

ADDING A LOT OF COHEN REALS BY ADDING A FEW II

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ABSTRACT. We study pairs (V, V_1) , $V \subseteq V_1$, of models of ZFC such that adding κ -many Cohen reals over V_1 adds λ -many Cohen reals over V for some $\lambda > \kappa$.

1. INTRODUCTION

We continue our study from [3]. We study pairs (V, V_1) , $V \subseteq V_1$, of models of ZFC with the same ordinals, such that adding κ -many Cohen reals over V_1 adds λ -many Cohen reals over V for some $\lambda > \kappa$ ¹. We are mainly interested when V and V_1 have the same cardinals and reals. We prove that for such models, adding κ -many Cohen reals over V_1 cannot produce more Cohen reals over V for κ below the first fixed point of the \aleph -function, but the situation at the first fixed point of the \aleph -function is different. We also reduce the large cardinal assumptions from [1, 3] to the optimal ones.

2. ADDING MANY COHEN REALS BY ADDING A FEW: A GENERAL RESULT

In this section we prove the following general result.

Theorem 2.1. *Suppose $\kappa < \lambda$ are infinite (regular or singular) cardinals, and let V_1 be an extension of V . Suppose that in V_1 :*

- (a) $\kappa < \lambda$ are still infinite cardinals²,
- (b) there exists an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals, cofinal in κ . In particular $cf(\kappa) = \omega$,
- (c) there is an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in the product $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$,
- (d) there is a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ into sets of size λ such that for every countable

¹By “ λ -many Cohen reals” we mean a generic object $\langle s_\alpha : \alpha < \lambda \rangle$ for the poset $\mathbb{C}(\lambda)$ of finite partial functions from $\lambda \times \omega$ to 2 ”.

² λ can be a regular or a singular cardinal, but by (b), κ is necessarily a singular cardinal of cofinality ω .

set $I \in V$ and every $\sigma < \kappa$ we have $|I \cap S_\sigma| < \aleph_0$.

Then adding κ -many Cohen reals over V_1 produces λ -many Cohen reals over V .

Remark 2.2. Condition (c) holds automatically for $\lambda = \kappa^+$; given any collection \mathcal{F} of κ -many elements of $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$, there exists f such that for each $g \in \mathcal{F}$, $f(n) > g(n)$ for all large n ³. Thus we can define by induction on $\alpha < \kappa^+$, an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ in $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$ ⁴.

Proof. Force to add κ -many Cohen reals over V_1 . Split them into $\langle r_{i,\sigma} : i, \sigma < \kappa \rangle$ and $\langle r'_\sigma : \sigma < \kappa \rangle$. Also in V , split κ into κ -blocks $B_\sigma, \sigma < \kappa$, each of size κ , and let $\langle f_\alpha : \alpha < \lambda \rangle \in V_1$ be an increasing (mod finite) sequence in $\prod_{n < \omega} (\kappa_{n+1} \setminus \kappa_n)$. Let $\alpha < \lambda$. We define a real s_α as follows. Pick $\sigma < \kappa$ such that $\alpha \in S_\sigma$. Let $k_\alpha = \min\{k < \omega : r'_\sigma(k)\} = 1$ and set

$$\forall n < \omega, s_\alpha(n) = r_{f_\alpha(n+k_\alpha), \sigma}(0).$$

The following lemma completes the proof.

Lemma 2.3. $\langle s_\alpha : \alpha < \lambda \rangle$ is a sequence of λ -many Cohen reals over V .

Notation 2.4. (a) For a forcing notion \mathbb{P} and $p, q \in \mathbb{P}$, we let $p \leq q$ mean p is stronger than q .

(b) For each set I , let $\mathbb{C}(I)$ be the Cohen forcing notion for adding I -many Cohen reals. Thus $\mathbb{C}(I) = \{p : p \text{ is a finite partial function from } I \times \omega \text{ into } 2\}$, ordered by $p \leq q$ iff $p \supseteq q$.

Proof. First note that $\langle \langle r_{i,\sigma} : i, \sigma < \kappa \rangle, \langle r'_\sigma : \sigma < \kappa \rangle \rangle$ is $\mathbb{C}(\kappa \times \kappa) \times \mathbb{C}(\kappa)$ -generic over V_1 . By the c.c.c. of $\mathbb{C}(\lambda)$ it suffices to show that for any countable set $I \subseteq \lambda$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I \rangle$ is $\mathbb{C}(I)$ -generic over V . Thus it suffices to prove the following

For every $(p, q) \in \mathbb{C}(\kappa \times \kappa) \times \mathbb{C}(\kappa)$ and every open dense subset $D \in V$

(*) of $\mathbb{C}(I)$, there is $(\bar{p}, \bar{q}) \leq (p, q)$ such that $(\bar{p}, \bar{q}) \Vdash \langle \check{s}_\alpha : \alpha \in I \rangle$ extends some element of D .

³To see this let $\mathcal{F} = \bigcup_{n < \omega} \mathcal{F}_n$, where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots$ and $|\mathcal{F}_n| < \kappa_{n+1}$, and define f so that $\sup\{g(n) : g \in \mathcal{F}_n\} < f(n) \in \kappa_{n+1} \setminus \kappa_n$.

⁴Let f_0 be arbitrary. Given $\alpha < \kappa^+$, we can apply the above to find f_α so that $f_\alpha(n) > f_\beta(n)$, for all large n , and all $\beta < \alpha$.

Let (p, q) and D be as above and for simplicity suppose that $p = q = \emptyset$. Let $b \in D$, and let $\alpha_1, \dots, \alpha_m$ be an enumeration of the components of b , i.e., those α such that $(\alpha, n) \in \text{dom}(b)$ for some n . Also let $\sigma_1, \dots, \sigma_m < \kappa$ be such that $\alpha_i \in S_{\sigma_i}, i = 1, \dots, m$. By (d) each $I \cap S_{\sigma_i}$ is finite, thus by (c) we can find $n^* < \omega$ such that for all $n \geq n^*, 1 \leq i \leq m$ and $\alpha_1^* < \alpha_2^*$ in $I \cap S_{\sigma_i}$ we have $f_{\alpha_1^*}(n) < f_{\alpha_2^*}(n)$. Let

$$\bar{q} = \{\langle \sigma_i, n, 0 \rangle : 1 \leq i \leq m, n < n^*\}.$$

Then $\bar{q} \in \mathbb{C}(\kappa)$ and $(\emptyset, \bar{q}) \Vdash^{-\Gamma} k_{\alpha_i} \geq n^{*\neg}$ for all $1 \leq i \leq m$. Let

$$\bar{p} = \{\langle f_{\alpha_i}(n + k_{\alpha_i}), \sigma_i, 0, b(\alpha_i, n) \rangle : 1 \leq i \leq m, (\alpha_i, n) \in \text{dom}(b)\}.$$

Then $\bar{p} \in \mathbb{C}(\kappa \times \kappa)$ is well-defined and for $(\alpha_i, n) \in \text{dom}(b), 1 \leq i \leq m$ we have

$$(\bar{p}, \bar{q}) \Vdash^{-\Gamma} \mathfrak{L}_{\alpha_i}(n) = \mathfrak{L}_{f_{\alpha_i}(n+k_{\alpha_i}), \sigma_i}(0) = \bar{p}(f_{\alpha_i}(n + k_{\alpha_i}), \sigma_i, 0) = b(\alpha_i, n)^{\neg}$$

and hence

$$(\bar{p}, \bar{q}) \Vdash^{-\Gamma} \langle \mathfrak{L}_{\alpha} : \alpha \in I \rangle \text{ extends } b^{\neg}.$$

(*) follows and we are done. □

The theorem follows. □

3. GETTING RESULTS FROM OPTIMAL HYPOTHESES

Theorem 3.1. *Suppose GCH holds and κ is a cardinal of countable cofinality and there are κ -many measurable cardinals below κ . Then there is a cardinal preserving not adding a real extension V_1 of V in which there is a splitting $\langle S_{\sigma} : \sigma < \kappa \rangle$ of κ^+ into sets of size κ^+ such that for every countable set $I \in V$ and every $\sigma < \kappa, |I \cap S_{\sigma}| < \aleph_0$.*

Proof. Let X be a set of measurable cardinals below κ of size κ which is discrete, i.e., contains none of its limit points, and for each $\xi \in X$ fix a normal measure U_{ξ} on ξ . For each $\xi \in X$ let \mathbb{P}_{ξ} be the Prikry forcing associated with the measure U_{ξ} and let \mathbb{P}_X be the Magidor iteration of \mathbb{P}_{ξ} 's, $\xi \in X$ (cf. [2, 5]). Since X is discrete, each condition in \mathbb{P}_X can be seen as $p = \langle \langle s_{\xi}, A_{\xi} \rangle : \xi \in X \rangle$ where for $\xi \in X, \langle s_{\xi}, A_{\xi} \rangle \in \mathbb{P}_{\xi}$ and $\text{supp}(p) = \{\xi \in X : s_{\xi} \neq \emptyset\}$ is finite. We may further suppose that for each $\xi \in X$ the Prikry sequence for ξ is contained in $(\text{sup}(X \cap \xi), \xi)$. Let G be \mathbb{P}_X -generic over V . Note that G is uniquely determined by a

sequence $(x_\xi : \xi \in X)$, where each x_ξ is an ω -sequence cofinal in ξ , V and $V[G]$ have the same cardinals, and GCH holds in $V[G]$.

Work in $V[G]$. We now force $\langle S_\sigma : \sigma < \kappa \rangle$ as follows. The set of conditions \mathbb{P} consists of pairs $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in V[G]$ such that:

- (1) $\tau < \kappa^+$,
- (2) $\langle s_\sigma : \sigma < \kappa \rangle$ is a splitting of τ ,
- (3) for every countable set $I \in V$ and every $\sigma < \kappa$, $|I \cap s_\sigma| < \aleph_0$.

Remark 3.2. (a) Given a condition $p \in \mathbb{P}$ as above, p decides an initial segment of S_σ , namely $S_\sigma \cap \tau$, to be s_σ . Condition (3) guarantees that each component in this initial segment has finite intersection with countable sets from the ground model.

(b) Let $t_0 = \bigcup_{\xi \in X} x_\xi$. By genericity arguments, it is easily seen that t_0 is a subset of κ of size κ such that for all countable sets $I \in V$, $|I \cap t_0| < \aleph_0$. For each $i < \kappa$ set $t_i = t_0 + i = \{\alpha + i : \alpha \in t_0\}$. Then again by genericity arguments, for every countable set $I \in V$, $|I \cap t_i| < \aleph_0$. Define $s_i, i < \kappa$ by recursion as $s_0 = t_0$ and $s_i = t_i \setminus \bigcup_{j < i} t_j$ for $i > 0$. Then $p = \langle \kappa, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in \mathbb{P}$ (since again by genericity arguments, $\langle s_\sigma : \sigma < \kappa \rangle$ is a splitting of κ), and hence \mathbb{P} is non-trivial.

We call τ the height of p and denote it by $ht(p)$. For $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle$ and $q = \langle \nu, \langle t_\sigma : \sigma < \kappa \rangle \rangle$ in \mathbb{P} we define $p \leq q$ iff

- (1) $\tau \geq \nu$,
- (2) for every $\sigma < \kappa$, $s_\sigma \cap \nu = t_\sigma$, i.e., each s_σ end extends t_σ .

Lemma 3.3. (a) \mathbb{P} satisfies the κ^{++} -c.c.,

(b) \mathbb{P} is $< \kappa$ -distributive.

Proof. (a) is trivial, as $|\mathbb{P}| \leq 2^\kappa = \kappa^+$. For (b), fix $\delta < \kappa$, δ regular, and let $p \in \mathbb{P}$ and $\underline{g} \in V[G]^\mathbb{P}$ be such that

$$p \Vdash \underline{g} : \delta \rightarrow On^\delta.$$

We find $q \leq p$ which decides \underline{g} . Fix in V a splitting of κ into δ -many sets of size κ , $\langle Z_i : i < \delta \rangle$ ⁵. Let θ be a large enough regular cardinal. Pick an increasing continuous sequence $\langle M_i : i \leq \delta \rangle$ of elementary submodels of $\langle H(\theta), \in \rangle$ of size κ such that⁶:

- (1) $\langle M_i : i \leq \delta \rangle \in V[G]$,
- (2) $p, \mathbb{P}, \underline{g}, \langle Z_i : i < \delta \rangle \in M_0$,
- (3) if $i < \delta$ is a limit ordinal, then $\langle M_j : j \leq i \rangle \in M_{i+1}$,
- (4) $cf(M_\delta \cap \kappa^+) = \delta$,
- (5) if i is not a limit ordinal, then $cf^V(M_{i+1} \cap \kappa^+) = \xi_i$ for a measurable ξ_i of V in X ,
- (6) $i < j \Rightarrow \xi_i < \xi_j$,
- (7) $\langle M_i \cap V : i \leq \delta \rangle \in V$.

For each non-limit $i < \delta$, $M_{i+1} \cap V$ is in V by clause (7), and so by clause (5), $cf^V(M_{i+1} \cap \kappa^+) = \xi_i$, where $\xi_i \in X$, so we can pick in V a cofinal in $M_{i+1} \cap \kappa^+$ sequence $\langle \eta_\alpha^i : \alpha < \xi_i \rangle$, where $\eta_\alpha^i > M_i \cap \kappa^+$, for all $\alpha < \xi_i$ ⁷.

Denote by ξ_i' the first element of the Prikry sequence of ξ_i . We define a descending sequence $p_i = \langle \tau_i, \langle s_{i,\sigma} : \sigma < \kappa \rangle \rangle$ of conditions by induction as follows:

i=0. Set $p_0 = p$.

i=j+1. Assume p_j is constructed such that $p_j \in M_j$ if j is not a limit ordinal, and $p_j \in M_{j+1}$ if j is a limit ordinal and p_j decides $\underline{g} \upharpoonright j$. Fix a bijection $f_j : Z_j \rightarrow (ht(p_j), \eta_{\xi_j'}^j)$ in M_{j+1} and set⁸

⁵Note that this is possible, as $\delta < \kappa$ are cardinals in V . The splitting can also be chosen in $V[G]$.

⁶Condition (5) can be guaranteed using the fact that the set $K = \{\alpha < \kappa^+ : cf^V(\alpha) \in X\}$ is a stationary subset of κ^+ in $V[G]$ (given M_i , build a suitable continuous increasing chain $\langle N_j : j < \kappa^+ \rangle$ consisting of models of size κ . Then $\langle \sup(N_j \cap \kappa^+) : j < \kappa^+ \rangle$ forms a club of κ^+ , and M_{i+1} can be chosen to be one of those N_j so that $\sup(N_j \cap \kappa^+) \in K$). (7) can be guaranteed by the fact that \mathbb{P}_X satisfies the κ^+ -c.c. and the models have size κ (use the fact that given any model N of size κ , there exists a model in V of the same size which contains $N \cap V$).

⁷Note that $\sup(M_{i+1} \cap \kappa^+) = M_{i+1} \cap \kappa^+$. This is because if $\xi < \kappa^+$, and $\xi \in M_{i+1}$, then since $\kappa \cup \{\kappa\} \subseteq M_{i+1}$, and $M_{i+1} \models |\xi| = \kappa$, we have $\xi \subseteq M_{i+1}$. Also, as the sequence of M_i 's is increasing continuous, $\sup(M_i \cap \kappa^+) = M_i \cap \kappa^+$ holds for limit ordinals i .

⁸It is easily seen by induction on $j \leq i$ that $ht(p_j) < \eta_{\xi_j'}^j$: if $j = 0$ or j is a successor ordinal, then $p_j \in M_j$, so $ht(p_j) \in M_j \cap \kappa^+ < \eta_{\xi_j'}^j$. If j is a limit ordinal, then $ht(p_j) = \sup_{k < j} ht(p_k) \leq \sup_{k < j} M_k \cap \kappa^+ = M_j \cap \kappa^+ < \eta_{\xi_j'}^j$.

$$p'_{j+1} = \langle \eta_{\xi'_j}^j, \langle s_{j,\sigma} \cup \{f_j(\sigma)\} : \sigma \in Z_j \rangle \wedge \langle s_{j,\sigma} : \sigma \in \kappa \setminus Z_j \rangle \rangle$$

Clearly $p'_{j+1} \in M_{j+1}$. Let $p_{j+1} \in M_{j+1}$ be an extension of p'_{j+1} which decides $g(j)$.

limit(i). Let $p_i = \langle \sup_{j < i} ht(p_j), \langle \bigcup_{j < i} s_{j,\sigma} : \sigma < \kappa \rangle \rangle$.

Let us show that the above sequence is well-defined. Thus we need to show that for each $i \leq \delta, p_i \in \mathbb{P}$. We prove this by induction on i . The successor case is trivial. Thus fix a limit ordinal $i \leq \delta$. If $p_i \notin \mathbb{P}$, we can find a countable set $I \in V, I \subseteq \kappa^+$, and $\sigma < \kappa$ such that $I \cap s_{i,\sigma}$ is infinite. Define the sequence $\langle \alpha(j) : j < i \rangle$ as follows:

- if $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$, then $\alpha(j) \in [\sup(X \cap \xi_j), \xi_j]$ is the least such that $\eta_{\alpha(j)}^j > \sup(I \cap (M_{j+1} \setminus M_j))$,
- $\alpha(j) = \sup(X \cap \xi_j)$ otherwise. Note that in this case $\alpha(j) < \xi'_j$ (because the Prikry sequence for ξ was chosen in the interval $(\sup(X \cap \xi), \xi)$).

Clearly $\langle \alpha(j) : j < i \rangle \in V$.

Lemma 3.4. *The set $K = \{j < i : \xi'_j \leq \alpha(j)\}$ is finite.*

Proof. Let $p \in \mathbb{P}_X, p = \langle \langle s_\xi, A_\xi \rangle : \xi \in X \rangle$. Extend p to $q = \langle \langle t_\xi, B_\xi \rangle : \xi \in X \rangle$ by setting

- $t_\xi = s_\xi$ and $B_\xi = A_\xi$ for $\xi \in \text{supp}(p)$,
- $t_\xi = \emptyset$ and $B_\xi = A_\xi \setminus (\alpha(j) + 1)$, if $\xi = \xi_j$ (some $j < i$) and $\xi \notin \text{supp}(p)$,
- $t_\xi = \emptyset$ and $B_\xi = A_\xi$, otherwise.

Then $q \leq p$ and $q \Vdash \ulcorner K \subseteq \{j < i : \xi_j \in \text{supp}(p)\} \urcorner$, so $q \Vdash \ulcorner K \urcorner$ is finite \urcorner . \square

Take $i_0 < i$ large enough so that no point $\geq i_0$ is in K . Then for all $j \geq i_0$ we have $\xi'_j > \alpha(j)$, hence $\eta_{\xi'_j}^j > \sup(I \cap (M_{j+1}))$ ⁹.

Claim 3.5. *We have*

$$I \cap s_{i,\sigma} \subseteq I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$$

where i_1 is the unique ordinal less than δ so that $\sigma \in Z_{i_1}$.

⁹This is trivial if $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$, as then $\eta_{\xi'_j}^j > \eta_{\alpha(j)}^j > \sup(I \cap (M_{j+1} \setminus M_j)) = \sup(I \cap M_{j+1})$. If $I \cap (M_{j+1} \setminus M_j) = \emptyset$, then $\eta_{\xi'_j}^j > M_j \cap \kappa^+ = \sup(M_j \cap \kappa^+) \geq \sup(I \cap M_j)$ (as $I \subseteq \kappa^+$) and $\sup(I \cap M_{j+1}) = \sup(I \cap M_j)$ (since I has no points in $M_{j+1} \setminus M_j$), and hence again $\eta_{\xi'_j}^j > \sup(I \cap (M_{j+1}))$.

Proof. Assume towards a contradiction that the inclusion fails, and let $t \in I \cap s_{i,\sigma}$ be such that $t \notin I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$. As i is a limit ordinal, $I \cap s_{i,\sigma} = I \cap \bigcup_{j < i} s_{j,\sigma}$. Let $j < i$ be the least such that $t \in s_{j+1,\sigma}$. Then as $t \in I \cap M_{j+1}$ and $j \geq i_0$ we have $t < \eta_{\xi_j}^j$, so that by our definition of p'_{j+1} , t must be of the form $f_j(\sigma)$, where $\sigma \in Z_j$. But then $j = i_1$ and hence $t = f_{i_1}(\sigma)$. This is a contradiction, and the result follows. \square

Thus, as $I \cap s_{i,\sigma}$ is infinite, we must have $I \cap s_{i_0,\sigma}$ is also infinite, and this is in contradiction with our inductive assumption.

It then follows that $q = p_\delta \in \mathbb{P}$ and it decides \underline{g} . \square

Let H be \mathbb{P} -generic over $V[G]$ and set $V_1 = V[G][H]$. It follows from Lemma 3.3 that all cardinals $\leq \kappa$ and $\geq \kappa^{++}$ are preserved. Also note that κ^+ is preserved, as otherwise it would have cofinality less than κ , which is impossible by the $< \kappa$ -distributivity of \mathbb{P} . Hence V_1 is a cardinal preserving and not adding reals forcing extension of $V[G]$ and hence of V . For $\sigma < \kappa$ set $S_\sigma = \bigcup_{\langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H} s_\sigma$.

Lemma 3.6. *The sequence $\langle S_\sigma : \sigma < \kappa \rangle$ is as required.*

Proof. For each $\tau < \kappa^+$, it is easily seen that the set of all conditions p such that $ht(p) \geq \tau$ is dense, so $\langle S_\sigma : \sigma < \kappa \rangle$ is a partition of κ^+ . Now suppose that $I \in V$ is a countable subset of κ^+ . Find $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H$ such that $\tau \supseteq I$. Then for all $\sigma < \kappa$, $S_\sigma \cap I = s_\sigma \cap I$, and hence $|S_\sigma \cap I| = |s_\sigma \cap I| < \aleph_0$. \square

Theorem 3.1 follows. \square

Remark 3.7. (a) *The size of a set I in V can be changed from countable to any fixed $\eta < \kappa$. Given such η , we start with the Magidor iteration of Prikry forcings above η .¹⁰ The rest of the conclusions are the same.*

¹⁰The reason for starting the iteration above η is to add no subsets of η . This will guarantee that if t_0 is defined as in Remark 3.2(b), then t_0 has finite intersection with sets from V of size η . Using this fact we can show as before that there is a splitting of κ into κ sets, each of them having finite intersection with ground model sets of size η . This makes the second step of the above forcing construction well-behaved.

(b) *It is possible to add a one element Prikry sequence to each $\xi \in X$.¹¹ Then V_1 will be a cofinality preserving generic extension of V .*

The next corollary follows from Theorem 3.1 and Remark 2.2.

Corollary 3.8. *Suppose that GCH holds in V , κ is a cardinal of countable cofinality and there are κ -many measurable cardinals below κ . Then there is a cardinal preserving not adding a real extension V_1 of V such that adding κ -many Cohen reals over V_1 produces κ^+ -many Cohen reals over V .*

Theorem 3.9. *Assume that there is no sharp for a strong cardinal. Suppose $V_1 \subseteq V_2$ have the same cardinals, same reals and there is an infinite set of ordinals S in V_2 which does not contain an infinite subset which is in V_1 . Then either*

(1) *S is countable, and then there is a measurable cardinal $\leq \sup(S)$ in \mathcal{K} ,*

or

(2) *S is uncountable, and then there is $\delta \leq \sup(S)$ which is a limit of $|S|$ -many or δ -many measurable cardinals of \mathcal{K} .*

Proof. Given a model V , let $\mathcal{K}(V)$ denote the core model of V below the strong cardinal. Note that $\mathcal{K}(V_1) = \mathcal{K}(V_2)$, since the models V_1 and V_2 agree about cardinals. We denote this common core model by \mathcal{K} .

Let us first assume that S is countable. Suppose otherwise, i.e., there are no measurable cardinals $\leq \sup(S)$ in \mathcal{K} . Then by the Covering Theorem (see [6]) there is $Y \in \mathcal{K}$, $|Y| = \aleph_1$ which covers S . Fix some $f : \aleph_1 \leftrightarrow Y$ in V_1 . Consider $Z = f^{-1} S$. Then Z also does not contain an infinite subset which is in V_1 . But Z is countable, hence there is $\eta < \omega_1$ with $Z \subseteq \eta$. Let $g : \omega \leftrightarrow \eta$ in V_1 . Consider $X = g^{-1} Z$. Then X also does not contain an infinite subset which is in V_1 . But this is impossible since V_1, V_2 have the same reals (and hence X itself is in V_1). Contradiction.

¹¹Conditions in the forcing are of the form $\langle p_\xi : \xi \in X \rangle$, where for each $\xi \in X$, p_ξ is either of the form A_ξ for some $A_\xi \in U_\xi$, or α_ξ for some $\alpha_\xi < \xi$. We also require that there are only finitely many p_ξ 's of the form α_ξ . When extending a condition, we allow either A_ξ to become thinner, or replace it by some ordinal $\alpha_\xi \in A_\xi$.

Let us deal now with the uncountable case. Suppose otherwise, i.e., there is no $\delta \leq \sup(S)$ which is a limit of $|S|$ -many or δ -many measurable cardinals of \mathcal{K} . Pick a counterexample S with $\sup(S)$ as small as possible. Denote $\sup(S)$ by δ . By minimality, δ is a cardinal. Also, the measurable cardinals of \mathcal{K} are unbounded in δ . For otherwise, let ξ be their supremum. Pick $S' \subseteq S$ of size ξ . By the Covering Theorem, S' can be covered by a set in \mathcal{K} of size $\xi < \delta$, and then we get a contradiction to the minimality of δ , as witnessed by ξ and S' ¹².

Clearly, δ must be a singular cardinal and by the above, δ is a limit of measurable cardinals in \mathcal{K} . Fix a cofinal sequence $\langle \delta_i : i < \text{cf}(\delta) \rangle$. Denote by η the cardinality of the set $\{\alpha < \delta : \alpha \text{ is a measurable cardinal in } \mathcal{K}\}$. By the assumption, $|S| > \eta \geq \text{cf}(\delta)$. But then there is $i^* < \text{cf}(\delta)$ such that $S \cap \delta_{i^*}$ has size $> \eta$. This is impossible by the minimality of δ . Contradiction. \square

The conclusions of the theorem are optimal. A Prikry sequence witnesses this in the countable case and the Magidor iteration of Prikry forcing witnesses this in the uncountable case.

Theorem 3.10. *Suppose that $V_1 \supseteq V$ are such that:*

- (a) V_1 and V have the same cardinals and reals,
- (b) $\kappa < \lambda$ are infinite cardinals of V_1 ,
- (c) there is no splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ in V_1 as in Theorem 2.1(d).

Then adding κ -many Cohen reals over V_1 cannot produce λ -many Cohen reals over V .

Proof. Suppose not. Let $\langle r_\alpha : \alpha < \lambda \rangle$ be a sequence of λ -many Cohen reals over V added after forcing with $\mathbb{C}(\kappa)$ over V_1 . Let G be $\mathbb{C}(\kappa)$ -generic over V_1 . For each $p \in \mathbb{C}(\kappa)$ set

$$C_p = \{\alpha < \lambda : p \text{ decides } r_\alpha(0)\}.$$

Then by genericity $\lambda = \bigcup_{p \in G} C_p$. Fix an enumeration $\langle p_\xi : \xi < \kappa \rangle$ of G , and define a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ in $V_1[G]$ by setting $S_\sigma = C_{p_\sigma} \setminus \bigcup_{\xi < \sigma} C_{p_\xi}$. By (a) and (c) we

¹²We then have $\sup(S') = \xi < \delta$ and S' is a counterexample to our assumption of smaller supremum.

can find a countable $I \in V$ and $\sigma < \kappa$ such that $I \subseteq S_\sigma$.¹³ Suppose for simplicity that $\forall \alpha \in S_\sigma, p_\sigma \Vdash \check{r}_\alpha(0) = 0^\top$. Let $q \in \mathbb{C}(\kappa)$ be such that

$$q \Vdash^V I \in V \text{ is countable and } \forall \alpha \in I, \check{r}_\alpha(0) = 0^\top.$$

Pick $\langle 0, \alpha \rangle \in \omega \times I$ such that $\langle 0, \alpha \rangle \notin \text{supp}(q)$. Let $\bar{q} = q \cup \{\langle \langle 0, \alpha \rangle, 1 \rangle\}$. Then $\bar{q} \in \mathbb{C}(\kappa)$, $\bar{q} \leq q$ and $\bar{q} \Vdash \check{r}_\alpha(0) = 1^\top$, which is a contradiction. \square

The following corollary answers a question from [1].

Corollary 3.11. *The following are equiconsistent:*

(a) *There exists a pair (V_1, V_2) , $V_1 \subseteq V_2$ of models of set theory with the same cardinals and reals and a cardinal κ of cofinality ω (in V_2) such that adding κ -many Cohen reals over V_2 adds more than κ -many Cohen reals over V_1 .*

(b) *There exists a cardinal δ which is a limit of δ -many measurable cardinals.*

Proof. Assume (a) holds for some pair (V_1, V_2) of models of set theory, $V_1 \subseteq V_2$ which have the same cardinals and reals. If there is a sharp for a strong cardinal, then clearly in \mathcal{K} , the core model for a strong cardinal, there is a cardinal δ which is a limit of δ -many measurable cardinals¹⁴. So assume there is no sharp for a strong cardinal. Then by Theorem 3.10 there exists a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ^+ in V_2 such that for every countable set $I \in V_1$ and $\sigma < \kappa$, $I \cap S_\sigma$ is finite. Take S to be one of the sets S_σ which has size κ^+ . So by Theorem 3.9, we get the consistency of (b)¹⁵.

Conversely if (b) is consistent, then by Corollary 3.8 the consistency of (a) follows¹⁶. \square

¹³In fact, by (c) there exist a countable $I \in V$ and some $\sigma < \kappa$ such that $I \cap S_\sigma$ is infinite. By (a), V and V_1 have the same reals, and hence $I \cap S_\sigma \in V$. So by replacing I with $I \cap S_\sigma$, if necessary, we can assume that $I \subseteq S_\sigma$.

¹⁴In fact there are many such cardinals δ .

¹⁵Note that necessarily case (b) of Theorem 3.9 happens.

¹⁶If $cf(\delta) > \omega$, then we can find $\delta^* < \delta$ of cofinality ω which is a limit of δ^* -many measurable cardinals, so that Corollary 3.8 can be applied. To see such a δ^* exists, define an increasing sequence $\delta_n, n < \omega$, of cardinals below δ , so that for any n , there are at least δ_n -many measurable cardinals below δ_{n+1} , and let $\delta^* = \sup_n \delta_n$. Also note that we can assume without loss of generality that *GCH* holds, so that Corollary 3.8 can be applied

4. BELOW THE FIRST FIXED POINT OF THE \aleph -FUNCTION

Theorem 4.1. *Suppose that $V_1 \supseteq V$ are such that V_1 and V have the same cardinals and reals. Suppose $\aleph_\delta <$ the first fixed point of the \aleph -function, $X \subseteq \aleph_\delta, X \in V_1$ and $|X| \geq \delta^+$ (in V_1). Then X has a countable subset which is in V .*

Proof. By induction on $\delta <$ the first fixed point of the \aleph -function.

Case 1. $\delta = 0$. Then $X \in V$ by the fact that V_1 and V have the same reals.

Case 2. $\delta = \delta' + 1$. We have $\delta' < \aleph_{\delta'}$, hence $\delta^+ < \aleph_\delta$, thus we may suppose that $|X| \leq \aleph_{\delta'}$. Let $\eta = \sup(X) < \aleph_\delta$. Pick $f_\eta : \aleph_{\delta'} \leftrightarrow \eta, f_\eta \in V$. Set $Y = f_\eta^{-1} X$. Then $Y \subseteq \aleph_{\delta'}, \delta' < \aleph_{\delta'}$ and $|Y| \geq \delta^+ = \delta'^+$. Hence by induction there is a countable set $B \in V$ such that $B \subseteq Y$. Let $A = f_\eta'' B$. Then $A \in V$ is a countable subset of X .

Case 3. $\text{limit}(\delta)$. Let $\langle \delta_\xi : \xi < cf\delta \rangle$ be increasing and cofinal in δ . Pick $\xi < cf\delta$ such that $|X \cap \aleph_{\delta_\xi}| \geq \delta^+$. By induction there is a countable set $A \in V$ such that $A \subseteq X \cap \aleph_{\delta_\xi} \subseteq X$. \square

The following corollary gives a negative answer to another question from [1].

Corollary 4.2. *Suppose V_1, V and δ are as in Theorem 4.1. Then adding \aleph_δ -many Cohen reals over V_1 cannot produce $\aleph_{\delta+1}$ -many Cohen reals over V .*

Proof. Towards a contradiction suppose that adding \aleph_δ -many Cohen reals over V_1 produces $\aleph_{\delta+1}$ -many Cohen reals over V . Then by Theorem 3.10, there exists $X \subseteq \aleph_{\delta+1}, X \in V_1$ such that $|X| = \aleph_{\delta+1} (\geq \delta^+)$ and X does not contain any countable subset from V ¹⁷, which is in contradiction with Theorem 4.1. \square

 5. AT THE FIRST FIXED POINT OF THE \aleph -FUNCTION

The next theorem shows that Theorem 4.1 does not extend to the first fixed point of the \aleph -function.

Theorem 5.1. *Suppose GCH holds and κ is the least singular cardinal of cofinality ω which is a limit of κ -many measurable cardinals. Then there is a pair $(V[G], V[H])$ of generic extensions of V with $V[G] \subseteq V[H]$ such that:*

¹⁷In fact, there exists a splitting $\langle S_\sigma : \sigma < \aleph_\delta \rangle$ of $\aleph_{\delta+1}$ in V_1 , consisting of sets of size $\aleph_{\delta+1}$ such that each S_σ has finite intersection with any countable set from V . The set X can be chosen to be any of the S_σ 's.

- (a) $V[G]$ and $V[H]$ have the same cardinals and reals,
- (b) κ is the first fixed point of the \aleph -function in $V[G]$ (and hence in $V[H]$),
- (c) in $V[H]$ there exists a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ into sets of size κ such that for every countable $I \in V[G]$ and $\sigma < \kappa$, $|I \cap S_\sigma| < \aleph_0$.

Proof. We first give a simple observation.

Claim 5.2. *Suppose there is $S \subseteq \kappa$ of size κ in $V[H] \supseteq V[G]$ such that for every countable $A \in V[G]$, $|A \cap S| < \aleph_0$. Then there is a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ as in (c).*

Proof. Let $\langle \alpha_i : i < \kappa \rangle$ be an increasing enumeration of S . We may further suppose that $\alpha_0 = 0$, each $\alpha_i, i > 0$ is measurable¹⁸ in V and is not a limit point of S .¹⁹ Note that for all $i < \kappa$, $\sup_{j < i} \alpha_j < \alpha_i \setminus \sup_{j < i} \alpha_j$. Now set:

$$S_0 = S,$$

$$S_\sigma = \{\alpha_l + \sigma : i \leq l < \kappa\}, \text{ for } 0 < \sigma \in [\sup_{j < i} \alpha_j, \alpha_i).$$

Then $\langle S_\sigma : \sigma < \kappa \rangle$ is as required (note that for $\sigma > 0$, $S_\sigma \subseteq S + \sigma = \{\alpha + \sigma : \alpha \in S\}$, and clearly $S + \sigma$, and hence S_σ , has finite intersection with countable sets from $V[G]$). \square

Thus it is enough to find a pair $(V[G], V[H])$ of generic extensions of V satisfying (a) and (b) with $V[G] \subseteq V[H]$ such that in $V[H]$ there is $S \subseteq \kappa$ of size κ composed of inaccessibles, such that for every countable $A \in V[G]$, $|A \cap S| < \aleph_0$.

Let X be a discrete set of measurable cardinals below κ of size κ , and for each $\xi \in X$ fix a normal measure U_ξ on ξ . For each $\xi \in X$ we define two forcing notions \mathbb{P}_ξ and \mathbb{Q}_ξ as follows.

Remark 5.3. *In the following definitions we let $\sup(X \cap \xi) = \omega$ for $\xi = \min X$.*

A condition in \mathbb{P}_ξ is of the form $p = \langle s_\xi, A_\xi, f_\xi \rangle$ where

- (1) $s_\xi \in [\xi \setminus \sup(X \cap \xi)]^{<2}$,
- (2) if $s_\xi \neq \emptyset$ then $s_\xi(0)$ is an inaccessible cardinal,

¹⁸In fact it suffices for each α_i to be inaccessible in V .

¹⁹Let $f \in V$ be such that $f : \kappa \rightarrow X$ is a bijection, where X is a discrete set of measurable cardinals of V below κ of size κ . Then if $S \subseteq \kappa$ satisfies the claim, so does $f[S]$, hence we can suppose all non-zero elements of S are measurable in V , and are not a limit point of S .

- (3) $A_\xi \in U_\xi$,
- (4) $\max s_\xi < \min A_\xi$,
- (5) $s_\xi = \emptyset \Rightarrow f_\xi \in \text{Col}(\text{sup}(X \cap \xi)^+, < \xi)$, where $\text{Col}(\text{sup}(X \cap \xi)^+, < \xi)$ is the Lévy collapse for collapsing all cardinals less than ξ to $\text{sup}(X \cap \xi)^+$, and making ξ become the successor of $\text{sup}(X \cap \xi)^+$,
- (6) $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f_\xi^1, f_\xi^2 \rangle$ where $f_\xi^1 \in \text{Col}(\text{sup}(X \cap \xi)^+, < s_\xi(0))$ and $f_\xi^2 \in \text{Col}((s_\xi(0))^+, < \xi)$.

For $p, q \in \mathbb{P}_\xi$, $p = \langle s_\xi, A_\xi, f_\xi \rangle$ and $q = \langle t_\xi, B_\xi, g_\xi \rangle$ we define $p \leq q$ iff

- (1) s_ξ end extends t_ξ ,
- (2) $A_\xi \cup (s_\xi \setminus t_\xi) \subseteq B_\xi$,
- (3) $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \leq g_\xi$,
- (4) $t_\xi = \emptyset$ and $s_\xi \neq \emptyset \Rightarrow \text{sup}(\text{ran}(g_\xi)) < s_\xi(0)$ and $f_\xi^1 \leq g_\xi$,
- (5) $t_\xi \neq \emptyset \Rightarrow f_\xi^1 \leq g_\xi^1$ and $f_\xi^2 \leq g_\xi^2$ (note that in this case we have $s_\xi = t_\xi$).

We also define $p \leq^* q$ (p is a Prikry or a direct extension of q) iff

- (1) $p \leq q$,
- (2) $s_\xi = t_\xi$.

The proof of the following lemma is essentially the same as in the proofs in [2, 5].

Lemma 5.4. (GCH) (a) \mathbb{P}_ξ satisfies the ξ^+ - c.c.

(b) Suppose $p = \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{P}_\xi$ and $l(s_\xi) = 1$ (where $l(s_\xi)$ is the length of s_ξ). Then $\mathbb{P}_\xi/p = \{q \in \mathbb{P}_\xi : q \leq p\}$ satisfies the ξ - c.c.

(c) $(\mathbb{P}_\xi, \leq, \leq^*)$ satisfies the Prikry property, i.e., given $p \in \mathbb{P}$ and a sentence σ of the forcing language for (\mathbb{P}, \leq) , there exists $q \leq^* p$ which decides σ .

(d) Let G_ξ be \mathbb{P}_ξ -generic over V and let $\langle s_\xi(0) \rangle$ be the one element sequence added by G_ξ . Then in $V[G_\xi]$, GCH holds, and the only cardinals which are collapsed are the cardinals in the intervals $(\text{sup}(X \cap \xi)^{++}, s_\xi(0))$ and $(s_\xi(0)^{++}, \xi)$, which are collapsed to $\text{sup}(X \cap \xi)^+$ and $s_\xi(0)^+$ respectively.

We now define the forcing notion \mathbb{Q}_ξ . A condition in \mathbb{Q}_ξ is of the form $p = \langle s_\xi, A_\xi, f_\xi \rangle$ where

- (1) $s_\xi \in [\xi \setminus \text{sup}(X \cap \xi)^+]^{<3}$,

- (2) if $s_\xi \neq \emptyset$ then for all $i < l(s_\xi)$, $s_\xi(i)$ is an inaccessible cardinal,
- (3) $A_\xi \in U_\xi$,
- (4) $\max s_\xi < \min A_\xi$,
- (5) $s_\xi = \emptyset \Rightarrow f_\xi \in \text{Col}(\text{sup}(X \cap \xi)^+, < \xi)$,
- (6) $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f_\xi^1, f_\xi^2 \rangle$ where, $f_\xi^1 \in \text{Col}(\text{sup}(X \cap \xi)^+, < s_\xi(0))$ and $f_\xi^2 \in \text{Col}((s_\xi(0))^+, < \xi)$.

For $p, q \in \mathbb{Q}_\xi$, $p = \langle s_\xi, A_\xi, f_\xi \rangle$ and $q = \langle t_\xi, B_\xi, g_\xi \rangle$ we define $p \leq q$ iff

- (1) s_ξ end extends t_ξ ,
- (2) $A_\xi \cup (s_\xi \setminus t_\xi) \subseteq B_\xi$,
- (3) $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \leq g_\xi$,
- (4) $t_\xi = \emptyset$ and $s_\xi \neq \emptyset \Rightarrow \text{sup}(\text{ran}(g_\xi)) < s_\xi(0)$ and $f_\xi^1 \leq g_\xi$,
- (5) $t_\xi \neq \emptyset$ and $s_\xi = t_\xi \Rightarrow f_\xi^1 \leq g_\xi^1$ and $f_\xi^2 \leq g_\xi^2$,
- (6) $t_\xi \neq \emptyset$ and $s_\xi \neq t_\xi \Rightarrow \text{sup}(\text{ran}(g_\xi^2)) < s_\xi(1)$, $f_\xi^1 \leq g_\xi^1$ and $f_\xi^2 \leq g_\xi^2$.

We also define $p \leq^* q$ iff

- (1) $p \leq q$,
- (2) $s_\xi = t_\xi$.

As above we have the following.

Lemma 5.5. *(GCH) (a) \mathbb{Q}_ξ satisfies the $\xi^+ - c.c.$*

(b) Suppose $p = \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{Q}_\xi$, $l(s_\xi) = 2$. Then $\mathbb{Q}_\xi/p = \{q \in \mathbb{Q}_\xi : q \leq p\}$ satisfies the $\xi - c.c.$.

(c) $(\mathbb{Q}_\xi, \leq, \leq^)$ satisfies the Prikry property.*

(d) Let H_ξ be \mathbb{Q}_ξ -generic over V and let $\langle s_\xi(0), s_\xi(1) \rangle$ be the two element sequence added by H_ξ . Then in $V[H_\xi]$, GCH holds, and the only cardinals which are collapsed are the cardinals in the intervals $(\text{sup}(X \cap \xi)^{++}, s_\xi(0))$ and $(s_\xi(0)^{++}, \xi)$, which are collapsed to $\text{sup}(X \cap \xi)^+$ and $s_\xi(0)^+$ respectively.

Now let \mathbb{P} be the Magidor iteration of the forcings $\mathbb{P}_\xi, \xi \in X$, and \mathbb{Q} be the Magidor iteration of the forcings $\mathbb{Q}_\xi, \xi \in X$. Since the set X is discrete we can view each condition in \mathbb{P} as a sequence $p = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$ where for each $\xi \in X$, $\langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{P}_\xi$ and $\text{supp}(p) = \{\xi : s_\xi \neq \emptyset\}$ is finite. Similarly each condition in \mathbb{Q} can be viewed as a sequence

$p = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$ where for each $\xi \in X$, $\langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{Q}_\xi$ and $\text{supp}(p) = \{\xi : s_\xi \neq \emptyset\}$ is finite (for more information see [2, 4, 5]).

Notation 5.6. *If p is as above, then we write $p(\xi)$ for $\langle s_\xi, A_\xi, f_\xi \rangle$.*

We also define

$$\pi : \mathbb{Q} \rightarrow \mathbb{P}$$

by

$$\pi(\langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle) = \langle \langle s_\xi \upharpoonright 1, A_\xi, f_\xi \rangle : \xi \in X \rangle.$$

It is clear that π is well-defined.

Lemma 5.7. *π is a projection, i.e.,*

- (a) $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$,
- (b) π is order preserving,
- (c) if $p \in \mathbb{Q}, q \in \mathbb{P}$ and $q \leq \pi(p)$ then there is $r \leq p$ in \mathbb{Q} such that $\pi(r) \leq q$.

Now let H be \mathbb{Q} -generic over V and let $G = \pi'' H$ be the filter generated by $\pi'' H$. Then G is \mathbb{P} -generic over V .

Lemma 5.8. (a) *if $\langle \tau_\xi : \xi \in X \rangle$ and $\langle \langle \eta_\xi^0, \eta_\xi^1 \rangle : \xi \in X \rangle$ are the Prikry sequences added by G and H respectively, then $\tau_\xi = \eta_\xi^0$ for all $\xi \in X$.*

(b) *The models $V[G]$ and $V[H]$ satisfy the GCH, have the same cardinals and reals, and furthermore the only cardinals of V below κ which are preserved are $\{\omega, \omega_1\} \cup \text{lim}(X) \cup \{\tau_\xi, \tau_\xi^+, \xi, \xi^+ : \xi \in X\}$.*

(c) *κ is the first fixed point of the \aleph -function in $V[G]$ (and hence in $V[H]$).*

Proof. (a) and (b) follow easily from Lemmas 5.4 and 5.5 and the definition of the projection π . Let's prove (c). It is clear that κ is a fixed point of the \aleph -function in $V[G]$. On the other hand, by (b), the only cardinals of V below κ which are preserved in $V[G]$ are $\{\omega, \omega_1\} \cup \text{lim}(X) \cup \{\tau_\xi, \tau_\xi^+, \xi, \xi^+ : \xi \in X\}$, and so if $\lambda < \kappa$ is a limit cardinal in $V[G]$, then $\lambda \in \text{lim}(X)$. But by our assumption on κ , if $\lambda \in \text{lim}(X)$, then $X \cap \lambda$ has order type less than λ , and hence $(\{\omega, \omega_1\} \cup \text{lim}(X) \cup \{\tau_\xi, \tau_\xi^+, \xi, \xi^+ : \xi \in X\}) \cap \lambda$ has order type less than \aleph_λ . Thus $\lambda < \aleph_\lambda$. \square

Let $\mathbb{Q}/G = \{p \in \mathbb{Q} : \pi(p) \in G\}$. Then $V[H]$ can be viewed as a generic extension of $V[G]$ by \mathbb{Q}/G .

Lemma 5.9. *\mathbb{Q}/G is cone homogenous: given p and q in \mathbb{Q}/G there exist $p^* \leq p, q^* \leq q$ and an isomorphism $\rho : (\mathbb{Q}/G)/p^* \rightarrow (\mathbb{Q}/G)/q^*$.*

Proof. Suppose $p, q \in \mathbb{Q}/G$. Extend p and q to $p^* = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$ and $q^* = \langle \langle t_\xi, B_\xi, g_\xi \rangle : \xi \in X \rangle$ respectively so that the following conditions are satisfied:

- (1) $\text{supp}(p^*) = \text{supp}(q^*)$. Call this common support K .
- (2) For every $\xi \in K, l(s_\xi) = l(t_\xi) = 2$. Note that then for every $\xi \in K, s_\xi(0) = t_\xi(0) = \tau_\xi, f_\xi = \langle f_\xi^1, f_\xi^2 \rangle$ and $g_\xi = \langle g_\xi^1, g_\xi^2 \rangle$ where $f_\xi^1, g_\xi^1 \in \text{Col}(\text{sup}(X \cap \xi)^+, < \tau_\xi)$ and $f_\xi^2, g_\xi^2 \in \text{Col}(\tau_\xi^+, < \xi)$.
- (3) For every $\xi \in K, A_\xi = B_\xi$.
- (4) For every $\xi \in K, \text{dom}(f_\xi^1) = \text{dom}(g_\xi^1)$ and $\text{dom}(f_\xi^2) = \text{dom}(g_\xi^2)$.
- (5) For every $\xi \in K$, there exists an automorphism ρ_ξ^1 of $\text{Col}(\text{sup}(X \cap \xi)^+, < \tau_\xi)$ such that $\rho_\xi^1(f_\xi^1) = g_\xi^1$.
- (6) For every $\xi \in K$, there exists an automorphism ρ_ξ^2 of $\text{Col}(\tau_\xi^+, < \xi)$ such that $\rho_\xi^2(f_\xi^2) = g_\xi^2$.

Note that clauses (5) and (6) are possible, as the corresponding forcing notions are homogeneous.

We now define $\rho : (\mathbb{Q}/G)/p^* \rightarrow (\mathbb{Q}/G)/q^*$ as follows. Suppose $r \in \mathbb{Q}/G, r \leq p^*$. Let $r = \langle \langle r_\xi, C_\xi, h_\xi \rangle : \xi \in X \rangle$. Then for every $\xi \in K, r_\xi = s_\xi$, and $h_\xi = \langle h_\xi^1, h_\xi^2 \rangle$ where $h_\xi^1 \in \text{Col}(\text{sup}(X \cap \xi)^+, < \tau_\xi)$ and $h_\xi^2 \in \text{Col}(\tau_\xi^+, < \xi)$. Let

$$\rho(r) = \langle \langle t_\xi, C_\xi, \langle \rho_\xi^1(h_\xi^1), \rho_\xi^2(h_\xi^2) \rangle \rangle : \xi \in K \rangle \frown \langle \langle r_\xi, C_\xi, h_\xi \rangle : \xi \in X \setminus K \rangle.$$

It is easily seen that ρ is an isomorphism from $(\mathbb{Q}/G)/p^*$ to $(\mathbb{Q}/G)/q^*$. □

The following lemma completes the proof.

Lemma 5.10. *Let $S = \{\eta_\xi^1 : \xi \in X\}$. Then S is a subset of κ of size κ and $|A \cap S| < \aleph_0$ for every countable set $A \in V[G]$.*

Remark 5.11. (a) Since $V[G]$ and $V[H]$ have the same reals, it suffices to prove the lemma for $A \subseteq S, A \in V[G]$. In fact suppose that the lemma is true for all countable $A \subseteq S, A \in V[G]$. If the lemma fails, then for some countable set $B \in V[G], |B \cap S| = \aleph_0$. Let $g : \omega \rightarrow B$ be a bijection in $V[G]$. Then $g^{-1}[B \cap S]$ is a subset of ω which is in $V[H]$, and hence in $V[G]$. Thus $B \cap S \in V[G]$. Hence we find a countable subset $A \subseteq S$ in $V[G]$, namely $B \cap S$, for which the lemma fails, which is in contradiction with our initial assumption.

(b) In what follows we say A codes ξ (for $\xi \in X$), if $\eta_\xi^1 \in A$.

Proof. Let \check{S} be a \mathbb{Q}/G -name for S . Also let $p_0 \in H \cap \mathbb{Q}/G$ be such that $p_0 \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A} \subseteq \check{S}$ is countable $^\neg$.

Claim 5.12. For every $p \in \mathbb{Q}/G$ and every $\xi \in X \setminus \text{supp}(p)$ there is $q \leq p$ in \mathbb{Q}/G such that $\xi \in \text{supp}(q)$ and if $q(\xi) = \langle s_\xi, A_\xi, f_\xi \rangle$, then $l(s_\xi) = 2$ and $q \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\neg$.

Proof. Let p and ξ be as in the claim. First pick $\langle \langle t_\xi(0) \rangle, A_\xi, f_\xi \rangle \in G$, and then let $q = p \frown \langle \langle s_\xi, A_\xi, f_\xi \rangle \rangle$, where $s_\xi(0) = t_\xi(0) = \tau_\xi$, $s_\xi(1) < \xi$ is large enough so that $s_\xi(1) \notin A$, $\text{sup}(\text{ran}(f_\xi^2)) < s_\xi(1)$ and $s_\xi(1)$ is inaccessible. Then $\pi(\langle \langle s_\xi, A_\xi, f_\xi \rangle \rangle) = \langle \langle t_\xi(0) \rangle, A_\xi, f_\xi \rangle \in G$. On the other hand $\pi(p) \in G$. Let $r \in G, r \leq \pi(p), \langle \langle t_\xi(0) \rangle, A_\xi, f_\xi \rangle$. Then $r \leq \pi(q)$, hence $\pi(q) \in G$. This implies that $q \in \mathbb{Q}/G$. Clearly q satisfies the requirements of the Claim. \square

It follows that the set

$$D = \{p \in \mathbb{Q}/G : \forall \xi \in X \setminus \text{supp}(p) \text{ there exists } q \leq p \text{ as in the above Claim}\}$$

is dense open in \mathbb{Q}/G . Let $p \in H \cap D$. We can assume that $p \leq p_0$. We show that $p \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A}$ codes ξ then $\xi \in \text{supp}(p)^\neg$. To see this suppose that $\xi \in X \setminus \text{supp}(p)$. Thus by Claim 5.12 we can find $q \leq p$ in \mathbb{Q}/G such that $\xi \in \text{supp}(q)$ and if $q(\xi) = \langle s_\xi, A_\xi, f_\xi \rangle$, then $l(s_\xi) = 2$ and $q \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\neg$. It then follows that $\sim p \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \in \check{A}^\neg$. But then by the cone homogeneity of \mathbb{Q}/G we have $p \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\neg$ ²⁰. Hence $p \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A}$ does not code ξ^\neg . This means that $p \Vdash_{\mathbb{Q}/G}^{V[G]} \check{A} \subseteq \{s_\xi(1) : \xi \in \text{supp}(p)\} = \{\eta_\xi^1 : \xi \in \text{supp}(p)\}^\neg$. Lemma 5.10

²⁰If not, then for some $p' \leq p, p' \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \in \check{A}^\neg$. By cone homogeneity of \mathbb{Q}/G we can find $q^* \leq p, p^* \leq p'$ and an isomorphism $\rho : (\mathbb{Q}/G)/p^* \rightarrow (\mathbb{Q}/G)/q^*$. But then by standard forcing arguments and the fact that $q^* \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\neg$, we can conclude that $p^* \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \notin \check{A}^\neg$, which is impossible, as $p^* \leq p'$ and $p' \Vdash_{\mathbb{Q}/G}^{V[G]} s_\xi(1) \in \check{A}^\neg$.

follows by noting that $p \in H$ and since the Magidor iteration is used, the support of any condition is finite. \square

Theorem 5.1 follows. \square

The following theorem can be proved by combining the methods of the proofs of Theorems 3.1 and 5.1.

Theorem 5.13. *Suppose GCH holds and κ is the least singular cardinal of cofinality ω which is a limit of κ -many measurable cardinals. Also let $V[G]$ and $V[H]$ be the models constructed in the proof of Theorem 5.1. Then there is a cardinal preserving, not adding a real generic extension $V[H][K]$ of $V[H]$ such that in $V[H][K]$ there exists a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ^+ into sets of size κ^+ such that for every countable set $I \in V[G]$ and $\sigma < \kappa$, $|I \cap S_\sigma| < \aleph_0$.*

Proof. Work over $V[H]$ and force the splitting $\langle S_\sigma : \sigma < \kappa \rangle$ as in the proof of Theorem 3.1, with $V, V[G]$ used there replaced by $V[G], V[H]$ here respectively. The role of the sequence $\bigcup_{\xi \in X} x_\xi$ in the proof of Theorem 3.1 is now played by the sequence $S = \{\eta_\xi^1 : \xi \in X\}$. \square

Corollary 5.14. *Suppose GCH holds and there exists a cardinal κ which is of cofinality ω and is a limit of κ -many measurable cardinals. Then there is pair (V_1, V_2) of models of ZFC, $V_1 \subseteq V_2$ such that:*

- (a) V_1 and V_2 have the same cardinals and reals.
- (b) κ is the first fixed point of the \aleph -function in V_1 (and hence in V_2).
- (c) Adding κ -many Cohen reals over V_2 adds κ^+ -many Cohen reals over V_1 .

Proof. Let $V_1 = V[G]$ and $V_2 = V[H][K]$, where $V[G], V[H][K]$ are as in Theorem 5.13. The result follows using Remark 2.2 and Theorem 5.13. \square

Acknowledgement

The authors would like to thank the referee of the paper for his/her many helpful suggestions.

REFERENCES

- [1] Gitik, Moti, Adding a lot of Cohen reals by adding a few. Unpublished paper.

- [2] Gitik, Moti, Prikry-type forcings. Handbook of set theory. Vols. 1, 2, 3, 1351-1447, Springer, Dordrecht, 2010.
- [3] Gitik, Moti; Golshani, Mohammad, Adding a lot of Cohen reals by adding a few I. Trans. Amer. Math. Soc. 367 (2015), no. 1, 209-229.
- [4] Krueger, John, Radin forcing and its iterations. Arch. Math. Logic 46 (2007), no. 3-4, 223-252.
- [5] Magidor, Menachem, How large is the first strongly compact cardinal? or A study on identity crises. Ann. Math. Logic 10 (1976), no. 1, 33-57.
- [6] Mitchell, William J. The covering lemma. Handbook of set theory. Vols. 1, 2, 3, 1497-1594, Springer, Dordrecht, 2010.