ADDING MANY RANDOM REALS MAY ADD MANY COHEN REALS

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Abstract. Let κ be an infinite cardinal. Then, forcing with $\mathbb{R}(\kappa)\times\mathbb{R}(\kappa)$ adds a generic filter for $\mathbb{C}(\kappa)$; where $\mathbb{R}(\kappa)$ and $\mathbb{C}(\kappa)$ are the forcing notions for adding κ-many random reals and adding κ-many Cohen reals respectively.

1. Introduction

For a cardinal $\kappa > 0$ let $\mathbb{R}(\kappa)$ be the forcing notion for adding κ-many random reals and let $\mathbb{C}(\kappa)$ be the Cohen forcing for adding κ-many Cohen reals[1].

It is a well-know fact that forcing with $\mathbb{R}(1)\times\mathbb{R}(1)$ adds a Cohen real; in fact, if r_1, r_2 are the added random reals, then $c = r_1 + r_2$ is Cohen [1]. This in turn implies all reals $c + a$, where $a \in \mathbb{R}^V$, are Cohen, and so, we have continuum many Cohen reals over V. However, the sequence $\langle c + a : a \in \mathbb{R}^V \rangle$ fails to be $\mathbb{C}((2^{\aleph_0})^V)$-generic over V. In fact, there is no sequence $\langle c_i : i < \omega_1 \rangle \in V[r_1, r_2]$ of Cohen reals which is $\mathbb{C}(\omega_1)$-generic over V.

In this paper, we extend the above mentioned result by showing that if we force with $\mathbb{R}(\kappa)\times\mathbb{R}(\kappa)$, then in the resulting extension, we can find a sequence $\langle c_i : i < \kappa \rangle$ of reals which is $\mathbb{C}(\kappa)$-generic over the ground model:

Theorem 1.1. *Let κ be an infinite cardinal. Then, forcing with $\mathbb{R}(\kappa)\times\mathbb{R}(\kappa)$ adds a generic filter for $\mathbb{C}(\kappa)$.*

In Section 2, we briefly review the forcing notions $\mathbb{C}(\kappa)$ and $\mathbb{R}(\kappa)$. Then in Section 3, we state some results from analysis which are needed for the proof of above theorem and in Section 4, we give a proof of Theorem 1.1.

The author's research has been supported by a grant from IPM (No. 95030417). He also thanks Moti Gitik for his useful comments and suggestions.

[1]See Section 2 for the definition of the forcing notions $\mathbb{R}(\kappa)$ and $\mathbb{C}(\kappa)$.

2. Cohen and Random forcings

In this section we briefly review the forcing notions $\mathbb{C}(\kappa)$ and $\mathbb{R}(\kappa)$, and present some of their properties.

2.1. **Cohen forcing.** Let I be a non-empty set. The forcing notion $\mathbb{C}(I)$, the Cohen forcing for adding $|I|$-many Cohen reals is defined by

$$\mathbb{C}(I) = \{p : I \times \omega \to 2 : |p| < \aleph_0\},$$

which is ordered by reverse inclusion.

Lemma 2.1. $\mathbb{C}(I)$ *is c.c.c.*

Assume G is $\mathbb{C}(I)$-generic over V, and set $F = \bigcup G : I \times \omega \to 2$. For each $i \in I$ set $c_i : \omega \to 2$ be defined by $c_i(n) = F(i, n)$. Then:

Lemma 2.2. *For each* $i \in I, c_i \in 2^\omega$ *is a new real and for* $i \neq j$ *in* $I, c_i \neq c_j$. *Further,* $V[G] = V[\langle c_i : i \in I\rangle]$.

The reals c_i are called Cohen reals. By κ-Cohen reals over V, we mean a sequence $\langle c_i : i < \kappa\rangle$ which is $\mathbb{C}(\kappa)$-generic over V.

2.2. **Random forcing.** In this subsection we briefly review random forcing. Suppose I is a non-empty set and consider the product measure space $2^{I\times\omega}$ with the standard product measure μ_I on it. Let $\mathbb{B}(I)$ denote the class of Borel subsets of $2^{I\times\omega}$. Recall that $\mathbb{B}(I)$ is the σ-algebra generated by the basic open sets

$$[s_p] = \{x \in 2^{I\times\omega} : x \supseteq p\},$$

where $p \in \mathbb{C}(I)$. Also $\mu_I([s_p]) = 2^{-|p|}$.

For Borel sets $S, T \in \mathbb{B}(I)$ set

$$S \sim T \iff S \bigtriangleup T \text{ is null},$$

where $S \bigtriangleup T$ denotes the symmetric difference of S and T. The relation $\sim$ is easily seen to be an equivalence relation on $\mathbb{B}(I)$. Then $\mathbb{R}(I)$, the forcing for adding $|I|$-many random reals, is defined as

$$\mathbb{R}(I) = \mathbb{B}(I)/\sim .$$

Thus elements of $\mathbb{R}(I)$ are equivalent classes $[S]$ of Borel sets modulo null sets. The order relation is defined by

$$[S] \leq [T] \iff \mu(S \setminus T) = 0.$$

The following fact is standard.

Lemma 2.3. *$\mathbb{R}(I)$ is c.c.c.*

Using the above lemma, we can easily show that $\mathbb{R}(I)$ is in fact a complete Boolean algebra. Let $\underset{\sim}{F}$ be an $\mathbb{R}(I)$-name for a function from $I \times \omega$ to 2 such that for each $i \in I, n \in \omega$ and $k < 2$, $\| \underset{\sim}{F}(i,n) = k \|_{\mathbb{R}(I)} = p^{i,n}_k$, where

$$p^{i,n}_k = [x \in 2^{I\times\omega} : x(i,n) = k].$$

This defines $\mathbb{R}(I)$-names $\underset{\sim}{r}_i \in 2^\omega, i \in I$, such that

$$\| \forall n < \omega,\ \underset{\sim}{r}_i(n) = \underset{\sim}{F}(i,n) \|_{\mathbb{R}(I)} = 1_{\mathbb{R}(I)} = [2^{I\times\omega}].$$

Lemma 2.4. *Assume G is $\mathbb{R}(I)$-generic over V, and for each $i \in I$ set $r_i = \underset{\sim}{r}_i[G]$. Then each $r_i \in 2^\omega$ is a new real and for $i \neq j$ in $I, r_i \neq r_j$. Further, $V[G] = V[\langle r_i : i \in I\rangle]$.*

The reals r_i are called random reals. By κ-random reals over V, we mean a sequence $\langle r_i : i < \kappa\rangle$ which is $\mathbb{R}(\kappa)$-generic over V.

3. Some results from analysis

A famous theorem of Steinhaus [2] from 1920 asserts that if $A, B \subseteq \mathbb{R}^n$ are measurable sets with positive Lebesgue measure, then $A + B$ has an interior point; see also [3]. Here, we need a version of Steinhaus theorem for the space $2^{\kappa\times\omega}$.

For $S, T \subseteq 2^{\kappa\times\omega}$, set $S + T = \{x + y : x \in S \text{ and } y \in T\}$, where $x + y : \kappa \times \omega \to 2$ is defined by

$$(x + y)(\alpha, n) = x(\alpha, n) + y(\alpha, n) \pmod 2.$$

Note that the above addition is continuous.

Lemma 3.1. *Suppose $S \subseteq 2^{\kappa\times\omega}$ is Borel and non-null. Then $S - S$ contains an open set around the zero function 0.*

Proof. We follow [3]. Set $\mu = \mu_\kappa$ be the product measure on $2^{\kappa\times\omega}$. As S is Borel and non-null, there is a compact subset of S of positive μ-measure, so may suppose that S itself is compact. Let $U \supseteq S$ be an open set with $\mu(U) < 2\cdot\mu(S)$. By continuity of addition, we can find an open set V containing the zero function 0 such that $V + S \subseteq U$.

We show that $V \subseteq S - S$. Thus suppose $x \in V$. Then $(x+S)\cap S \neq \emptyset$, as otherwise we will have $(x+S)\cup S \subseteq U$, and hence $\mu(U) \geq 2\cdot\mu(S)$, which is in contradiction with our choice of U. Thus let $y_1, y_2 \in S$ be such that $x + y_1 = y_2$. Then $x = y_2 - y_1 \in S - S$ as required. □

Similarly, we have the following:

Lemma 3.2. *Suppose $S, T \subseteq 2^{\kappa\times\omega}$ are Borel and non-null. Then $S + T$ contains an open set.*

Suppose $S, T \subseteq 2^{\kappa\times\omega}$ are Borel and non-null. It follows from Lemma 3.2 that for some $p \in \mathbb{C}(\kappa)$, $[s_p] \subseteq S + T$. Thus, by continuity of the addition, we can find $x \in S$ and $y \in T$ such that:

- $(x + y) \restriction \mathrm{dom}(p) = p$.
- The sets $S \cap [s_{x\restriction \mathrm{dom}(p)}]$ and $T \cap [s_{y\restriction \mathrm{dom}(p)}]$ are Borel and non-null.

4. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. Thus force with $\mathbb{R}(\kappa)\times\mathbb{R}(\kappa)$ and let $G \times H$ be generic over V. Let $\langle\langle r_\alpha : \alpha < \kappa\rangle, \langle s_\alpha : \alpha < \kappa\rangle\rangle$ be the sequence of random reals added by $G \times H$.

For $\alpha < \kappa$ set $c_\alpha = r_\alpha + s_\alpha$. The following completes the proof:

Lemma 4.1. *The sequence $\langle c_\alpha : \alpha < \kappa\rangle$ is a sequence of κ-Cohen reals over V.*

Proof. It suffices to prove the following:

For every $([S],[T]) \in \mathbb{R}(\kappa)\times\mathbb{R}(\kappa)$, and every open dense subset $D \in V$

(∗) of $\mathbb{C}(\kappa)$, there is $([\bar{S}],[\bar{T}]) \leq ([S],[T])$ such that $([\bar{S}],[\bar{T}]) \Vdash$ "$\langle \underset{\sim}{c}_\alpha : \alpha \in \kappa\rangle$

extends some element of D".

Thus fix $([S],[T]) \in \mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$ and $D \in V$ as above, where $S, T \subseteq 2^{\kappa\times\omega}$ are Borel and non-null. By Lemma 3.2 and the remarks after it, we can find $p \in \mathbb{C}(\kappa)$ and $(x, y) \in S \times T$ such that:

(1) $[s_p] \subseteq S + T$.

(2) $(x + y) \restriction \mathrm{dom}(p) = p$.

(3) The sets $S \cap [s_{x\restriction \mathrm{dom}(p)}]$ and $T \cap [s_{y\restriction \mathrm{dom}(p)}]$ are Borel and non-null.

Now let $q \in D$ be such that

$$([S \cap [s_{x\restriction \mathrm{dom}(p)}]], [T \cap [s_{y\restriction \mathrm{dom}(p)}]]) \Vdash \text{“}q \leq_{\mathbb{C}(\kappa)} p\text{”}.$$

Using continuity of the addition and further application of Lemma 3.2 and the remarks after it, we can find x', y' such that:

(4) $x' \in S \cap [s_{x\restriction \mathrm{dom}(p)}]$ and $y' \in T \cap [s_{y\restriction \mathrm{dom}(p)}]$.

(5) $(x' + y') \restriction \mathrm{dom}(q) = q$.

(6) The sets $S \cap [s_{x'\restriction \mathrm{dom}(q)}]$ and $T \cap [s_{y'\restriction \mathrm{dom}(q)}]$ are Borel and non-null.

It is now clear that

$$([S \cap [s_{x'\restriction \mathrm{dom}(q)}]], [T \cap [s_{y'\restriction \mathrm{dom}(q)}]]) \Vdash \text{“}\langle \underset{\sim}{c}_\alpha : \alpha \in \kappa\rangle \text{ extends } q\text{”}.$$

The result follows. □

References

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