ADDING MANY RANDOM REALS MAY ADD MANY COHEN REALS

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Abstract. Let \( \kappa \) be an infinite cardinal. Then, forcing with \( \mathbb{R}(\kappa) \times \mathbb{R}(\kappa) \) adds a generic filter for \( C(\kappa) \); where \( \mathbb{R}(\kappa) \) and \( C(\kappa) \) are the forcing notions for adding \( \kappa \)-many random reals and adding \( \kappa \)-many Cohen reals respectively.

1. Introduction

For a cardinal \( \kappa > 0 \) let \( \mathbb{R}(\kappa) \) be the forcing notion for adding \( \kappa \)-many random reals and let \( C(\kappa) \) be the Cohen forcing for adding \( \kappa \)-many Cohen reals\(^1\).

It is a well-know fact that forcing with \( \mathbb{R}(1) \times \mathbb{R}(1) \) adds a Cohen real; in fact, if \( r_1, r_2 \) are the added random reals, then \( c = r_1 + r_2 \) is Cohen [1]. This in turn implies all reals \( c + a \), where \( a \in \mathbb{R}^V \), are Cohen, and so, we have continuum many Cohen reals over \( V \). However, the sequence \( \langle c + a : a \in \mathbb{R}^V \rangle \) fails to be \( C((2^{\aleph_0})^V) \)-generic over \( V \). In fact, there is no sequence \( \langle c_i : i < \omega_1 \rangle \in V[r_1, r_2] \) of Cohen reals which is \( C(\omega_1) \)-generic over \( V \).

In this paper, we extend the above mentioned result by showing that if we force with \( \mathbb{R}(\kappa) \times \mathbb{R}(\kappa) \), then in the resulting extension, we can find a sequence \( \langle c_i : i < \kappa \rangle \) of reals which is \( C(\kappa) \)-generic over the ground model:

**Theorem 1.1.** Let \( \kappa \) be an infinite cardinal. Then, forcing with \( \mathbb{R}(\kappa) \times \mathbb{R}(\kappa) \) adds a generic filter for \( C(\kappa) \).

In Section 2, we briefly review the forcing notions \( C(\kappa) \) and \( \mathbb{R}(\kappa) \). Then in Section 3, we state some results from analysis which are needed for the proof of above theorem and in Section 4, we give a proof of Theorem 1.1.

\(^1\)See Section 2 for the definition of the forcing notions \( \mathbb{R}(\kappa) \) and \( C(\kappa) \).

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2. COHEN AND RANDOM FORCINGS

In this section we briefly review the forcing notions $C(\kappa)$ and $R(\kappa)$, and present some of their properties.

2.1. Cohen forcing. Let $I$ be a non-empty set. The forcing notion $C(I)$, the Cohen forcing for adding $|I|$-many Cohen reals is defined by

$$C(I) = \{ p : I \times \omega \to 2 : |p| < \aleph_0 \},$$

which is ordered by reverse inclusion.

**Lemma 2.1.** $C(I)$ is c.c.c.

Assume $G$ is $C(I)$-generic over $V$, and set $F = \bigcup G : I \times \omega \to 2$. For each $i \in I$ set $c_i : \omega \to 2$ be defined by $c_i(n) = F(i, n)$. Then:

**Lemma 2.2.** For each $i \in I, c_i \in 2^\omega$ is a new real and for $i \neq j$ in $I, c_i \neq c_j$. Further, $V[G] = V[\langle c_i : i \in I \rangle]$.

The reals $c_i$ are called Cohen reals. By $\kappa$-Cohen reals over $V$, we mean a sequence $\langle c_i : i < \kappa \rangle$ which is $C(\kappa)$-generic over $V$.

2.2. Random forcing. In this subsection we briefly review random forcing. Suppose $I$ is a non-empty set and consider the product measure space $2^{I \times \omega}$ with the standard product measure $\mu_I$ on it. Let $B(I)$ denote the class of Borel subsets of $2^{I \times \omega}$. Recall that $B(I)$ is the $\sigma$-algebra generated by the basic open sets

$$[s_p] = \{ x \in 2^{I \times \omega} : x \supseteq p \},$$

where $p \in C(I)$. Also $\mu_I([s_p]) = 2^{-|p|}$.

For Borel sets $S, T \in B(I)$ set

$$S \sim T \iff S \triangle T \text{ is null},$$

where $S \triangle T$ denotes the symmetric difference of $S$ and $T$. The relation $\sim$ is easily seen to be an equivalence relation on $B(I)$. Then $R(I)$, the forcing for adding $|I|$-many random reals, is defined as
\[ \mathbb{R}(I) = \mathbb{B}(I) / \sim . \]

Thus elements of \(\mathbb{R}(I)\) are equivalent classes \([S]\) of Borel sets modulo null sets. The order relation is defined by

\[ [S] \leq [T] \iff \mu(S \setminus T) = 0. \]

The following fact is standard.

**Lemma 2.3.** \(\mathbb{R}(I)\) is c.c.c.

Using the above lemma, we can easily show that \(\mathbb{R}(I)\) is in fact a complete Boolean algebra. Let \(\mathcal{F}\) be an \(\mathbb{R}(I)\)-name for a function from \(I \times \omega\) to \(2\) such that for each \(i \in I, n \in \omega\) and \(k < 2\), \(\| F(i, n) = k \|_{\mathbb{R}(I)} = p_{k}^{i,n}\), where

\[ p_{k}^{i,n} = [x \in 2^{I \times \omega} : x(i, n) = k]. \]

This defines \(\mathbb{R}(I)\)-names \(\mathcal{F}_{i} \in 2^{\omega}, i \in I\), such that

\[ \| \forall n < \omega, \mathcal{F}_{i}(n) = F(i, n) \|_{\mathbb{R}(I)} = 1_{\mathbb{R}(I)} = [2^{I \times \omega}]. \]

**Lemma 2.4.** Assume \(G\) is \(\mathbb{R}(I)\)-generic over \(V\), and for each \(i \in I\) set \(r_{i} = \mathcal{F}_{i}[G]\). Then each \(r_{i} \in 2^{\omega}\) is a new real and for \(i \neq j\) in \(I, r_{i} \neq r_{j}\). Further, \(V[G] = V[\langle r_{i} : i \in I \rangle]\).

The reals \(r_{i}\) are called random reals. By \(\kappa\)-random reals over \(V\), we mean a sequence \(\langle r_{i} : i < \kappa \rangle\) which is \(\mathbb{R}(\kappa)\)-generic over \(V\).

3. Some results from analysis

A famous theorem of Steinhaus [2] from 1920 asserts that if \(A, B \subseteq \mathbb{R}^{n}\) are measurable sets with positive Lebesgue measure, then \(A + B\) has an interior point; see also [3]. Here, we need a version of Steinhaus theorem for the space \(2^{\kappa \times \omega}\).

For \(S, T \subseteq 2^{\kappa \times \omega}\), set \(S + T = \{ x + y : x \in S \text{ and } y \in T \}\), where \(x + y : \kappa \times \omega \rightarrow 2\) is defined by

\[ (x + y)(\alpha, n) = x(\alpha, n) + y(\alpha, n) \pmod{2}. \]

Note that the above addition is continuous.
Lemma 3.1. Suppose $S \subseteq 2^{\kappa \times \omega}$ is Borel and non-null. Then $S - S$ contains an open set around the zero function 0.

Proof. We follow [3]. Set $\mu = \mu_\kappa$ be the product measure on $2^{\kappa \times \omega}$. As $S$ is Borel and non-null, there is a compact subset of $S$ of positive $\mu$-measure, so may suppose that $S$ itself is compact. Let $U \supseteq S$ be an open set with $\mu(U) < 2 \cdot \mu(S)$. By continuity of addition, we can find an open set $V$ containing the zero function 0 such that $V + S \subseteq U$.

We show that $V \subseteq S - S$. Thus suppose $x \in V$. Then $(x + S) \cap S \neq \emptyset$, as otherwise we will have $(x + S) \cup S \subseteq U$, and hence $\mu(U) \geq 2 \cdot \mu(S)$, which is in contradiction with our choice of $U$. Thus let $y_1, y_2 \in S$ be such that $x + y_1 = y_2$. Then $x = y_2 - y_1 \in S - S$ as required.

Similarly, we have the following:

Lemma 3.2. Suppose $S, T \subseteq 2^{\kappa \times \omega}$ are Borel and non-null. Then $S + T$ contains an open set.

Suppose $S, T \subseteq 2^{\kappa \times \omega}$ are Borel and non-null. It follows from Lemma 3.2 that for some $p \in C(\kappa)$, $[s_p] \subseteq S + T$. Thus, by continuity of the addition, we can find $x \in S$ and $y \in T$ such that:

- $(x + y) \upharpoonright \text{dom}(p) = p$.
- The sets $S \cap [s_x\upharpoonright\text{dom}(p)]$ and $T \cap [s_y\upharpoonright\text{dom}(p)]$ are Borel and non-null.

4. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. Thus force with $\mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$ and let $G \times H$ be generic over $V$. Let $\langle \langle r_\alpha : \alpha < \kappa \rangle, \langle s_\alpha : \alpha < \kappa \rangle \rangle$ be the sequence of random reals added by $G \times H$.

For $\alpha < \kappa$ set $c_\alpha = r_\alpha + s_\alpha$. The following completes the proof:

Lemma 4.1. The sequence $\langle c_\alpha : \alpha < \kappa \rangle$ is a sequence of $\kappa$-Cohen reals over $V$.

Proof. It suffices to prove the following:

For every $([S], [T]) \in \mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$, and every open dense subset $D \in V$

($*$) of $C(\kappa)$, there is $([\bar{S}], [\bar{T}]) \leq ([S], [T])$ such that $([\bar{S}], [\bar{T}])] - \mathcal{L}_\alpha : \alpha \in \kappa)$
extends some element of $D''$.

Thus fix $([S], [T]) \in \mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$ and $D \in V$ as above, where $S, T \subseteq 2^{\kappa \times \omega}$ are Borel and non-null. By Lemma 3.2 and the remarks after it, we can find $p \in C(\kappa)$ and $(x, y) \in S \times T$ such that:

1. $[s_p] \subseteq S + T$.
2. $(x + y) \upharpoonright \text{dom}(p) = p$.
3. The sets $S \cap [s_x \upharpoonright \text{dom}(p)]$ and $T \cap [s_y \upharpoonright \text{dom}(p)]$ are Borel and non-null.

Now let $q \in D$ be such that

$$([S \cap [s_x \upharpoonright \text{dom}(p)]], [T \cap [s_y \upharpoonright \text{dom}(p)]]) \Vdash \text{"} q \leq C(\kappa) p \text{"}.$$ 

Using continuity of the addition and further application of Lemma 3.2 and the remarks after it, we can find $x', y'$ such that:

4. $x' \in S \cap [s_x \upharpoonright \text{dom}(q)]$ and $y' \in T \cap [s_y \upharpoonright \text{dom}(q)]$.
5. $(x' + y') \upharpoonright \text{dom}(q) = q$.
6. The sets $S \cap [s_{x'} \upharpoonright \text{dom}(q)]$ and $T \cap [s_{y'} \upharpoonright \text{dom}(q)]$ are Borel and non-null.

It is now clear that

$$([S \cap [s_{x'} \upharpoonright \text{dom}(q)]], [T \cap [s_{y'} \upharpoonright \text{dom}(q)]]) \Vdash \text{"} \langle \xi_\alpha : \alpha \in \kappa \rangle \text{ extends } q' \text{"}.$$ 

The result follows. \qed

**References**


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