag 1,2] we can show that forcing with \mathbb{P} preserves cardinals and the *GCH*. Let G be \mathbb{P} -generic over V , and let $g_n^* : \kappa_n^+ \rightarrow \kappa_{n+1}$ be the generic function added by the part of the forcing corresponding to \mathbb{P}_n , for $0 < n < \omega$. Let $X = \bigcup_{0 < n < \omega} ((\text{ran}(g_n^*) \setminus \kappa_n^+) \cup \{\kappa_{n+1}\})$ and let $g^* : \kappa \rightarrow \kappa$ be an enumeration of X in increasing order. Then $X = \text{ran}(g^*)$ is club in κ and consists entirely of measurable cardinals of V . Also $V[G] = V[g^*]$.

Working in $V[G]$, let \mathbb{Q} be the usual forcing notion for adding a club subset of κ^+ which avoids points of countable V -cofinality. Thus $\mathbb{Q} = \{p : p \text{ is a closed bounded subset of } \kappa^+ \text{ and avoids points of countable } V\text{-cofinality}\}$, ordered by end extension. Let H be \mathbb{Q} -generic over $V[G]$ and $C = \bigcup \{p : p \in H\}$.

Lemma 2.6. (a) (\mathbb{Q}, \leq) satisfies the κ^{++} -c.c.,

(b) (\mathbb{Q}, \leq) is $< \kappa^+$ -distributive,

(c) C is a club subset of κ^+ which avoids points of countable V -cofinality.

⁷In fact if $\alpha > \max(\text{dom}(g))$, then $G(\alpha)$ belongs to a normal measure on κ_{n+1} , and if $\alpha < \max(\text{dom}(g))$, then $G(\alpha)$ belongs to a normal measure on $g(\beta)$ where $\beta = \min(\text{dom}(g) \setminus \alpha)$.

⁸This is because any two conditions $\langle \langle g_n, G_n \rangle : n < \omega \rangle$ and $\langle \langle g_n, H_n \rangle : n < \omega \rangle$ in \mathbb{P} are compatible.

(a) and (c) of the above lemma are trivial. For use later we prove a more general version of (b).

Lemma 2.7. *Let $V \subseteq W$, let ν be regular in W and suppose that:*

(a) *W is a ν -c.c. extension of V ,*

(b) *For every $\lambda < \nu$ which is regular in W , there is $\tau < \nu$ so that $cf^W(\tau) = \lambda$ and τ has a club subset in W which avoids points of countable V -cofinality.*

In W let $\mathbb{Q} = \{p \subseteq \nu : p \text{ is closed and bounded in } \nu \text{ and avoids points of countable } V\text{-cofinality}\}$. Then in W , \mathbb{Q} is $< \nu$ -distributive.

Proof. This lemma first appeared in [G-N-S]. We prove it for completeness. Suppose that $W = V[G]$, where G is \mathbb{P} -generic over V for a ν -c.c. forcing notion \mathbb{P} . Let $\lambda < \nu$ be regular, $q \in \mathbb{Q}$, $\check{f} \in W^{\mathbb{Q}}$ and

$$q \Vdash \check{f} : \lambda \longrightarrow ON.$$

We find an extension of q which decides \check{f} . By (b) we can find $\tau < \nu$ and $g : \lambda \longrightarrow \tau$ such that $cf^W(\tau) = \lambda$, g is normal and $C = \text{ran}(g)$ is a club of τ which avoids points of countable V -cofinality.

In W , let $\theta > \nu$ be large enough regular. Working in V , let $\bar{H} \prec V_\theta$ and $R : \tau \longrightarrow ON$ be such that

- $\text{Card}(\bar{H}) < \nu$,
- \bar{H} has $\lambda, \tau, \nu, \mathbb{P}$ and \mathbb{P} -names for $p, \mathbb{Q}, \check{f}, g$ and C as elements,
- $\text{ran}(R)$ is cofinal in $\text{sup}(\bar{H} \cap \nu)$,
- $R \upharpoonright \beta \in \bar{H}$ for each $\beta < \tau$.

Let $H = \bar{H}[G]$. Then $\text{sup}(H \cap \nu) = \text{sup}(\bar{H} \cap \nu)$, since \mathbb{P} is ν -c.c., $H \prec V_\theta^W$ and if $\gamma = \text{sup}(H \cap \nu)$, then $cf^W(\gamma) = cf^W(\tau) = \lambda$. For $\alpha < \lambda$ let $\gamma_\alpha = R(g(\alpha))$. Then

- $\langle \gamma_\alpha : \alpha < \lambda \rangle \in W$ is a normal sequence cofinal in γ ,
- $\langle \gamma_\alpha : \alpha < \beta \rangle \in H$ for each $\beta < \lambda$, since $R \upharpoonright g(\beta) \in \bar{H}$,
- $cf^V(\gamma_\alpha) = cf^V(g(\alpha)) \neq \omega$ for each $\alpha < \lambda$, since R is normal and $g(\alpha) \in C$.

Let $D = \{\gamma_\alpha : \alpha < \lambda\}$. We define by induction a sequence $\langle q_\eta : \eta < \lambda \rangle$ of conditions in \mathbb{Q} such that for each $\eta < \lambda$

- $q_0 = q$,
- $q_\eta \in H$,
- $q_{\eta+1} \leq q_\eta$,
- $q_{\eta+1}$ decides $\underset{\sim}{f}(\eta)$,
- $D \cap (\max(q_\eta), \max(q_{\eta+1})) \neq \emptyset$,
- $q_\eta = \bigcup_{\rho < \eta} q_\rho \cup \{\delta_\eta\}$, where $\delta_\eta = \sup_{\rho < \eta} (\max(q_\rho))$, if η is a limit ordinal.

We may further suppose that

- q_η 's are chosen in a uniform way (say via a well-ordering which is built in to \bar{H}).

We can define such a sequence using the facts that H contains all initial segments of D and that $\delta_\eta \in D$ for every limit ordinal $\eta < \lambda$ (and hence $cf^V(\delta_\eta) \neq \omega$).

Finally let $q_\lambda = \bigcup_{\eta < \lambda} q_\eta \cup \{\delta_\lambda\}$, where $\delta_\lambda = \sup_{\eta < \lambda} (\max(q_\eta))$. Then $\delta_\lambda \in D \cup \{\gamma\}$, hence $cf^V(\delta_\lambda) \neq \omega$. It follows that $q_\lambda \in \mathbb{Q}$ is well-defined. Trivially $q_\lambda \leq q$ and q_λ decides $\underset{\sim}{f}$. The lemma follows. \square

Let $V_1 = V[G * H]$. The following is obvious

Lemma 2.8. (a) V and V_1 have the same cardinals and reals,

(b) $V_1 \models GCH$,

It follows from Theorem 2.1 that adding κ -many Cohen reals over V_1 adds κ^+ -many Cohen reals over V . This concludes the proof of Theorem 2.5. \square

Let us show that some large cardinals are needed for the previous result.

Theorem 2.9. *Assume that $V_1 \supseteq V$ and V_1 and V have the same cardinals and reals. Suppose that for some uncountable cardinal κ of V_1 , adding κ -many Cohen reals to V_1 produces κ^+ -many Cohen reals to V . Then in V_1 there is an inner model with a measurable cardinal.*

Proof. Suppose on the contrary that in V_1 there is no inner model with a measurable cardinal. Thus by Dodd-Jensen covering lemma (see [D-J 1,2]) $(\mathcal{K}(V_1), V_1)$ satisfies the covering lemma, where $\mathcal{K}(V_1)$ is the Dodd-Jensen core model as computed in V_1 .

Claim 2.10. $\mathcal{K}(V) = \mathcal{K}(V_1)$

Proof. The claim is well-known and follows from the fact that V and V_1 have the same cardinals. We present a proof for completeness⁹. Suppose not. Clearly $\mathcal{K}(V) \subseteq \mathcal{K}(V_1)$, so let $A \subseteq \alpha$, $A \in \mathcal{K}(V_1)$, $A \notin \mathcal{K}(V)$. Then there is a mouse of $\mathcal{K}(V_1)$ to which A belongs, hence there is such a mouse of $\mathcal{K}(V_1)$ -power α . It then follows that for every limit cardinal $\lambda > \alpha$ of V_1 there is a mouse with critical point λ to which A belongs, and the filter is generated by end segments of

$$\{\chi : \chi < \lambda, \chi \text{ a cardinal in } V_1\}.$$

As V and V_1 have the same cardinals, this mouse is in V , hence in $\mathcal{K}(V)$. □

Let us denote this common core model by \mathcal{K} . Then $\mathcal{K} \subseteq V$, and hence (V, V_1) satisfies the covering lemma. It follows that $([\kappa^+]^{\leq \omega_1})^V$ is unbounded in $([\kappa^+]^{\leq \omega})^{V_1}$ and since $\omega_1^V = \omega_1^{V_1}$, we can easily show that $([\kappa^+]^{\leq \omega})^V$ is unbounded in $([\kappa^+]^{\leq \omega})^{V_1}$. Since V_1 and V have the same reals, $([\kappa^+]^{\leq \omega})^V = ([\kappa^+]^{\leq \omega})^{V_1}$ and we get a contradiction. □

If we relax our assumptions, and allow some cardinals to collapse, then no large cardinal assumptions are needed.

Theorem 2.11. (a) *Suppose V is a model of GCH. Then there is a generic extension V_1 of V satisfying GCH so that the only cardinal of V which is collapsed in V_1 is \aleph_1 and such that adding \aleph_ω -many Cohen reals to V_1 produces $\aleph_{\omega+1}$ -many of them over V .*

(b) *Suppose V satisfies GCH. Then there is a generic extension V_1 of V satisfying GCH and having the same reals as V does, so that the only cardinals of V which are collapsed in V_1 are \aleph_2 and \aleph_3 and such that adding \aleph_ω -many Cohen reals to V_1 produces $\aleph_{\omega+1}$ -many of them over V .*

Proof. (a) Working in V , let $\mathbb{P} = \text{Col}(\aleph_0, \aleph_1)$ and let G be \mathbb{P} -generic over V . Also let $S = \{\alpha < \omega_2 : \text{cf}^V(\alpha) = \omega_1\}$. Then S remains stationary in $V[G]$. Working in $V[G]$, let \mathbb{Q} be the standard forcing notion for adding a club subset of S with countable conditions, and let H be \mathbb{Q} -generic over $V[G]$. Let $C = \bigcup H$. Then C is a club subset of $\omega_1^{V[G]} = \omega_2^V$ such that $C \subseteq S$, and in particular C avoids points of countable V -cofinality. Working in $V[G * H]$, let

⁹Our proof is the same as in the proof of [Sh 2, Theorem VII. 4.2(1)].

$$\mathbb{R} = \langle \langle \mathbb{P}_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+2}, \nu \text{ regular} \rangle, \langle \mathbb{Q}_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+1}, \nu \text{ regular} \rangle \rangle$$

be the Easton support iteration by letting \mathbb{Q}_ν name the poset $\{p \subset \nu : p \text{ is closed and bounded in } \nu \text{ and avoids points of countable } V\text{-cofinality}\}$ as defined in $V[G * H]^{\mathbb{P}_\nu}$. Let

$$K = \langle \langle G_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+2}, \nu \text{ regular} \rangle, \langle H_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+1}, \nu \text{ regular} \rangle \rangle$$

be \mathbb{R} -generic over $V[G * H]$ (i.e G_ν is \mathbb{P}_ν -generic over $V[G * H]$ and H_ν is $\mathbb{Q}_\nu = \mathbb{Q}_\nu[G_\nu]$ -generic over $V[G * H * G_\nu]$). Then

Lemma 2.12. (a) \mathbb{P}_ν adds a club disjoint from $\{\alpha < \lambda : cf^V(\alpha) = \omega\}$ for each regular $\lambda \in (\aleph_1, \nu)$,

(b) (By 2.7) $V[G * H * G_\nu] \models \text{“}\mathbb{Q}_\nu \text{ is } < \nu\text{-distributive”}$,

(c) $V[G * H]$ and $V[G * H * K]$ have the same cardinals and reals, and satisfy *GCH*,

(d) In $V[G * H * K]$ there is a club subset C of $\aleph_{\omega+1}$ which avoids points of countable *V*-cofinality.

Let $V_1 = V[G * H * K]$. By above results, V_1 satisfies *GCH* and the only cardinal of V which is collapsed in V_1 is \aleph_1 . The proof of the fact that adding \aleph_ω -many Cohen reals over V_1 produces $\aleph_{\omega+1}$ -many of them over V follows from Theorem 2.1.

(b) Working in V , let \mathbb{P} be the following version of Namba forcing:

$$\mathbb{P} = \{T \subseteq \omega_2^{<\omega} : T \text{ is a tree and for every } s \in T, \text{ the set } \{t \in T : t \supset s\} \text{ has size } \aleph_2\}$$

ordered by inclusion. Let G be \mathbb{P} -generic over V . It is well-known that forcing with \mathbb{P} adds no new reals, preserves cardinals $\geq \aleph_4$ and that $|\aleph_2^V|^{V[G]} = |\aleph_3^V|^{V[G]} = \aleph_1^{V[G]} = \aleph_1^V$ (see [Sh 1]). Let $S = \{\alpha < \omega_3 : cf^V(\alpha) = \omega_2\}$.

Lemma 2.13. S remains stationary in $V[G]$.

Proof. See [Ve-W, Lemma 3]. □

Now the rest of the proof is exactly as in (a).

The Theorem follows □

By the same line but using stronger initial assumptions, adding κ -many Cohen reals may produce λ -many of them for λ much larger than κ^+ .

Theorem 2.14. *Suppose that κ is a strong cardinal, $\lambda \geq \kappa$ is regular and GCH holds. Then there exists a cardinal preserving generic extension V_1 of V having the same reals as V does, so that adding κ -many Cohen reals over V_1 produces λ -many of them over V .*

Proof. Working in V , build for each δ a measure sequence \vec{u}_δ from a j witnessing “ κ is δ -strong” out to the first weak repeat point. Find \vec{u} such that $\vec{u} = \vec{u}_\delta$ for unboundedly many δ . Let $\mathbb{R}_{\vec{u}}$ be the corresponding Radin forcing notion and let G be $\mathbb{R}_{\vec{u}}$ -generic over V . Then

Lemma 2.15. (a) *Forcing with $\mathbb{R}_{\vec{u}}$ preserves cardinals and the GCH and adds no new reals,*
 (b) *In $V[G]$, there is a club $C_\kappa \subseteq \kappa$ consisting of inaccessible cardinals of V and $V[G] = V[C_\kappa]$,*
 (c) *κ remains strong in $V[G]$.*

Proof. See [Git 2] and [Cu]. □

Working in $V[G]$, let

$$E = \langle \langle U_\alpha : \alpha < \lambda \rangle, \langle \pi_{\alpha\beta} : \alpha \leq_E \beta \rangle \rangle$$

be a nice system satisfying conditions (0)-(9) in [Git 2, page 37]. Also let

$$\mathbb{R} = \langle \langle \mathbb{P}_\nu : \kappa^+ \leq \nu \leq \lambda^+, \nu \text{ regular} \rangle, \langle \mathbb{Q}_\nu : \kappa^+ \leq \nu \leq \lambda, \nu \text{ regular} \rangle \rangle$$

be the Easton support iteration by letting \mathbb{Q}_ν name the poset $\{p \subseteq \nu : p \text{ is closed and bounded in } \nu \text{ and avoids points of countable } V\text{-cofinality}\}$ as defined in $V[G]^{\mathbb{P}_\nu}$. Let

$$K = \langle \langle G_\nu : \kappa^+ \leq \nu \leq \lambda^+, \nu \text{ regular} \rangle, \langle H_\nu : \kappa^+ \leq \nu \leq \lambda, \nu \text{ regular} \rangle \rangle$$

be \mathbb{R} -generic over $V[G]$. Then

Lemma 2.16. (a) \mathbb{P}_ν *adds a club disjoint from $\{\alpha < \delta : cf^V(\alpha) = \omega\}$ for each regular $\delta \in (\kappa, \nu)$,*

(b) *(By 2.7) $V[G * G_\nu] \models \text{“}\mathbb{Q}_\nu = \mathbb{Q}_\nu[G_\nu] \text{ is } < \nu\text{-distributive”}$,*

(c) *$V[G]$ and $V[G * K]$ have the same cardinals, and satisfy GCH,*

(d) *\mathbb{R} is $\leq \kappa$ -distributive, hence forcing with \mathbb{R} adds no new κ -sequences,*

(e) *In $V[G * K]$, for each regular cardinal $\kappa \leq \nu \leq \lambda$ there is a club $C_\nu \subseteq \nu$ such that C_ν avoids points of countable V -cofinality.*

By 2.16.(d), E remains a nice system in $V[G * K]$, except that the condition (0) is replaced by (λ, \leq_E) is κ^+ -directed closed. Hence working in $V[G * K]$, by results of [Git-Mag 1,2] and [Mer], we can find a forcing notion S such that if L is S -generic over $V[G * K]$ then

- $V[G * K]$ and $V[G * K * L]$ have the same cardinals and reals,
- In $V[G * K * L]$, $2^\kappa = \lambda$, $cf(\kappa) = \aleph_0$ and there is an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals cofinal in κ and an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ in $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$.

Let $V_1 = V[G * K * L]$. Then V_1 and V have the same cardinals and reals. The fact that adding κ -many Cohen reals over V_1 produces λ -many Cohen reals over V follows from Theorem 2.1. \square

If we allow many cardinals between V and V_1 to collapse, then using [Git-Mag 1,Sec 2] one can obtain the following

Theorem 2.17. *Suppose that there is a strong cardinal and GCH holds. Let $\alpha < \omega_1$. Then there is a model $V_1 \supset V$ having the same reals as V and satisfying GCH below $\aleph_\omega^{V_1}$ such that adding $\aleph_\omega^{V_1}$ -many Cohen reals to V_1 produces $\aleph_{\alpha+1}^{V_1}$ -many of them over V .*

Proof. Proceed as in Theorem 2.14 to produce the model $V[G * K]$. Then working in $V[G * K]$, we can find a forcing notion S such that if L is S -generic over $V[G * K]$ then

- $V[G * K]$ and $V[G * K * L]$ have the same reals,
- In $V[G * K * L]$, cardinals $\geq \kappa$ are preserved, $\kappa = \aleph_\omega$, GCH holds below \aleph_ω , $2^\kappa = \aleph_{\alpha+1}$ and there is an increasing (mod finite) sequence $\langle f_\beta : \beta < \aleph_{\alpha+1} \rangle$ in $\prod_{n < \omega} (\aleph_{n+1} \setminus \aleph_n)$.

Let $V_1 = V[G * K * L]$. Then V_1 and V have the same reals. The fact that adding $\aleph_\omega^{V_1}$ -many Cohen reals over V_1 produces $\aleph_{\alpha+1}^{V_1}$ -many Cohen reals over V follows from Theorem 2.1. \square

3. MODELS WITH THE SAME COFINALITY FUNCTION BUT DIFFERENT REALS

This section is completely devoted to the proof of the following theorem.

Theorem 3.1. *Suppose that V satisfies GCH. Then there is a cofinality preserving generic extension V_1 of V satisfying GCH so that adding a Cohen real over V_1 produces \aleph_1 -many Cohen reals over V .*

The basic idea of the proof will be to split ω_1 into ω sets such that none of them will contain an infinite set of V . Then something like in section 2 will be used for producing Cohen reals. It turned out however that just not containing an infinite set of V is not enough. We will use a stronger property. As a result the forcing turns out to be more complicated. We are now going to define the forcing sufficient for proving the theorem. Fix a nonprincipal ultrafilter U over ω .

Definition 3.2. *Let $(\mathbb{P}_U, \leq, \leq^*)$ be the Prikry (or in this context Mathias) forcing with U , i.e.*

- $\mathbb{P}_U = \{\langle s, A \rangle \in [\omega]^{<\omega} \times U : \max(s) < \min(A)\}$,
- $\langle t, B \rangle \leq \langle s, A \rangle \iff t \text{ end extends } s \text{ and } (t \setminus s) \cup B \subseteq A$,
- $\langle t, B \rangle \leq^* \langle s, A \rangle \iff t = s \text{ and } B \subseteq A$.

We call \leq^* a direct or $*$ -extension. The following are the basic facts on this forcing that will be used further.

Lemma 3.3. (a) *The generic object of \mathbb{P}_U is generated by a real,*

(b) *(\mathbb{P}_U, \leq) satisfies the c.c.c.,*

(c) *If $\langle s, A \rangle \in \mathbb{P}_U$ and $b \subseteq \omega \setminus (\max(s) + 1)$ is finite, then there is a $*$ -extension of $\langle s, A \rangle$, forcing the generic real to be disjoint to b .*

Proof. (a) If G is \mathbb{P}_U -generic over V , then let $r = \bigcup \{s : \exists A, \langle s, A \rangle \in G\}$. r is a real and $G = \{\langle s, A \rangle \in \mathbb{P}_U : r \text{ end extends } s \text{ and } r \setminus s \subseteq A\}$.

(b) Trivial using the fact that for $\langle s, A \rangle, \langle t, B \rangle \in \mathbb{P}_U$, if $s = t$ then $\langle s, A \rangle$ and $\langle t, B \rangle$ are compatible.

(c) Consider $\langle s, A \setminus (\max(b) + 1) \rangle$. □

We now define our main forcing notion.

Definition 3.4. $p \in \mathbb{P}$ iff $p = \langle p_0, \underline{p}_1 \rangle$ where

(1) $p_0 \in \mathbb{P}_U$,

(2) \underline{p}_1 is a \mathbb{P}_U -name such that for some $\alpha < \omega_1$, $p_0 \Vdash \underline{p}_1 : \alpha \rightarrow \omega$ and such that the following hold

(2a) For every $\beta < \alpha$, $\underline{p}_1(\beta) \subseteq \mathbb{P}_U \times \omega$ is a \mathbb{P}_U -name for a natural number such

that

- $\underline{p}_1(\beta)$ is partial function from \mathbb{P}_U into ω ,
- for some fixed $l < \omega$, $\text{dom } \underline{p}_1(\beta) \subseteq \{ \langle s, \omega \setminus \max(s) + 1 \rangle : s \in [\omega]^l \}$,
- for all $\beta_1 \neq \beta_2 < \alpha$, $\text{range}(\underline{p}_1(\beta_1)) \cap \text{range}(\underline{p}_1(\beta_2))$ is finite ¹⁰.

(2b) for every $I \subseteq \alpha$, $I \in V$, $p'_0 \leq p_0$ and finite $J \subseteq \omega$ there is a finite set

$a \subseteq \alpha$ such that for every finite set $b \subseteq I \setminus a$ there is $p''_0 \leq^* p'_0$ such

that $p''_0 \Vdash (\forall \beta \in b, \forall k \in J, \underline{p}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underline{p}_1(\beta_1) \neq \underline{p}_1(\beta_2))$.

Notation 3.5. (1) Call α the length of p (or \underline{p}_1) and denote it by $l(p)$ (or $l(\underline{p}_1)$).

(2) For $n < \omega$ let $\underline{I}_{p,n}$ be a \mathbb{P}_U -name such that $p_0 \Vdash \underline{I}_{p,n} = \{ \beta < \alpha : \underline{p}_1(\beta) = n \}$. Then we can identify \underline{p}_1 with $\langle \underline{I}_{p,n} : n < \omega \rangle$.

Remark 3.6. (2a) will guarantee that for $\beta < \alpha$, $p_0 \Vdash \underline{p}_1(\beta) \in \omega$. The last condition in (2a) is a technical fact that will be used in several parts of the argument. The condition (2b) appears technical but it will be crucial for producing numerous Cohen reals.

Definition 3.7. For $p = \langle p_0, \underline{p}_1 \rangle, q = \langle q_0, \underline{q}_1 \rangle \in \mathbb{P}$, define

(1) $p \leq q$ iff

- $p_0 \leq_{\mathbb{P}_U} q_0$,
- $l(q) \leq l(p)$,
- $p_0 \Vdash \forall n < \omega, \underline{I}_{q,n} = \underline{I}_{p,n} \cap l(q)$.

(2) $p \leq^* q$ iff

- $p_0 \leq_{\mathbb{P}_U}^* q_0$,
- $p \leq q$.

¹⁰Thus if G and r are as in the proof of Lemma 3.3 with $p_0 \in G$, then $p_0 \Vdash \underline{p}_1(\beta)$ is the l -th element of r .

we call \leq^* a direct or *-extension.

Remark 3.8. In the definition of $p \leq q$, we can replace the last condition by $p_0 \parallel_{\sim} q_1 = p_1 \upharpoonright l(q)$.

Lemma 3.9. Let $\langle p_0, p_1 \rangle \parallel_{\sim} \text{“}\alpha \text{ is an ordinal”}$. Then there are \mathbb{P}_U -names $\tilde{\beta}$ and \tilde{q}_1 such that $\langle p_0, \tilde{q}_1 \rangle \leq^* \langle p_0, p_1 \rangle$ and $\langle p_0, \tilde{q}_1 \rangle \parallel_{\sim} \tilde{\alpha} = \tilde{\beta}$.

Proof. Suppose for simplicity that $\langle p_0, p_1 \rangle = \langle \langle \langle \cdot \rangle, \omega \rangle, \phi \rangle$. Let θ be large enough regular and let $\langle N_n : n < \omega \rangle$ be an increasing sequence of countable elementary submodels of H_θ such that $\mathbb{P}, \mathfrak{Q} \in N_0$ and $N_n \in N_{n+1}$ for each $n < \omega$. Let $N = \bigcup_{n < \omega} N_n$, $\delta_n = N_n \cap \omega_1$ for $n < \omega$ and $\delta = \bigcup_{n < \omega} \delta_n = N \cap \omega_1$. Let $\langle J_n : n < \omega \rangle \in N_0$ be a sequence of infinite subsets of $\omega \setminus \{0\}$ such that $\bigcup_{n < \omega} J_n = \omega \setminus \{0\}$, $J_n \subseteq J_{n+1}$, and $J_{n+1} \setminus J_n$ is infinite for each $n < \omega$. Also let $\langle \alpha_i : 0 < i < \omega \rangle$ be an enumeration of δ such that for every $n < \omega$, $\{\alpha_i : i \in J_n\} \in N_{n+1}$ is an enumeration of δ_n and $\{\alpha_i : i \in J_{n+1}\} \cap \delta_n = \{\alpha_i : i \in J_n\}$.

We define by induction on the length of s , a sequence $\langle p^s : s \in [\omega]^{<\omega} \rangle$ of conditions such that

- $p^s = \langle p_0^s, p_1^s \rangle = \langle \langle s, A_s \rangle, p_1^s \rangle$,
- $p^s \in N_{s(l(s)-1)+1}$,
- $l(p^s) = \delta_{s(l(s)-1)+1}$,
- if t does not contradict p_0^s (i.e if t end extends s and $t \setminus s \subseteq A_s$) then $p^t \leq p^s$.

For $s = \langle \cdot \rangle$, let $p^{\langle \cdot \rangle} = \langle \langle \langle \cdot \rangle, \omega \rangle, \phi \rangle$. Suppose that $\langle \cdot \rangle \neq s \in [\omega]^{<\omega}$ and $p^{s \upharpoonright l(s)-1}$ is defined. We define p^s . First we define $t^{s \upharpoonright l(s)-1} \leq^* p^{s \upharpoonright l(s)-1}$ as follows: If there is no *-extension of $p^{s \upharpoonright l(s)-1}$ deciding \mathfrak{Q} then let $t^{s \upharpoonright l(s)-1} = p^{s \upharpoonright l(s)-1}$. Otherwise let $t^{s \upharpoonright l(s)-1} \in N_{s(l(s)-2)+1}$ be such an extension. Note that $l(t^{s \upharpoonright l(s)-1}) \leq \delta_{s(l(s)-2)+1}$.

Let $t^{s \upharpoonright l(s)-1} = \langle t_0, t_1 \rangle$, $t_0 = \langle s \upharpoonright l(s) - 1, A \rangle$. Let $C \subseteq \omega$ be an infinite set almost disjoint to $\langle \text{range}(t_1(\beta)) : \beta < l(t_1) \rangle$. Split C into ω infinite disjoint sets C_i , $i < \omega$. Let $\langle c_{ij} : j < \omega \rangle$ be an increasing enumeration of C_i , $i < \omega$. We may suppose that all of these is done in $N_{s(l(s)-1)+1}$. Let $p^s = \langle p_0^s, p_1^s \rangle$, where

- $p_0^s = \langle s, A \setminus (\max(s) + 1) \rangle$,
- for $\beta < l(\underline{t}_1)$, $p_1^s(\beta) = \underline{t}_1(\beta)$,
- for $i \in J_{s(l(s)-1)}$ such that $\alpha_i \in \delta_{s(l(s)-1)} \setminus l(\underline{t}_1)$

$$p_1^s(\alpha_i) = \{ \langle \langle s \frown \langle r_1, \dots, r_i \rangle, \omega \setminus (r_i + 1) \rangle, c_{ir_i} \rangle : r_1 > \max(s), \langle r_1, \dots, r_i \rangle \in [\omega]^i \}.$$

Trivially $p^s \in N_{s(l(s)-1)+1}$, $l(p^s) = \delta_{s(l(s)-1)}$, and if $s(l(s) - 1) \in A$, then $p^s \leq t^{s(l(s)-1)}$.

Claim 3.10. $p^s \in \mathbb{P}$.

Proof. We check conditions in Definition 3.4.

(1) i.e. $p_0^s \in \mathbb{P}_U$ is trivial.

(2) It is clear that $p_0^s \Vdash p_1^s : \delta_{s(l(s)-1)} \longrightarrow \omega$ and that (2a) holds. Let us prove (2b). Thus suppose that $I \subseteq \delta_{s(l(s)-1)}$, $I \in V$, $p \leq p_0^s$ and $J \subseteq \omega$ is finite. First we apply (2b) to $\langle p, \underline{t}_1 \rangle, I \cap l(\underline{t}_1)$, p and J to find a finite set $a' \subseteq l(\underline{t}_1)$ such that

- (*) For every finite set $b \subseteq I \cap l(\underline{t}_1) \setminus a'$ there is $p' \leq^* p$ such that
- $$p' \Vdash (\forall \beta \in b, \forall k \in J, \underline{t}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underline{t}_1(\beta_1) \neq \underline{t}_1(\beta_2)).$$

Let $p = \langle s \frown \langle r_1, \dots, r_m \rangle, B \rangle$. Suppose that $\delta_{s(l(s)-1)} \setminus l(\underline{t}_1) = \{\alpha_{J_1}, \dots, \alpha_{J_i}, \dots\}$ where $J_1 < J_2 < \dots$ are in $J_{s(l(s)-1)}$. Let

$$a = a' \cup \{\alpha_{J_1}, \dots, \alpha_{J_m}\}.$$

We show that a is as required. Thus suppose that $b \subseteq I \setminus a$ is finite. Apply (*) to $b \cap l(\underline{t}_1)$ to find $p' = \langle s \frown \langle r_1, \dots, r_m \rangle, B' \rangle \leq^* p$ such that

$$p' \Vdash (\forall \beta \in b \cap l(\underline{t}_1), \forall k \in J, \underline{t}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b \cap l(\underline{t}_1), \underline{t}_1(\beta_1) \neq \underline{t}_1(\beta_2)).$$

Also note that

$$p' \Vdash \forall \beta \in b \cap l(\underline{t}_1), p_1^s(\beta) = \underline{t}_1(\beta).$$

Pick $k < \omega$ such that

$$\forall \beta \in b \cap l(\underline{t}_1), \forall \alpha_i \in b \setminus l(\underline{t}_1), \text{range}(p_1^s(\beta_1)) \cap (\text{range}(p_1^s(\alpha_i)) \setminus k) = \phi.$$

Let $q = \langle s \frown \langle r_1, \dots, r_m \rangle, B \rangle = \langle s \frown \langle r_1, \dots, r_m \rangle, B' \setminus (\max(J) + k + 1) \rangle$. Then $q \leq^* p' \leq^* p$.

We show that q is as required. We need to show that

- (1) $q \Vdash \forall \beta \in b \setminus l(\underline{t}_1), \forall k \in J, p_1^s(\beta) \neq k$,
- (2) $q \Vdash \forall \beta_1 \neq \beta_2 \in b \setminus l(\underline{t}_1), p_1^s(\beta_1) \neq p_1^s(\beta_2)$,
- (3) $q \Vdash \forall \beta_1 \in b \cap l(\underline{t}_1), \forall \beta_2 \in b \setminus l(\underline{t}_1), p_1^s(\beta_1) \neq p_1^s(\beta_2)$.

Now (1) follows from the fact that $q \Vdash "p_1^s(\alpha_i) \geq (i - m)\text{-th element of } B > \max(J)"$.

(2) follows from the fact that for $i \neq j < \omega$, $C_i \cap C_j = \emptyset$, and $\text{range}(p_1^s(\alpha_i)) \subseteq C_i$. (3)

follows from the choice of k . The claim follows. \square

This completes our definition of the sequence $\langle p^s : s \in [\omega]^{<\omega} \rangle$. Let

$$q_1 = \{ \langle p_0^s, \langle \beta, p_1^s(\beta) \rangle \rangle : s \in [\omega]^{<\omega}, \beta < l(p^s) \}.$$

Then q_1 is a \mathbb{P}_U -name and for $s \in [\omega]^{<\omega}$, $p_0^s \Vdash p_1^s = q_1 \upharpoonright l(p_1^s)$.

Claim 3.11. $\langle \langle \cdot \rangle, \omega \rangle, q_1 \in \mathbb{P}$.

Proof. We check conditions in Definition 3.4.

(1) i.e. $\langle \langle \cdot \rangle, \omega \rangle \in \mathbb{P}_U$ is trivial.

(2) It is clear from our definition that

$$\langle \langle \cdot \rangle, \omega \rangle \Vdash "q_1 \text{ is a well-defined function into } \omega".$$

Let us show that $l(q_1) = \delta$. By the construction it is trivial that $l(q_1) \leq \delta$. We show that $l(q_1) \geq \delta$. It suffices to prove the following

(*) For every $\tau < \delta$ and $p \in \mathbb{P}_U$ there is $q \leq p$ such that $q \Vdash "q_1(\tau) \text{ is defined}"$.

Fix $\tau < \delta$ and $p = \langle s, A \rangle \in \mathbb{P}_U$ as in (*). Let t be an end extension of s such that $t \setminus s \subseteq A$ and $\delta_{t(l(t)-1)} > \tau$. Then p_0^t and p are compatible and $p_0^t \Vdash \check{q}_1(\tau) = \check{p}_1^t(\tau)$ is defined". Let $q \leq p_0^t, p$. Then $q \Vdash \check{q}_1(\tau)$ is defined" and (*) follows. Thus $l(q_1) = \delta$.

(2a) is trivial. Let us prove (2b). Thus suppose that $I \subseteq \delta$, $I \in V$, $p \leq \langle \langle \rangle, \omega \rangle$ and $J \subseteq \omega$ is finite. Let $p = \langle s, A \rangle$.

First we consider the case where $s = \langle \rangle$. Let $a = \emptyset$. We show that a is as required. Thus let $b \subseteq I$ be finite. Let $n \in A$ be such that $n > \max(J) + 1$ and $b \subseteq \delta_n$. Let $t = s \hat{\ } \langle n \rangle$. Note that

$$\forall \beta_1 \neq \beta_2 \in b, \text{range}(p_1^t(\beta_1)) \cap \text{range}(p_1^t(\beta_2)) = \emptyset.$$

Let $q = \langle \langle \rangle, B \rangle = \langle \langle \rangle, A \setminus (\max(J) + 1) \rangle$. Then $q \leq^* p$ and q is compatible with p_0^t .

We show that q is as required. We need to show that

- (1) $q \Vdash \forall \beta \in b, \forall k \in J, \check{q}_1(\beta) \neq k$,
- (2) $q \Vdash \forall \beta_1 \neq \beta_2 \in b, \check{q}_1(\beta_1) \neq \check{q}_1(\beta_2)$.

For (1), if it fails, then we can find $\langle r, D \rangle \leq q, p_0^t$, $\beta \in b$ and $k \in J$ such that $\langle r, D \rangle \leq^* p_0^r$ and $\langle r, D \rangle \Vdash \check{q}_1(\beta) = k$. But $\langle r, D \rangle \Vdash \check{q}_1(\beta) = \check{p}_1^r(\beta) = \check{p}_1^t(\beta)$, hence $\langle r, D \rangle \Vdash \check{p}_1^t(\beta) = k$. This is impossible since $\min(D) \geq \min(B) > \max(J)$. For (2), if it fails, then we can find $\langle r, D \rangle \leq q, p_0^t$ and $\beta_1 \neq \beta_2 \in b$ such that $\langle r, D \rangle \leq^* p_0^r$ and $\langle r, D \rangle \Vdash \check{q}_1(\beta_1) = \check{q}_1(\beta_2)$. As above it follows that $\langle r, D \rangle \Vdash \check{p}_1^t(\beta_1) = \check{p}_1^t(\beta_2)$. This is impossible since for $\beta_1 \neq \beta_2 \in b$, $\text{range}(p_1^t(\beta_1)) \cap \text{range}(p_1^t(\beta_2)) = \emptyset$. Hence q is as required and we are done.

Now consider the case $s \neq \langle \rangle$. First we apply (2b) to t^s , $I \cap l(t^s)$, p and J to find a finite set $a' \subseteq l(t^s)$ such that

$$(**) \quad \text{For every finite set } b \subseteq I \cap l(t^s) \setminus a' \text{ there is } p' \leq^* p \text{ such that } p' \Vdash (\forall \beta \in b, \forall k \in J, \check{p}_1^s(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \check{p}_1^s(\beta_1) \neq \check{p}_1^s(\beta_2))$$

Let $t^s = \langle t_0, \check{t}_1 \rangle, \delta_{s(l(s)-1)+1} \setminus \delta_{s(l(s)-1)} = \{\alpha_{J_1}, \alpha_{J_2}, \dots\}$, where $J_1 < J_2 < \dots$ are in $J_{s(l(s)-1)+1}$. Define

$$a = a' \cup \{\alpha_1, \alpha_2, \dots, \alpha_{J_{l(s)+1}}\}.$$

We show that a is as required. First apply $(**)$ to $b \cap l(t^s)$ to find $p' = \langle s, A' \rangle \leq^* p$ such that

$$p' \Vdash (\forall \beta \in b \cap l(t^s), \forall k \in J, \dot{\surd}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b \cap l(t^s), \dot{\surd}_1(\beta_1) \neq \dot{\surd}_1(\beta_2)).$$

Pick $n \in A'$ such that $n > \max(J) + 1$ and $b \subseteq \delta_n$ and let $r = s \frown \langle n \rangle$. Then

$$\forall \beta_1 \neq \beta_2 \in b \setminus l(t^s), \text{range}(p_1^r(\beta_1)) \cap \text{range}(p_1^r(\beta_2)) = \emptyset.$$

Pick $k < \omega$ such that $k > n$ and

$$\forall \beta_1 \in b \cap l(t^s), \forall \beta_2 \in b \setminus l(t^s), \text{range}(p_1^r(\beta_1)) \cap (\text{range}(p_1^r(\beta_2)) \setminus k) = \emptyset.$$

Let $q = \langle s, B \rangle = \langle s, A' \setminus (\max(J) + k + 1) \cup \{n\} \rangle$. Then $q \leq^* p' \leq^* p$ and q is compatible with p_0^r (since $n \in B$). We show that q is as required. We need to prove the following

- (1) $q \Vdash \forall \beta \in b, \forall k \in J, q_1(\beta) \neq k,$
- (2) $q \Vdash \forall \beta_1 \neq \beta_2 \in b \setminus l(t^s), q_1(\beta_1) \neq q_1(\beta_2),$
- (3) $q \Vdash \forall \beta_1 \in b \cap l(t^s), \forall \beta_2 \in b \setminus l(t^s), q_1(\beta_1) \neq q_1(\beta_2).$

The proofs of (1) and (2) are as in the case $s = \langle \rangle$. Let us prove (3). Suppose that (3) fails. Thus we can find $\langle u, D \rangle \leq q, p_0^r, \beta_1 \in b \cap l(t^s)$ and $\beta_2 \in b \setminus l(t^s)$ such that $\langle u, D \rangle \leq^* p_0^u$ and $\langle u, D \rangle \Vdash q_1(\beta_1) = q_1(\beta_2)$. But for $\beta \in b$, $\langle u, D \rangle \Vdash q_1(\beta) = p_1^u(\beta) = p_1^r(\beta)$, hence $\langle u, D \rangle \Vdash p_1^r(\beta_1) = p_1^r(\beta_2)$. Now note that $\beta_2 = \alpha_i$ for some $i > l(s) + 1$, $\min(D) \geq n$ and $\min(D \setminus \{n\}) > k$, hence by the construction of p^r

$$\langle u, D \rangle \Vdash \text{“} p_1^r(\beta_2) \geq (i - l(s)\text{-th element of } D > k\text{”}.$$

By our choice of k , $\text{range}(p_1^r(\beta_1)) \cap (\text{range}(p_1^r(\beta_2)) \setminus k) = \emptyset$ and we get a contradiction.

(3) follows. Thus q is as required, and the claim follows. \square

Let

$$\beta = \{\langle p_0^s, \delta \rangle : s \in [\omega]^{<\omega}, \exists \gamma (\delta < \gamma, p^s \Vdash \alpha = \gamma)\}.$$

Then β is a \mathbb{P}_U -name of an ordinal.

Claim 3.12. $\langle \langle \rangle, \omega \rangle, q_1 \Vdash \alpha = \beta$.

Proof. Suppose not. There are two cases to be considered.

Case 1. There are $\langle r_0, \mathcal{r}_1 \rangle \leq \langle \langle \langle \rangle, \omega \rangle, q_1 \rangle$ and δ such that $\langle r_0, \mathcal{r}_1 \rangle \Vdash \text{“}\delta \in \mathcal{Q} \text{ and } \delta \notin \mathcal{B}\text{”}$. We may suppose that for some ordinal α , $\langle r_0, \mathcal{r}_1 \rangle \Vdash \mathcal{Q} = \alpha$. Then $\delta < \alpha$. Let $r_0 = \langle s, A \rangle$. Consider $p^s = \langle p_0^s, p_1^s \rangle$. Then p_0^s is compatible with r_0 and there is a $*$ -extension of p^s deciding \mathcal{Q} . Let $t \in N_{s(1(s)-1)+1}$ be the $*$ -extension of p^s deciding \mathcal{Q} chosen in the proof of Lemma 3.9. Let $t = \langle t_0, \mathcal{t}_1 \rangle, t_0 = \langle s, B \rangle$, and let γ be such that $\langle t_0, \mathcal{t}_1 \rangle \Vdash \mathcal{Q} = \gamma$. Let $n \in A \cap B$. Then

- $p_0^{s \frown \langle n \rangle}, t_0$ and p_0^s are compatible and $\langle s \frown \langle n \rangle, A \cap B \cap A_{s \frown \langle n \rangle} \rangle$ extends them,
- $p^{s \frown \langle n \rangle} \leq t$.

Thus $p^{s \frown \langle n \rangle} \Vdash \mathcal{Q} = \gamma$. Let $u = \langle s \frown \langle n \rangle, A \cap B \cap A_{s \frown \langle n \rangle} \setminus (n+1) \rangle$.

Then $u \leq p_0^{s \frown \langle n \rangle}$ and $u \Vdash \mathcal{r}_1$ extends $p_1^{s \frown \langle n \rangle}$ which extends \mathcal{t}_1 . Thus $\langle u, \mathcal{r}_1 \rangle \leq t, \langle r_0, \mathcal{r}_1 \rangle, p^{s \frown \langle n \rangle}$. It follows that $\alpha = \gamma$. Now $\delta < \gamma$ and $p^{s \frown \langle n \rangle} \Vdash \mathcal{Q} = \gamma$. Hence $\langle p_0^{s \frown \langle n \rangle}, \delta \rangle \in \mathcal{B}$ and $p^{s \frown \langle n \rangle} \Vdash \delta \in \mathcal{B}$. This is impossible since $\langle r_0, \mathcal{r}_1 \rangle \Vdash \delta \notin \mathcal{B}$.

Case 2. There are $\langle r_0, \mathcal{r}_1 \rangle \leq \langle \langle \langle \rangle, \omega \rangle, q_1 \rangle$ and δ such that $\langle r_0, \mathcal{r}_1 \rangle \Vdash \text{“}\delta \in \mathcal{B} \text{ and } \delta \notin \mathcal{Q}\text{”}$. We may further suppose that for some ordinal α , $\langle r_0, \mathcal{r}_1 \rangle \Vdash \mathcal{Q} = \alpha$. Thus $\delta \geq \alpha$. Let $r = \langle s, A \rangle$. Then as above p_0^s is compatible with r and there is a $*$ -extension of p^s deciding \mathcal{Q} . Choose t as in Case 1, $t = \langle t_0, \mathcal{t}_1 \rangle, t_0 = \langle s, B \rangle$ and let γ be such that $\langle t_0, \mathcal{t}_1 \rangle \Vdash \mathcal{Q} = \gamma$. Let $n \in A \cap B$. Then as in Case 1, $\alpha = \gamma$ and $p^{s \frown \langle n \rangle} \Vdash \mathcal{Q} = \gamma$. On the other hand since $\langle r_0, \mathcal{r}_1 \rangle \Vdash \delta \in \mathcal{B}$, we can find \bar{s} such that \bar{s} does not contradict $p_0^{s \frown \langle n \rangle}, \langle p_0^{\bar{s}}, p_1^{\bar{s}} \rangle \Vdash \mathcal{Q} = \bar{\gamma}$, for some $\bar{\gamma} > \delta$ and $\langle p_0^{\bar{s}}, \delta \rangle \in \mathcal{B}$. Now $\bar{\gamma} = \gamma = \alpha > \delta$ which is in contradiction with $\delta \geq \alpha$. The claim follows. \square

This completes the proof of Lemma 3.9. \square

Lemma 3.13. *Let $\langle p_0, p_1 \rangle \Vdash \mathcal{f} : \omega \rightarrow ON$. Then there are \mathbb{P}_U -names g and q_1 such that $\langle p_0, q_1 \rangle \leq^* \langle p_0, p_1 \rangle$ and $\langle p_0, q_1 \rangle \Vdash \mathcal{f} = g$.*

Proof. For simplicity suppose that $\langle p_0, p_1 \rangle = \langle \langle \langle \rangle, \omega \rangle, \emptyset \rangle$. Let θ be large enough regular and let $\langle N_n : n < \omega \rangle$ be an increasing sequence of countable elementary submodels of H_θ

such that $\mathbb{P}, \underline{f} \in N_0$ and $N_n \in N_{n+1}$ for every $n < \omega$. Let $N = \bigcup_{n < \omega} N_n$, $\delta_n = N_n \cap \omega_1$ for $n < \omega$ and $\delta = \bigcup_{n < \omega} \delta_n = N \cap \omega_1$. Let $\langle J_n : n < \omega \rangle \in N_0$ and $\langle \alpha_i : 0 < i < \omega \rangle$ be as in Lemma 3.9.

We define by induction a sequence $\langle p^s : s \in [\omega]^{<\omega} \rangle$ of conditions and a sequence $\langle \underline{\beta}_s : s \in [\omega]^{<\omega} \rangle$ of \mathbb{P}_U -names for ordinals such that

- $p^s = \langle p_0^s, \underline{p}_1^s \rangle = \langle \langle s, \omega \setminus (\max(s) + 1) \rangle, \underline{p}_1^s \rangle$,
- $p^s \in N_{s \upharpoonright (l(s)-1)+1}$,
- $l(p^s) \geq \delta_{s \upharpoonright (l(s)-1)}$,
- $p^s \Vdash \underline{f}(l(s) - 1) = \underline{\beta}_s$,
- if t end extends s , then $p^t \leq p^s$.

For $s = \langle \rangle$, let $p^{\langle \rangle} = \langle \langle \langle \rangle, \omega \rangle, \emptyset \rangle$. Now suppose that $s \neq \langle \rangle$ and $p^{s \upharpoonright (l(s)-1)}$ is defined. We define p^s . Let $C_{s \upharpoonright (l(s)-1)}$ be an infinite subset of ω almost disjoint to $\langle \text{range}(p_1^{s \upharpoonright (l(s)-1)}(\beta)) : \beta < l(p^{s \upharpoonright (l(s)-1)}) \rangle$. Split $C_{s \upharpoonright (l(s)-1)}$ into ω infinite disjoint sets $\langle C_{s \upharpoonright (l(s)-1, t} : t \in [\omega]^{<\omega}$ and t end extends $s \upharpoonright (l(s) - 1)$. Again split $C_{s \upharpoonright (l(s)-1, s}$ into ω infinite disjoint sets $\langle C_i : i < \omega \rangle$. Let $\langle c_{ij} : j < \omega \rangle$ be an increasing enumeration of C_i , $i < \omega$. We may suppose that all of these is done in $N_{s \upharpoonright (l(s)-1)+1}$. Let $q^s = \langle q_0^s, \underline{q}_1^s \rangle$, where

- $q_0^s = \langle s, \omega \setminus (\max(s) + 1) \rangle$,
- for $\beta < l(p^{s \upharpoonright (l(s)-1)})$, $\underline{q}_1^s(\beta) = \underline{p}_1^{s \upharpoonright (l(s)-1)}(\beta)$,
- for $i \in J_{s \upharpoonright (l(s)-1)}$ such that $\alpha_i \in \delta_{s \upharpoonright (l(s)-1)} \setminus l(p^{s \upharpoonright (l(s)-1)})$

$$\underline{q}_1^s(\alpha_i) = \{ \langle \langle s \frown \langle r_1, \dots, r_i \rangle, \omega \setminus (r_i + 1) \rangle, c_{ir_i} \rangle : r_1 > \max(s), \langle r_1, \dots, r_i \rangle \in [\omega]^i \}.$$

Then $q^s \in N_{s \upharpoonright (l(s)-1)+1}$ and as in the proof of claim 3.10, $q^s \in \mathbb{P}$. By Lemma 3.9, applied inside $N_{s \upharpoonright (l(s)-1)+1}$, we can find \mathbb{P}_U -names $\underline{\beta}_s$ and \underline{p}_1^s such that $\langle q_0^s, \underline{p}_1^s \rangle \leq \langle q_0^s, \underline{q}_1^s \rangle$ and $\langle q_0^s, \underline{p}_1^s \rangle \Vdash \underline{f}(l(s) - 1) = \underline{\beta}_s$. Let $p^s = \langle p_0^s, \underline{p}_1^s \rangle = \langle q_0^s, \underline{p}_1^s \rangle$. Then $p^s \leq p^{s \upharpoonright (l(s)-1)}$ and $p^s \Vdash \underline{f} \upharpoonright l(s) = \{ \langle i, \underline{\beta}_{s \upharpoonright i+1} \rangle : i < l(s) \}$.

This completes our definition of the sequences $\langle p^s : s \in [\omega]^{<\omega} \rangle$ and $\langle \underline{\beta}_s : s \in [\omega]^{<\omega} \rangle$. Let

$$\underline{q}_1 = \{ \langle p_0^s, \langle \beta, \underline{p}_1^s(\beta) \rangle \rangle : s \in [\omega]^{<\omega}, \beta < l(p^s) \},$$

$$\underline{g} = \{ \langle p_0^s, \langle i, \underline{\beta}_{s \upharpoonright i+1} \rangle \rangle : s \in [\omega]^{<\omega}, i < l(s) \}.$$

Then \underline{q}_1 and \underline{g} are \mathbb{P}_U -names.

Claim 3.14. $\langle\langle \cdot \rangle, \omega \rangle, \underline{q}_1 \in \mathbb{P}$.

Proof. We check conditions in Definition 3.4.

(1) i.e $\langle\langle \cdot \rangle, \omega \rangle \in \mathbb{P}_U$ is trivial.

(2) It is clear by our construction that

$$\langle\langle \cdot \rangle, \omega \rangle \Vdash \text{“} \underline{q}_1 \text{ is a well-defined function”}$$

and as in the proof of claim 3.11, we can show that $l(\underline{q}_1) = \delta$. (2a) is trivial. Let us prove (2b). Thus suppose that $I \subseteq \delta$, $I \in V$, $p \leq \langle\langle \cdot \rangle, \omega \rangle$ and $J \subseteq \omega$ is finite. Let $p = \langle s, A \rangle$. If $s = \langle \cdot \rangle$, then as in the proof of 3.11, we can show that $a = \emptyset$ is a required. Thus suppose that $s \neq \langle \cdot \rangle$. First we apply (2b) to p^s , $I \cap l(p^s)$, p and J to find $a' \subseteq l(p^s)$ such that

$$(*) \quad \text{For every finite } b \subseteq I \cap l(p^s) \setminus a' \text{ there is } p' \leq^* p \text{ such that } p' \\ \Vdash (\forall \beta \in b, \forall k \in J, \underline{p}_1^s(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underline{p}_1^s(\beta_1) \neq \underline{p}_1^s(\beta_2)).$$

Let $\delta_{s(l(s)-1)+1} \setminus \delta_{s(l(s)-1)} = \{\alpha_{J_1}, \dots, \alpha_{J_i}, \dots\}$ where $J_1 < J_2 < \dots$ are in $J_{s(l(s)-1)+1}$. Let

$$a = a' \cup \{\alpha_1, \alpha_2, \dots, \alpha_{J_{l(s)}}\}.$$

We show that a is as required. Let $b \subseteq I \setminus a$ be finite. First we apply (*) to $b \cap l(p^s)$ to find $p' = \langle s, A' \rangle \leq^* p$ such that

$$p' \Vdash (\forall \beta \in b \cap l(p^s), \forall k \in J, \underline{p}_1^s(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b \cap l(p^s), \underline{p}_1^s(\beta_1) \neq \underline{p}_1^s(\beta_2)).$$

Also note that for $\beta \in b \cap l(p^s)$, $p' \Vdash \underline{q}_1(\beta) = \underline{p}_1^s(\beta)$. Pick m such that $\max(s) + \max(J) + 1 < m < \omega$ and if t end extends s and $m < \max(t)$, then $C_{s,t}$ is disjoint to J and to $\text{range}(\underline{p}_1^s(\beta))$ for $\beta \in b \cap l(p^s)$. Then pick $n > m, n \in A'$ such that $b \subseteq \delta_n$, and let $t = s \frown \langle n \rangle$. Then

- $\forall \beta_1 \neq \beta_2 \in b \setminus l(p^s)$, $\text{range}(\underline{p}_1^t(\beta_1)) \cap \text{range}(\underline{p}_1^t(\beta_2)) = \emptyset$,
- $\forall \beta_1 \in b \cap l(p^s)$, $\forall \beta_2 \in b \setminus l(p^s)$, $\text{range}(\underline{p}_1^t(\beta_1)) \cap \text{range}(\underline{p}_1^t(\beta_2)) = \emptyset$,
- $\forall \beta \in b \setminus l(p^s)$, $\text{range}(\underline{p}_1^t(\beta)) \cap J = \emptyset$.

Let $q = \langle s, B \rangle = \langle s, A' \setminus (n+1) \rangle$. Then $q \leq^* p' \leq^* p$ and using the above facts we can show that

$$q \Vdash (\forall \beta \in b, \forall k \in J, \underset{\sim}{q}_1(\beta) = \underset{\sim}{p}_1^t(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{q}_1(\beta_1) = \underset{\sim}{p}_1^t(\beta_1) \neq \underset{\sim}{p}_1^t(\beta_2) = \underset{\sim}{q}_1(\beta_2)).$$

Thus q is as required and the claim follows. \square

Claim 3.15. $\langle \langle \langle \rangle, \omega \rangle, \underset{\sim}{q}_1 \rangle \Vdash \underset{\sim}{f} = \underset{\sim}{g}$.

Proof. Suppose not. Then we can find $\langle r_0, \underset{\sim}{r}_1 \rangle \leq \langle \langle \langle \rangle, \omega \rangle, \underset{\sim}{q}_1 \rangle$ and $i < \omega$ such that $\langle r_0, \underset{\sim}{r}_1 \rangle \Vdash \underset{\sim}{f}(i) \neq \underset{\sim}{g}(i)$. Let $r_0 = \langle s, A \rangle$. Then r_0 is compatible with p_0^s and $r_0 \Vdash \text{“}\underset{\sim}{r}_1 \text{ extends } p_1^s\text{”}$. Hence $\langle r_0, \underset{\sim}{r}_1 \rangle \leq \langle p_0^s, \underset{\sim}{p}_1^s \rangle = p^s$. Now $p^s \Vdash \underset{\sim}{g}(i) = \underset{\sim}{\beta}_{s \upharpoonright i+1} = \underset{\sim}{f}(i)$, and we get a contradiction. The claim follows. \square

This completes the proof of Lemma 3.13. \square

The following is now immediate.

Lemma 3.16. *The forcing (\mathbb{P}, \leq) preserves cofinalities.*

Proof. By Lemma 3.13, \mathbb{P} preserves cofinalities $\leq \omega_1$. On the other hand by a Δ -system argument, \mathbb{P} satisfies the ω_2 -c.c. and hence it preserves cofinalities $\geq \omega_2$. \square

Lemma 3.17. *Let G be (\mathbb{P}, \leq) -generic over V . Then $V[G] \models GCH$.*

Proof. By Lemma 3.13, $V[G] \models CH$. Now let $\kappa \geq \omega_1$. Then

$$(2^\kappa)^{V[G]} \leq ((|\mathbb{P}^{\omega_1}|)^\kappa)^V \leq (2^\kappa)^V = \kappa^+.$$

The result follows. \square

Now we return to the proof of Theorem 3.1. Suppose that G is (\mathbb{P}, \leq) -generic over V , and let $V_1 = V[G]$. Then V_1 is a cofinality and GCH preserving generic extension of V . We show that adding a Cohen real over V_1 produces \aleph_1 -many Cohen reals over V . Thus force to add a Cohen real over V_1 . Split it into ω Cohen reals over V_1 . Denote them by $\langle r_{n,m} : n, m < \omega \rangle$. Also let $\langle f_i : i < \omega_1 \rangle \in V$ be a sequence of almost disjoint functions from ω into ω . First we define a sequence $\langle s_{n,i} : i < \omega_1 \rangle$ of reals by

$$\forall k < \omega, s_{n,i}(k) = r_{n,f_i(k)}(0).$$

Let $\langle I_n : n < \omega \rangle$ be the partition of ω_1 produced by G . For $\alpha < \omega_1$ let

- $n(\alpha) =$ that $n < \omega$ such that $\alpha \in I_n$,
- $i(\alpha) =$ that $i < \omega_1$ such that α is the i -th element of $I_{n(\alpha)}$.

We define a sequence $\langle t_\alpha : \alpha < \omega_1 \rangle$ of reals by $t_\alpha = s_{n(\alpha),i(\alpha)}$. The following lemma completes the proof of Theorem 3.1.

Lemma 3.18. $\langle t_\alpha : \alpha < \omega_1 \rangle$ is a sequence of \aleph_1 -many Cohen reals over V .

Proof. First note that $\langle r_{n,m} : n, m < \omega \rangle$ is $\mathbb{C}(\omega \times \omega)$ -generic over V_1 . By c.c.c. of $\mathbb{C}(\omega_1)$ it suffices to show that for every countable $I \subseteq \omega_1$, $I \in V$, $\langle t_\alpha : \alpha \in I \rangle$ is $\mathbb{C}(I)$ -generic over V .

Thus it suffices to prove the following

$$(*) \quad \text{For every } \langle \langle p_0, \underline{p}_1 \rangle, q \rangle \in \mathbb{P} * \mathbb{C}(\omega \times \omega) \text{ and every open dense subset } D \in V \text{ of } \mathbb{C}(I), \text{ there is } \langle \langle q_0, \underline{q}_1 \rangle, r \rangle \leq \langle \langle p_0, \underline{p}_1 \rangle, q \rangle \text{ such that } \langle \langle q_0, \underline{q}_1 \rangle, r \rangle \Vdash \text{“} \langle \underline{t}_\nu : \nu \in I \rangle \text{ extends some element of } D \text{”}$$

Let $\langle \langle p_0, \underline{p}_1 \rangle, q \rangle$ and D be as above. Let $\alpha = \sup(I)$. We may suppose that $l(\underline{p}_1) \geq \alpha$. Let $J = \{n : \exists m, k, \langle n, m, k \rangle \in \text{dom}(q)\}$. We apply (2b) to $\langle p_0, \underline{p}_1 \rangle, I, p_0$ and J to find a finite set $a \subseteq I$ such that:

$$(**) \quad \text{For every finite } b \subseteq I \setminus a \text{ there is } p'_0 \leq^* p_0 \text{ such that } p'_0 \Vdash (\forall \beta \in b, \forall k \in J, \underline{p}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underline{p}_1(\beta_1) \neq \underline{p}_1(\beta_2)).$$

Let

$$S = \{ \langle \nu, k, j \rangle : \nu \in a, k < \omega, j < 2, \langle n(\nu), f_{i(\nu)}(k), 0, j \rangle \in q \}.$$

Then $S \in \mathbb{C}(\omega_1)$. Pick $k_0 < \omega$ such that for all $\nu_1 \neq \nu_2 \in a$, and $k \geq k_0$, $f_{i(\nu_1)}(k) \neq f_{i(\nu_2)}(k)$.

Let

$$S^* = S \cup \{ \langle \nu, k, 0 \rangle : \nu \in a, k < \kappa_0, \langle \nu, k, 1 \rangle \notin S \}.$$

The reason for defining S^* is to avoid possible collisions. Then $S^* \in \mathbb{C}(\omega_1)$. Pick $S^{**} \in D$ such that $S^{**} \leq S^*$. Let $b = \{\nu : \exists k, j, \langle \nu, k, j \rangle \in S^{**}\} \setminus q$. By $(**)$ there is $p'_0 \leq^* p_0$ such that

$$p'_0 \Vdash (\forall \nu \in b, \forall k \in J, \underline{p}_1(\nu) \neq k) \& (\forall \nu_1 \neq \nu_2 \in b, \underline{p}_1(\nu_1) \neq \underline{p}_1(\nu_2)).$$

Let $p''_0 \leq p'_0$ be such that $\langle p''_0, \underline{p}_1 \rangle$ decides all the colors of elements of $a \cup b$. Let

$$q^* = q \cup \{\langle n(\nu), f_{i(\nu)}(k), 0, S^{**}(\nu, k) \rangle : \langle \nu, k \rangle \in \text{dom}(S^{**})\}.$$

Then q^* is well defined and $q^* \in \mathbb{C}(\omega \times \omega)$. Now $q^* \leq q$, $\langle \langle p''_0, \underline{p}_1 \rangle, q^* \rangle \leq \langle \langle p_0, \underline{p}_1 \rangle, q \rangle$ and for $\langle \nu, k \rangle \in \text{dom}(S^{**})$

$$\langle \langle p''_0, \underline{p}_1 \rangle, q^* \rangle \Vdash S^{**}(\nu, k) = q^*(n(\nu), f_{i(\nu)}(k), 0) = \underline{r}_{n(\nu), f_{i(\nu)}(k)}(0) = \underline{t}_\nu(k).$$

It follows that

$$\langle \langle p''_0, \underline{p}_1 \rangle, q^* \rangle \Vdash \text{“}\langle \underline{t}_\nu : \nu \in I \rangle \text{ extends } S^{**}\text{”}.$$

(*) and hence Lemma 3.18 follows. \square

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REFERENCES

- [Cu] J. Cummings, A model in which GCH holds at successors but fails at limits, *Trans. Amer. Math. Soc.* 329 (1992), no. 1, 139. MR1041044 (92h:03076)
- [D-J 1] A. Dodd, R. Jensen, The covering lemma for $L[U]$. *Ann. Math. Logic* 22 (1982), no. 2, 127135. MR0667224 (83i:03082b)
- [D-J 2] A. Dodd, R. Jensen, The covering lemma for K . *Ann. Math. Logic* 22 (1982), no. 1, 130. MR0661475 (83i:03082a)
- [Git 1] M. Gitik, Adding a lot of Cohen reals by adding a few, Unpublished paper.
- [Git 2] M. Gitik, Prikry type forcings, *Handbook of Set Theory*, (2010), 1351-1447.
- [Git-Mag 1] M. Gitik, M. Magidor, The singular cardinal hypothesis revisited. *Set theory of the continuum* (Berkeley, CA, 1989), 243279, *Math. Sci. Res. Inst. Publ.*, 26, Springer, New York, 1992. MR1233822 (95c:03131)
- [Git-Mag 2] M. Gitik, M. Magidor, Extender based forcings, *J. Symbolic Logic* 59 (1994), no. 2, 445460. MR1276624 (95k:03079)

- [G-N-S] M. Gitik, I. Neeman, D. Sinapova, A cardinal preserving extension making the set of points of countable VV cofinality nonstationary. *Arch. Math. Logic* 46 (2007), no. 5-6, 451456. MR2321586 (2008e:03085)
- [Mag 1] M. Magidor, Changing cofinality of cardinals, *Fund. Math.* 99 (1978), no. 1, 6171. MR0465868 (57:5754)
- [Mag 2] M. Magidor, How large is the first strongly compact cardinal, *Ann. Math. Logic* 10 (1976), no. 1, 3357. MR0429566 (55:2578)
- [Mer] C. Merimovich, Prikry on extenders, revisited. *Israel J. Math.* 160 (2007), 253280. MR2342498 (2008j:03075)
- [Sh 1] S. Shelah, Proper forcing, *Lecture Notes in Mathematics*, 940. Springer-Verlag, Berlin-New York, 1982. MR0675955(84h:03002)
- [Sh 2] S. Shelah, Cardinal arithmetic, *Oxford Logic Guides*, 29. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994. MR1318912 (96e:03001)
- [Ve-W] B. Veličković, H. Woodin, Complexity of reals in inner models of set theory, *Ann. Pure Appl. Logic* 92 (1998), no. 3, 283295. MR1640916 (99f:03067)