

ALL UNCOUNTABLE REGULAR CARDINALS CAN BE INACCESSIBLE IN HOD ¹

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ABSTRACT. Assuming the existence of a supercompact cardinal and an inaccessible above it, we construct a model of ZFC, in which all uncountable regular cardinals are inaccessible in HOD.

1. INTRODUCTION

An important development in large cardinal theory is the construction of inner models M all of whose sets are definable from ordinals and which serve as good approximations to the entire universe V . The former means that M is contained in HOD, the universe of hereditarily ordinal definable sets, and the latter can be interpreted in a number of ways:

- (α) V covers M ; in the sense that every uncountable set of ordinals in V is covered by a set of ordinals in M of the same V -cardinality.
- (β) V weakly covers M ; in the sense that α^+ of M equals α^+ of V for every singular cardinals α of V . This is for example the case if V does not contain 0^\sharp and M equals L [3], or if V does not contain an inner model with a Woodin cardinal and M is the core model K for a Woodin cardinal [7].

It is easily seen that if V covers M , then V weakly covers M . In [1], it is shown that we can force, in a certain sense, the ultimate failure of weak covering:

Theorem 1.1. ([1]) *Suppose GCH holds and κ is a κ^{+3} -supercompact cardinal. Then there is a generic extension W of V in which κ remains inaccessible and for all infinite cardinals $\alpha < \kappa$, $(\alpha^+)^{HOD} < \alpha^+$. In particular the rank-initial segment W_κ is a model of ZFC in which for all infinite cardinals α , $(\alpha^+)^{HOD} < \alpha^+$.*

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The result of this paper is motivated by a suggestion of Moti Gitik, to whom the author is very thankful.

¹ A strengthening of the result of this paper is proved by Gitik-Merimovich [5], where they produced a model in which all uncountable regular cardinals are measurable in HOD.

The problem of finding a model of ZFC in which all uncountable regular cardinals are inaccessible in HOD remained open in [1].

On the other hand HOD plays an important role in Woodin's recent work. The following is an important result in these directions:

Theorem 1.2. *(The HOD Dichotomy theorem) Assume that δ is an extendible cardinal. Then exactly one of the following holds:*

- (1) *For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$,*
- (2) *Every regular cardinal greater than δ is measurable in HOD.*

A cardinal κ is called ω -strongly measurable in HOD if there exists $\lambda < \kappa$ such that $(2^\lambda)^{HOD} < \kappa$ and such that there is no partition of $S = \{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$ into λ many sets $\langle S_\alpha : \alpha < \lambda \rangle \in \text{HOD}$ such each set S_α is stationary in V . One of the major open problems related to Woodin's work in the following, which is known as HOD conjecture.

The HOD conjecture. There is a proper class of regular cardinals that are not ω -strongly measurable in HOD.

We refer to [2] for more information about Woodin's work. It turns out that if δ is an extendible cardinal, then the HOD Conjecture is equivalent to the failure of clause (2) of the dichotomy theorem 1.2. Also note that if HOD is correct about singular cardinals and computes their successors correctly, then the HOD Conjecture holds, as

$$\{\gamma^+ : \gamma \text{ is a singular cardinal}\}$$

is a proper class of regular cardinals which are not ω -strongly measurable in HOD. On the other hand, in his talk [13] presented at the Bristol University, Woodin has introduced the following version of HOD conjecture:

The HOD conjecture (strong version). There is a proper class of uncountable regular cardinals κ which are not measurable cardinals in HOD.

We may note that if κ is ω -strongly measurable in HOD, then it is measurable in HOD, and hence strong HOD conjecture implies HOD conjecture. In this paper, we address these problems, and prove the following theorem

Theorem 1.3. *Suppose GCH holds, κ is a supercompact cardinal and $\lambda > \kappa$ is inaccessible. Then there is a generic extension W of V in which κ remains inaccessible and for all infinite*

cardinals $\alpha < \kappa$, α^+ is inaccessible in HOD. In particular the rank-initial segment W_λ , where $\lambda \leq \kappa$ is the least inaccessible cardinal of W is a model of ZFC in which all uncountable regular cardinals α , are inaccessible in HOD.

The theorem answers the question left open in [1], and proves the consistency of the negation of a weak version of Woodin's strong HOD conjecture, where measurable cardinals are replaced by inaccessible cardinals.

We may mention that a stronger result, which completely answer Woodin's strong HOD conjecture was proved recently by Gitik-Merimovich [5]; however our proof of Theorem 1.3 has differences with the one given in [5], though the main forcing notion used in both of the proofs, i.e., supercompact extender based Radin forcing, is the same.

The structure of the paper is as follows. In section 2, we give some preliminaries and results which appear later in our work. In section 3, we consider a simpler problem, namely the consistency of the existence of a singular cardinal κ such that κ^+ is inaccessible in HOD. We give the consistency of this problem in some details, because it gives some motivations for the proof of the main theorem, whose proof is much more complicated. Finally in section 4, we complete the proof of the above mentioned theorem.

2. SOME PRELIMINARIES

In this section, we present some definitions and results which appear in next sections. Let's start with the definition of a projection map between forcing notions.

Definition 2.1. *Let \mathbb{P}, \mathbb{Q} be two forcing notions. π is a projection from \mathbb{P} into \mathbb{Q} if $\pi : \mathbb{P} \rightarrow \mathbb{Q}$, and it satisfies the following conditions:*

- (1) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$,
- (2) π is order preserving; i.e., $p \leq_{\mathbb{P}} q \Rightarrow \pi(p) \leq_{\mathbb{Q}} \pi(q)$,
- (3) If $p \in \mathbb{P}, q \in \mathbb{Q}$ and $q \leq_{\mathbb{Q}} \pi(p)$, then there exists $p^* \leq_{\mathbb{P}} p$ such that $\pi(p^*) \leq_{\mathbb{Q}} q$.

It is clear that if $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ is a projection from \mathbb{P} into \mathbb{Q} , then $\pi[\mathbb{P}]$ is dense in \mathbb{Q} . The next lemma shows that if \mathbb{P} projects into \mathbb{Q} , then a generic filter for \mathbb{P} yields a generic filter for \mathbb{Q} .

Lemma 2.2. *Let $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ be a projection from \mathbb{P} into \mathbb{Q} , let G be \mathbb{P} -generic over V , and let $H \subseteq \mathbb{Q}$ be the filter generated by $\pi[G]$. Then H is \mathbb{Q} -generic over V and $V[H] \subseteq V[G]$.*

Prikry type forcing notions arise in our work.

Definition 2.3. $\langle \mathbb{P}, \leq, \leq^* \rangle$ is of Prikry type, iff

- (1) $\leq^* \subseteq \leq$,
- (2) For any $p \in \mathbb{P}$ and any statement σ in the forcing language $\langle \mathbb{P}, \leq \rangle$, there exists $q \leq^* p$ which decides σ .

The relation \leq^* is usually called the Prikry relation. The following is well-known.

Lemma 2.4. *Assume $\langle \mathbb{P}, \leq, \leq^* \rangle$ is of Prikry type, and suppose $\langle \mathbb{P}, \leq^* \rangle$ is κ -closed, where κ is regular uncountable. Then Forcing with $\langle \mathbb{P}, \leq \rangle$ does not add new bounded subsets to κ .*

Projection between Prikry type forcing notions arises in our work in several places. So let's present a new definition, and give an application of it.

Definition 2.5. Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, \leq_{\mathbb{P}}^* \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \leq_{\mathbb{Q}}^* \rangle$ be two forcing notions with $\leq_{\mathbb{P}}^* \subseteq \leq_{\mathbb{P}}$ and $\leq_{\mathbb{Q}}^* \subseteq \leq_{\mathbb{Q}}$. A map $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ is a “projection of Prikry type” iff

- (1) π is a projection from $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ into $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$,
- (2) π preserves the \leq^* -relation, i.e., $p \leq_{\mathbb{P}}^* q \Rightarrow \pi(p) \leq_{\mathbb{Q}}^* \pi(q)$,
- (3) If $p \in \mathbb{P}, q \in \mathbb{Q}$ and $q \leq_{\mathbb{Q}} \pi(p)$, then there exists $p^* \leq_{\mathbb{P}} p$ such that $\pi(p^*) \leq_{\mathbb{Q}} q$.

It is clear that if $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ is a projection of Prikry type from \mathbb{P} into \mathbb{Q} , then $\pi[\mathbb{P}]$ is dense in \mathbb{Q} , with respect to both \leq and \leq^* relations. Note that in the above definition we did not require $\langle \mathbb{P}, \leq_{\mathbb{P}}, \leq_{\mathbb{P}}^* \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \leq_{\mathbb{Q}}^* \rangle$ be Prikry type forcing notions. The following lemma shows the importance of Prikry type projections.

Lemma 2.6. *Assume $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ is a projection of Prikry type, and assume $\langle \mathbb{P}, \leq_{\mathbb{P}}, \leq_{\mathbb{P}}^* \rangle$ satisfies the Prikry property. Then $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \leq_{\mathbb{Q}}^* \rangle$ is also of Prikry type.*

Proof. First we show that $\langle \pi[\mathbb{P}], \leq_{\mathbb{Q}}, \leq_{\mathbb{Q}}^* \rangle$ satisfies the Prikry property. Let $b \in R.O(\pi[\mathbb{P}])$ and $q \in \pi[\mathbb{P}]$. Let $p \in \mathbb{P}$ be such that $q = \pi(p)$. Then there is $p^* \leq_{\mathbb{P}}^* p$ such that p^* decides

$\|b \in \pi[\dot{G}]\|_{R.O(\pi[\mathbb{P}])}$ where \dot{G} is the canonical name for a generic filter over \mathbb{P} . Let $q^* = \pi(p^*)$. Then $q^* \leq_{\mathbb{Q}}^* q$ and it decides b .

But $\pi[\mathbb{P}]$ is in fact $\leq_{\mathbb{Q}}^*$ -dense in \mathbb{Q} , and hence $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \leq_{\mathbb{Q}}^* \rangle$ satisfies the Prikry property. \square

3. THE CONSISTENCY OF κ IS SINGULAR AND κ^+ IS INACCESSIBLE IN HOD

In this section, we prove the following result, which is a very weak version of our main theorem. We have decided to bring it, because it motivates the main construction of the paper, without going into many complications we arrive in the proof of main theorem.

Theorem 3.1. *Assume GCH holds, κ is a supercompact cardinal and $\lambda > \kappa$ is inaccessible. Then there is a generic extension $V[G]$ of the universe V , in which κ is singular of cofinality ω , $(\kappa^+)^{V[G]} = \lambda$, and λ is strongly inaccessible in $HOD^{V[G]}$.*

The rest of this section is devoted to the proof of the above theorem. We assume the following hold in V :

- (1) GCH,
- (2) $j : V \rightarrow M$ is an elementary embedding with $\text{crit}(j) = \kappa$ and $M \supseteq {}^{<\lambda}M$,
- (3) $\lambda > \kappa$ is strongly inaccessible.

In subsection 3.1, we use Merimovich's "supercompact extender based Prikry forcing" \mathbb{P}_E [11], to find a generic extension $V[G]$, which makes κ singular of cofinality ω , and collapses all cardinals in (κ, λ) into κ , while preserves λ , so that $(\kappa^+)^{V[G]} = \lambda$. Then in subsection 3.2, we present a cardinal preserving forcing notion \mathbb{P}_E^π , that we call it "projected supercompact extender based Prikry forcing", where in its generic extension, κ becomes singular of countable cofinality and $\lambda > \kappa$ remains inaccessible, and next we show that there is a projection from \mathbb{P}_E into \mathbb{P}_E^π , so that we can find a generic G^π for \mathbb{P}_E^π with $V[G^\pi] \subseteq V[G]$. In subsection 3.5 we show that the resulting quotient forcing has enough homogeneity properties so that $HOD^{V[G]} \subseteq V[G^\pi]$, and from it we get the result.

3.1. Supercompact extender based Prikry forcing. In this subsection, we present Merimovich's "supercompact extender based Prikry forcing". For each $\alpha < j(\lambda)$ let λ_α be minimal $\eta < \lambda$ such that $\alpha < j(\eta)$, and let $E(\alpha) \subseteq P(\lambda)$ be defined by

$$A \in E(\alpha) \Leftrightarrow \alpha \in j(A).$$

Note that each $E(\alpha)$ is a κ -complete ultrafilter on λ and it has concentrated on λ_α . Also let

$$i_\alpha : V \rightarrow N_\alpha \simeq \text{Ult}(V, E(\alpha)).$$

Finally let

$$E = \langle \langle E(\alpha) : \alpha < j(\lambda) \rangle, \langle \pi_{\beta, \alpha} : \beta, \alpha < j(\lambda), \alpha \in \text{range}(i_\beta) \rangle \rangle$$

be the extender derived from j , where $\pi_{\beta, \alpha} : \lambda \rightarrow \lambda$ is such that $j(\pi_{\beta, \alpha})(\beta) = \alpha$ (such a $\pi_{\beta, \alpha}$ exists as $\alpha \in \text{range}(i_\beta)$). Let $i : V \rightarrow N \simeq \text{Ult}(V, E)$ be the resulting extender embedding.

We may assume that $j = i$.

Definition 3.2. Let $d \in [j(\lambda)]^{<\lambda}$ be such that $\kappa, |d| \in d$. Then $\nu \in OB(d)$ iff:

- (1) $\nu : \text{dom}(\nu) \rightarrow \lambda$, where $\text{dom}(\nu) \subseteq d$,
- (2) $\kappa, |d| \in \text{dom}(\nu)$,
- (3) $|\nu| \leq \nu(|d|)$,
- (4) $\forall \alpha < \lambda (j(\alpha) \in \text{dom}(\nu) \Rightarrow \nu(j(\alpha)) = \alpha)$,
- (5) $\alpha \in \text{dom}(\nu) \Rightarrow \nu(\alpha) < \lambda_\alpha$,
- (6) $\alpha < \beta$ in $\text{dom}(\nu) \Rightarrow \nu(\alpha) < \nu(\beta)$.

Also for $\nu_0, \nu_1 \in OB(d)$, set $\nu_0 < \nu_1$ iff

- (6) $\text{dom}(\nu_0) \subseteq \text{dom}(\nu_1)$,
- (7) For all $\alpha \in \text{dom}(\nu_0) \setminus j[\lambda]$, $\nu_0(\alpha) < \nu_1(\alpha)$.

We now define the forcing notion \mathbb{P}_E^* .

Definition 3.3. \mathbb{P}_E^* consists of all functions $f : d \rightarrow \lambda^{<\omega}$, where $d \in [j(\lambda)]^{<\lambda}$, $\kappa, |d| \in d$, and such that

- (1) For any $j(\alpha) \in d$, $f(j(\alpha)) = \langle \alpha \rangle$,
- (2) For any $\alpha \in d \setminus j[\lambda]$, there is some $k < \omega$ such that

$$f(\alpha) = \langle f_0(\alpha), \dots, f_{k-1}(\alpha) \rangle \subseteq \lambda_\alpha$$

is a finite increasing subsequence of λ_α . For $f, g \in \mathbb{P}_E^*$,

$$f \leq_{\mathbb{P}_E^*}^* g \Leftrightarrow f \supseteq g.$$

Remark 3.4. $\langle \mathbb{P}_E^*, \leq_{\mathbb{P}_E^*}^* \rangle \approx \text{Add}(\lambda, |j(\lambda)|)$.

Definition 3.5. Assume $d \in [j(\lambda)]^{<\lambda}$ and $\kappa, |d| \in d$. Let $T \subseteq OB(d)^{<\xi}$ ($1 < \xi \leq \omega$) and $n < \omega$. Then

- (1) $Lev_n(T) = T \cap OB(d)^{n+1}$,
- (2) $Suc_T(\langle \rangle) = Lev_0(T)$,
- (3) $Suc_T(\langle \nu_0, \dots, \nu_{n-1} \rangle) = \{\mu \in OB(d) : \langle \nu_0, \dots, \nu_{n-1}, \mu \rangle \in T\}$.

Definition 3.6. Assume $d \in [j(\lambda)]^{<\lambda}$ and $\kappa, |d| \in d$. Let $T \subseteq OB(d)^{<\xi}$ ($1 < \xi \leq \omega$). For $\langle \nu \rangle \in T$, let

$$T_{\langle \nu \rangle} = \{\langle \nu_0, \dots, \nu_{k-1} \rangle : k < \omega, \langle \nu, \nu_0, \dots, \nu_{k-1} \rangle \in T\}$$

and define by recursion for $\langle \nu_0, \dots, \nu_{n-1} \rangle \in T$,

$$T_{\langle \nu_0, \dots, \nu_{n-1} \rangle} = (T_{\langle \nu_0, \dots, \nu_{n-2} \rangle})_{\langle \nu_{n-1} \rangle}.$$

Definition 3.7. Assume $d \in [j(\lambda)]^{<\lambda}$ and $\kappa, |d| \in d$. We define the measure $E(d)$ on $OB(d)$ by

$$E(d) = \{X \subseteq OB(d) : mc(d) \in j(X)\},$$

where $mc(d) = \{\langle j(\alpha), \alpha \rangle : \alpha \in d\}$.

Definition 3.8. Assume $d \in [j(\lambda)]^{<\lambda}$ and $\kappa, |d| \in d$. Let $T \subseteq OB(d)^{<\omega}$ be a tree. T is called an $E(d)$ -tree, if

- (1) $\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in T$ ($\nu_0 < \dots < \nu_{n-1}$),
- (2) $\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in T$ ($Suc_T(\langle \nu_0, \dots, \nu_{n-1} \rangle) \in E(d)$).

Definition 3.9. Assume $d \in [j(\lambda)]^{<\lambda}$ and $A \subseteq OB(d)^{<\omega}$. Then

$$A \upharpoonright c = \{\langle \nu_0 \upharpoonright c, \dots, \nu_{n-1} \upharpoonright c \rangle : n < \omega, \langle \nu_0, \dots, \nu_{n-1} \rangle \in A\}.$$

Remark 3.10. For $f \in \mathbb{P}_E^*$, we use $OB(f)$, $E(f)$ and $mc(f)$ to denote $OB(\text{dom}(f))$, $E(\text{dom}(f))$ and $mc(\text{dom}(f))$ respectively.

We are now ready to define our main forcing notion, \mathbb{P}_E .

Definition 3.11. $p \in \mathbb{P}_E$ iff $p = \langle f^p, A^p \rangle$ where

- (1) $f^p \in \mathbb{P}_E^*$,
- (2) A^p is an $E(f^p)$ -tree.

Definition 3.12. Let $p, q \in \mathbb{P}_E$. Then $p \leq^* q$ (p is a Prikry extension of q) iff:

- (1) $f^p \leq_{\mathbb{P}_E^*}^* f^q$,
- (2) $A^p \upharpoonright \text{dom}(f^q) \subseteq A^q$.

Definition 3.13. Let $f \in \mathbb{P}_E^*$, $\nu \in OB(f)$ and suppose $\nu(\kappa) > \max(f(\kappa))$. Then $f_{\langle \nu \rangle} \in \mathbb{P}_E^*$ has the same domain as f and

$$f_{\langle \nu \rangle}(\alpha) = \begin{cases} f(\alpha) \frown \langle \nu(\alpha) \rangle & \text{if } \alpha \in \text{dom}(f), \nu(\alpha) > \max(f(\alpha)), \\ f(\alpha) & \text{Otherwise.} \end{cases}$$

Given $\langle \nu_0, \dots, \nu_{n-1} \rangle \in OB(f)^n$ such that $\nu_0(\kappa) > \max(f(\kappa))$ and $\nu_0 < \dots < \nu_{n-1}$, define $f_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$ by recursion as

$$f_{\langle \nu_0, \dots, \nu_{n-1} \rangle} = (f_{\langle \nu_0, \dots, \nu_{n-2} \rangle})_{\langle \nu_{n-1} \rangle}.$$

Let $p \in \mathbb{P}_E$, and suppose $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^p$ is such that $\nu_0(\kappa) > \max(f^p(\kappa))$ and $\nu_0 < \dots < \nu_{n-1}$. Then

$$p_{\langle \nu_0, \dots, \nu_{n-1} \rangle} = \langle f_{\langle \nu_0, \dots, \nu_{n-1} \rangle}^p, A_{\langle \nu_0, \dots, \nu_{n-1} \rangle}^p \rangle.$$

Remark 3.14. Whenever the notation $\langle \nu_0, \dots, \nu_{n-1} \rangle$ is used, where $\nu_0, \dots, \nu_{n-1} \in OB(f)$, it is implicitly assumed $\nu_0(\kappa) > \max(f(\kappa))$ and $\nu_0 < \dots < \nu_{n-1}$.

Definition 3.15. Let $p, q \in \mathbb{P}_E$. Then

$$p \leq q \Leftrightarrow \exists \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^q \text{ (} p \leq^* q_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \text{)}.$$

Let's state the main properties of the forcing notion \mathbb{P}_E . The proof can be found in [11].

Theorem 3.16. Let G be \mathbb{P}_E -generic over V . Then

- (1) $\langle \mathbb{P}_E, \leq \rangle$ satisfies the λ^+ -c.c.,
- (2) $\langle \mathbb{P}_E, \leq, \leq^* \rangle$ satisfies the Prikry property,
- (3) $\langle \mathbb{P}_E, \leq^* \rangle$ is κ -closed,
- (4) $cf^{V[G]}(\kappa) = \omega$,
- (5) All V -cardinals in the interval (κ, λ) are collapsed,
- (6) λ is preserved in $V[G]$.

It follows that V and $V[G]$ have the same bounded subsets of κ , $cf^{V[G]}(\kappa) = \omega$, and $(\kappa^+)^{V[G]} = \lambda$.

It is possible to show that the forcing notion \mathbb{P}_E has enough homogeneity properties to guarantee that $\text{HOD}^{\text{V[G]}} \subseteq \text{V}$, from which we can conclude that $(\kappa^+)^{\text{V[G]}} = \lambda$ is inaccessible in $\text{HOD}^{\text{V[G]}}$. However, the forcing we define for our main theorem does not have this homogeneity property, and to prove the main theorem, we need some extra work to find some intermediate model, which is a cardinal preserving extension of the universe V , so that the tail construction is homogeneous. To motivate that construction, we prove the fact $(\kappa^+)^{\text{V[G]}} = \lambda$ is inaccessible in $\text{HOD}^{\text{V[G]}}$ in a more complicated way, which is similar to the proof of the main theorem, and so can give us some motivation for that construction.

3.2. Projected supercompact extender based Prikry forcing. In this subsection we define our projected forcing.

Definition 3.17. $\mathbb{P}_E^{*,\pi}$ consists of all functions $f : d \rightarrow \lambda^{<\omega}$, where $d \in [j(\lambda)]^{<\lambda}$ and $\kappa, |d| \in d$, such that

- (1) For some $k < \omega$, $f(\kappa) = \langle f_0(\kappa), \dots, f_{k-1}(\kappa) \rangle \subseteq \kappa$ is a finite increasing subsequence of κ .
- (2) For all $\kappa < \alpha \in d$, $f(\alpha) = \langle 0 \rangle$.

For $f, g \in \mathbb{P}_E^{*,\pi}$,

$$f \leq_{\mathbb{P}_E^{*,\pi}}^* g \Leftrightarrow f \supseteq g.$$

Remark 3.18. $\langle \mathbb{P}_E^{*,\pi}, \leq_{\mathbb{P}_E^{*,\pi}}^* \rangle$ is the trivial forcing notion.

Lemma 3.19. $\mathbb{P}_E^{*,\pi}$ satisfies the κ^+ -c.c.

Proof. This follows from the fact that for $f, g \in \mathbb{P}_E^{*,\pi}$, if $f(\kappa) = g(\kappa)$, then f and g are compatible, and there are only κ possibilities for the choice of $f(\kappa)$'s. \square

Definition 3.20. \mathbb{P}_E^π consists of all pair $p = \langle f^p, A^p \rangle$, where

- (1) $f^p \in \mathbb{P}_E^{*,\pi}$,
- (2) A^p is an $E(f^p)$ -tree.

Definition 3.21. Let $p, q \in \mathbb{P}_E^\pi$. Then $p \leq^{*,\pi} q$ (p is a Prikry extension of q) iff:

- (1) $f^p \leq_{\mathbb{P}_E^{*,\pi}}^* f^q$,
- (2) $A^p \upharpoonright \text{dom}(f^q) \subseteq A^q$.

Definition 3.22. Let $f \in \mathbb{P}_E^{*,\pi}$, $\nu \in OB(f)$ and suppose $\nu(\kappa) > \max(f(\kappa))$. Then $f_{\langle \nu \rangle} \in \mathbb{P}_E^{*,\pi}$ has the same domain as f and

$$f_{\langle \nu \rangle}(\alpha) = \begin{cases} f(\kappa) \smallfrown \langle \nu(\kappa) \rangle & \text{if } \alpha = \kappa, \\ \langle 0 \rangle & \text{Otherwise.} \end{cases}$$

Given $\langle \nu_0, \dots, \nu_{n-1} \rangle \in OB(f)^n$ such that $\nu_0(\kappa) > \max(f(\kappa))$ and $\nu_0 < \dots < \nu_{n-1}$, define $f_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$ by recursion as

$$f_{\langle \nu_0, \dots, \nu_{n-1} \rangle} = (f_{\langle \nu_0, \dots, \nu_{n-2} \rangle})_{\langle \nu_{n-1} \rangle}.$$

Let $p \in \mathbb{P}_E^\pi$, and suppose $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^p$ is such that $\nu_0(\kappa) > \max(f^p(\kappa))$ and $\nu_0 < \dots < \nu_{n-1}$. Then

$$p_{\langle \nu_0, \dots, \nu_{n-1} \rangle} = \langle f_{\langle \nu_0, \dots, \nu_{n-1} \rangle}^p, A_{\langle \nu_0, \dots, \nu_{n-1} \rangle}^p \rangle.$$

Definition 3.23. Let $p, q \in \mathbb{P}_E^\pi$. Then

$$p \leq^\pi q \Leftrightarrow \exists \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^q (p \leq^{*,\pi} q_{\langle \nu_0, \dots, \nu_{n-1} \rangle}).$$

3.3. Projecting \mathbb{P}_E onto \mathbb{P}_E^π . In this subsection, we show that there is a projection from \mathbb{P}_E into \mathbb{P}_E^π . For $f \in \mathbb{P}_E^*$, let f^π be defined as follows: $\text{dom}(f^\pi) = \text{dom}(f)$, and for $\alpha \in \text{dom}(f^\pi)$

$$f^\pi(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha = \kappa, \\ \langle 0 \rangle & \text{Otherwise.} \end{cases}$$

It is clear that $f^\pi \in \mathbb{P}_E^{*,\pi}$. Now define $\pi : \mathbb{P}_E \rightarrow \mathbb{P}_E^\pi$ by

$$\pi(\langle f, A \rangle) = \langle f^\pi, A \rangle.$$

π is well-defined, and it is clear that it is order preserving with respect to both \leq and \leq^* relations, in the sense that

$$p \leq^* q \Rightarrow \pi(p) \leq^{*,\pi} \pi(q),$$

and

$$p \leq q \Rightarrow \pi(p) \leq^\pi \pi(q).$$

Theorem 3.24. π is a projection of Prikry type.

Proof. Let $p = \langle f, A \rangle \in \mathbb{P}_E$, $p^* = \langle f^*, A^* \rangle \in \mathbb{P}_E^\pi$ and $p^* \leq^\pi \pi(p) = \langle f^\pi, A \rangle$. Let $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A$ be such that $p^* \leq^{*,\pi} \pi(p)_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$. Let $q = \langle g, B \rangle$ where

- $g = f_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \cup f^* \upharpoonright (\text{dom}(f^*) \setminus \text{dom}(f))$,
- $B = A^*$.

Clearly $q \leq p$ (as $q \leq^* p_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$) and $\pi(q) = \langle g^\pi, B \rangle \leq^{*,\pi} p^*$, which gives the result. \square

3.4. More on \mathbb{P}_E^π . Let's state the main properties of the forcing notion \mathbb{P}_E^π .

Lemma 3.25. \mathbb{P}_E^π satisfies the κ^+ -c.c.

Proof. This follows from Lemma 3.19, and the fact that for $p, q \in \mathbb{P}_E^\pi$, if f^p, f^q are compatible in $\mathbb{P}_E^{*,\pi}$, then p, q are compatible in \mathbb{P}_E^π . \square

Theorem 3.26. Let H be \mathbb{P}_E^π -generic over V . Then

- (1) $\langle \mathbb{P}_E^\pi, \leq^* \rangle$ is κ -closed,
- (2) $cf^{V[H]}(\kappa) = \omega$,
- (3) $\langle \mathbb{P}_E^\pi, \leq, \leq^* \rangle$ satisfies the Prikry property.

Proof. (1) and (2) are clear, and (3) follows from Lemma 2.6 and Theorem 3.24. \square

It follows that V and $V[H]$ have the same cardinals and the same bounded subsets of κ , and λ remains an inaccessible cardinal in $V[H]$.

3.5. Homogeneity of the quotient forcing. Let G be \mathbb{P}_E -generic over V , and let G^π be the filter generated by $\pi[G]$. It follows from Theorem 2.27 that G^π is \mathbb{P}_E^π -generic over V and that $V \subseteq V[G^\pi] \subseteq V[G]$. By standard forcing theorems, $V[G]$ is itself a forcing extension of $V[G^\pi]$. We show that this forcing has enough homogeneity properties, which guarantees that $\text{HOD}^{V[G]} \subseteq V[G^\pi]$.

For a forcing notion \mathbb{P} and a condition $p \in \mathbb{P}$, set $\mathbb{P} \downarrow p = \{q \in \mathbb{P} : q \leq p\}$ consists of all extensions of p in \mathbb{P} . The homogeneity of our quotient forcing follows from the next theorem.

Lemma 3.27. (*Homogeneity lemma*) For all $p, q \in \mathbb{P}_E$, if $\pi(p) = \pi(q)$, then

$$\mathbb{P}_E \downarrow p \simeq \mathbb{P}_E \downarrow q.$$

In particular, if $p \Vdash \text{“}\phi(\alpha, \vec{\gamma})\text{”}$, where $\alpha, \vec{\gamma}$ are ordinals, then it is not the case that $q \Vdash \text{“}\neg\phi(\alpha, \vec{\gamma})\text{”}$.

Remark 3.28. In fact, it suffices $\pi(p)$ and $\pi(q)$ to be compatible in the $\leq^{*,\pi}$ -ordering relation.

Proof. Since $\pi(p) = \pi(q)$, we have $\text{dom}(f^p) = \text{dom}(f^q) = \Delta$, $f^p(\kappa) = f^q(\kappa)$ and $A^p = A^q = A$. Let $p^* \leq p$, and find $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A$ such that $p^* \leq^* p_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$. Let $\Phi(f^{p^*}) : \text{dom}(p^*) \rightarrow \lambda^{<\omega}$ be defined as follows:

$$\Phi(f^{p^*})(\alpha) = \begin{cases} f_{\langle \nu_0, \dots, \nu_{n-1} \rangle}^q(\alpha) & \text{if } \alpha \in \Delta, \\ p^*(\alpha) & \alpha \in \text{dom}(p^*) \setminus \Delta \end{cases}$$

It is clear that $\Phi(f^{p^*}) \in \mathbb{P}_E^*$ and that $\Phi(f^{p^*}) \leq_{\mathbb{P}_E^*}^* f^q$. Finally set

$$\Phi(p^*) = \langle \Phi(f^{p^*}), A^{p^*} \rangle.$$

Trivially $\Phi(p^*) \leq q$, and so $\Phi(f^{p^*}) \in \mathbb{P}_E \downarrow q$. It is also easily seen that

$$\Phi : \mathbb{P}_E \downarrow p \rightarrow \mathbb{P}_E \downarrow q$$

is an isomorphism. Let us just define its converse. So let $q^* \leq q$, and find $\langle \gamma_0, \dots, \gamma_{m-1} \rangle \in A$ such that $q^* \leq^* q_{\langle \gamma_0, \dots, \gamma_{m-1} \rangle}$. Let $\Psi(f^{q^*}) : \text{dom}(q^*) \rightarrow \lambda^{<\omega}$ be defined as follows:

$$\Psi(f^{q^*})(\alpha) = \begin{cases} f_{\langle \gamma_0, \dots, \gamma_{m-1} \rangle}^p(\alpha) & \text{if } \alpha \in \Delta, \\ q^*(\alpha) & \alpha \in \text{dom}(q^*) \setminus \Delta \end{cases}$$

Then set

$$\Psi(q^*) = \langle \Psi(f^{q^*}), A^{q^*} \rangle.$$

Then we have

$$\Psi : \mathbb{P}_E \downarrow q \rightarrow \mathbb{P}_E \downarrow p,$$

and it is easily seen to be the converse of Φ . □

Theorem 3.29. *G be \mathbb{P}_E -generic over V , and let G^π be the filter generated by $\pi[G]$. Then $HOD^{V[G]} \subseteq V[G^\pi]$.*

Proof. By the above homogeneity result. □

3.6. Completing the proof of Theorem 3.1. Finally we are ready to complete the proof of Theorem 3.1. Let \mathbb{P}_E and \mathbb{P}_E^π be as above, and let G be \mathbb{P}_E -generic over V . Let G^π be the filter generated by $\pi[G]$. It follows from Lemma 2.2 and Theorem 3.24 that G^π is \mathbb{P}_E^π -generic over V and $V[G^\pi] \subseteq V[G]$. By Theorem 3.26, λ remains inaccessible in $V[G^\pi]$. By Theorem 3.29, $HOD^{V[G]} \subseteq V[G^\pi]$, hence λ remains inaccessible in $HOD^{V[G]}$.

4. ALL UNCOUNTABLE REGULAR CARDINALS CAN BE INACCESSIBLE IN HOD

In this section we give a proof of Theorem 1.3. Our strategy of the proof is similar to [6], but here we deal with a different forcing notion. As the proof of the theorem is long and complicated, we first give an idea of the proof, and then go into the details of the proof. First note that by the results from [1] and [4], it suffices to prove the following:

Theorem 4.1. *Assume GCH holds, κ is a supercompact cardinal and λ is an inaccessible cardinal above κ . Then there is a generic extension $V[G]$ of V in which:*

- (1) κ remains inaccessible,
- (2) There exists a club $C = \{\kappa_\xi : \xi < \kappa\}$ of κ consisting of V -measurable cardinals,
- (3) For all limit $\xi < \kappa$, κ_ξ^+ is inaccessible in $HOD^{V[G]}$.

So we start with $GCH+$ a suitable Mitchell increasing sequence of extenders $\bar{E} = \langle E_\xi : \xi < o(\bar{E}) \rangle$ on $\kappa + \lambda$ is the least inaccessible above κ . We define a forcing notion $\mathbb{P}_{\bar{E}}$, called the “supercompact extender based Radin forcing”, due to Merimovich [12], which has the following properties:

- (1) $\mathbb{P}_{\bar{E}}$ preserves the inaccessibility of κ ,
- (2) It collapses all cardinals in (κ, λ) , and preserves all cardinals $\geq \lambda$, so $\kappa^+ = \lambda$ in the extension by $\mathbb{P}_{\bar{E}}$,
- (3) It adds a club $C = \{\kappa_\xi : \xi < \kappa\}$ of κ consisting of V -measurable cardinals,
- (4) If $\xi < \kappa$ is a limit ordinal, and λ_ξ is the least inaccessible above κ_ξ , then for each regular $\mu \in (\kappa_\xi, \lambda_\xi)$, there exists a cofinal sequence into μ of order type $\leq \kappa_\xi$, in particular all cardinals $\mu \in (\kappa_\xi, \lambda_\xi)$ are collapsed.
- (5) The forcing preserves λ_ξ , so for limit $\xi < \kappa$, $\kappa_\xi^+ = \lambda_\xi$ in the extension by $\mathbb{P}_{\bar{E}}$. All other cardinals $< \kappa$ are preserved.

Let G be $\mathbb{P}_{\bar{E}}$ -generic over V . We will show that $V[G]$ is as required. To this end, we define a new forcing notion $\mathbb{P}_{\bar{E}}^\pi$, called the “projected supercompact extender based Radin forcing”, and a Prikry type projection $\pi : \mathbb{P}_{\bar{E}} \rightarrow \mathbb{P}_{\bar{E}}^\pi$. The forcing notion $\mathbb{P}_{\bar{E}}^\pi$ adds the club C , and does not collapse any cardinals. Furthermore, the resulting quotient forcing has enough homogeneity properties to guarantee that $HOD^{V[G]} \subseteq V[G^\pi]$, where G^π is the filter generated by $\pi[G]$. From this results it follows that for all limit ordinals $\xi < \kappa$, $(\kappa_\xi^+)^{V[G]} = \lambda_\xi$

is inaccessible in $HOD^{V[G]}$, which will give us a proof of Theorem 4.1 and hence of Theorem 1.3.

The rest of this section is devoted to the proof of Theorem 4.1. We proceed the proof in the same way as in the proof of Theorem 3.1.

4.1. Supercompact extender based Radin forcing. “supercompact extender based Radin forcing” was defined by Merimovich [12]. Our proof of Theorem 1.4. is based on this forcing notion, so we give the basic facts about it and present some of its main properties. All of the results of this section are due to Merimovich. We assume that GCH holds in the ground model, κ is a supercompact cardinal and λ is the least strongly inaccessible above κ .

Definition 4.2. *Assume $j : V \rightarrow M$ is an elementary embedding with $\text{crit}(j) = \kappa, j(\kappa) > \lambda$ such that $M \supseteq {}^{<\lambda}M$.*

- (1) *For each α let λ_α be minimal η such that $\alpha < j(\eta)$.*
- (2) *The generators of the embedding j , $\mathfrak{g}(j) = \{\kappa_\xi : \xi \in ON\}$, are defined by induction by*

$$\kappa_\xi = \min\{\alpha \in ON : \forall \zeta < \xi \forall \eta \in ON \forall f : \eta \rightarrow ON (j(f)(\kappa_\zeta) \neq \alpha)\}.$$

If $\mathfrak{g}(j)$ is a set, then we can code j by an extender $E = \langle E(\alpha) : \alpha \in \mathfrak{g}(j) \rangle$, where for each $\alpha \in \mathfrak{g}(j)$, $E(\alpha)$ is a measure on λ_α defined by

$$\forall A \subseteq \lambda_\alpha (A \in E(\alpha)) \Leftrightarrow \alpha \in j(A).$$

Remark 4.3. *In this paper, we only deal with embeddings having their generators below $j(\lambda)$ (and hence a set), and consider the natural elementary embedding $j_E : V \rightarrow \text{Ult}(V, E)$. We may further assume that $j = j_E$.*

Assume $\bar{E} = \langle E_\xi : \xi < o(\bar{E}) \rangle$ is a sequence of extenders on κ such that:

- (1) \bar{E} is Mitchell increasing, i.e., for each $\xi < o(\bar{E})$ $\langle E_\zeta : \zeta < \xi \rangle \in E_\xi$,
- (2) If $j_{E_\xi} : V \rightarrow M_\xi \simeq \text{Ult}(V, E_\xi)$ is the corresponding elementary embedding, then
 - (2-1) $\text{crit}(j_{E_\xi}) = \kappa$ and $j_{E_\xi}(\kappa) \geq \lambda$,
 - (2-2) λ is minimal such that $M_\xi \not\supseteq {}^\lambda M_\xi$ (and hence $M_\xi \supseteq {}^{<\lambda} M_\xi$),
 - (2-3) $\forall \xi < o(\bar{E}) \mathfrak{g}(j_{E_\xi}) \subseteq \sup j_{E_\xi}[\lambda]$.

Note that for $\xi_1 < \xi_2 < o(\bar{E})$, $j_{E_{\xi_1}}(\lambda) < j_{E_{\xi_2}}(\lambda)$. Let $\lambda \leq \epsilon \leq \sup\{j_{E_\xi}(\lambda) : \xi < o(\bar{E})\}$.

Definition 4.4. An extender sequence $\bar{\nu}$ has the form $\langle \tau, e_0, \dots, e_\xi, \dots \rangle_{\xi < \mu}$, where $\langle e_\xi : \xi < \mu \rangle$ is a Mitchell increasing sequence of extenders with identical critical points and closure points, and $\text{crit}(e_0) \leq \tau < j_{e_0}(\alpha)$, where α is the closure point of M_{e_0} . The order of the extender sequence $\bar{\nu}$ is μ , which we denote by $o(\bar{\nu}) = \mu$. We write $\bar{\nu}_0$ for τ and naturally $\bar{\nu}_{1+\xi}$ for e_ξ .

Note that formally the Mitchell order function $o(\dots)$ is defined on different type of objects. The first object is of the form $\langle E_\xi : \xi < \mu \rangle$, and the second is of the form $\langle \tau, e_0, \dots, e_\xi, \dots \rangle_{\xi < \mu}$. In either case only the extenders are considered, thus there is no confusion.

Definition 4.5. The set \mathfrak{D} is a base set used in the domain of functions. For each $\kappa \leq \alpha < \sup\{j_{E_\xi}(\lambda) : \xi < o(\bar{E})\}$, $\alpha \notin j[\lambda]$, define

$$\bar{\alpha} = \langle \alpha \rangle \frown \langle E_\zeta : \zeta < o(\bar{E}), \alpha < j_{E_\zeta}(\lambda) \rangle.$$

Then define

$$\mathfrak{D} = \{\bar{\alpha} : \kappa \leq \alpha < \epsilon\}.$$

On \mathfrak{D} the order $<$ is defined by $\bar{\alpha} < \bar{\beta} \Leftrightarrow \alpha < \beta$. The set \mathfrak{R} is used as the base for range of functions

$$\mathfrak{R} = \{\bar{\nu} \in V_\lambda : \bar{\nu} \text{ is an extender sequence}\}.$$

On \mathfrak{R} the order $<$ is defined by $\bar{\nu} < \bar{\mu} \Leftrightarrow \bar{\nu}_0 < \bar{\mu}_0$. For some technical reasons appearing in the next subsection, we assume $\langle \rangle \in \mathfrak{R}$.

Definition 4.6. Assume $d \in P_\lambda(\mathfrak{D})$. Then $\nu \in OB(d)$ iff;

- (1) $\nu : \text{dom}(\nu) \rightarrow \mathfrak{R}$,
- (2) $\bar{\kappa} \in \text{dom}(\nu) \subseteq d \cup j[\lambda]$,
- (3) $|\nu| < \nu(\bar{\kappa})_0$,
- (4) $\forall \alpha < \lambda (j(\alpha) \in \text{dom}(\nu) \Rightarrow \nu(j(\alpha)) = \langle \alpha \rangle)$,
- (5) $\forall \bar{\alpha} \in \text{dom}(\nu) \setminus j[\lambda] (o(\nu(\bar{\alpha})) < o(\bar{\alpha}))$,
- (6) For each $\bar{\alpha} \in \text{dom}(\nu) \setminus j[\lambda]$ such that $\bar{\alpha} \neq \bar{\kappa}$, the following is satisfied: Assume

$$\nu(\bar{\kappa}) = \langle \tau, e_0, \dots, e_\xi, \dots \rangle_{\xi < \zeta_\kappa} \text{ (where } \text{crit}(e_0) = \tau \text{)}$$

and

$$\nu(\bar{\alpha}) = \langle \tau', e'_0, \dots, e'_\xi, \dots \rangle_{\xi < \zeta_\alpha}.$$

Then $\langle e_{\zeta+\xi} : \xi < \zeta_\kappa \rangle = \langle e'_\xi : \xi < \zeta_\alpha \rangle$, where $\zeta < \zeta_\kappa$ is minimal such that $\tau' \in$

$[\sup_{\zeta' < \zeta} j_{e_{\zeta'}}(\tau), j_{e_\zeta}(\sigma)]$, where σ is the closure point of e_ζ .

$$(7) \quad \forall \bar{\alpha}, \bar{\beta} \in \text{dom}(\nu) \setminus j[\lambda] \quad (\bar{\alpha} < \bar{\beta} \Rightarrow \nu(\bar{\alpha}) < \nu(\bar{\beta})).$$

On $OB(d)$ the partial order $<$ is defined by $\mu < \nu$ iff either

$$\forall \bar{\alpha} \in \text{dom}(\mu) \cap \text{dom}(\nu) \quad (o(\mu(\bar{\alpha})) > o(\nu(\bar{\alpha})) \text{ and } \mu(\bar{\alpha}) < \nu(\bar{\alpha}))$$

or

$$\text{dom}(\mu) \subseteq \text{dom}(\nu) \text{ and } \forall \bar{\alpha} \in \text{dom}(\mu) \quad (o(\mu(\bar{\alpha})) \leq o(\nu(\bar{\alpha})) \text{ and } \mu(\bar{\alpha}) < \nu(\bar{\alpha})).$$

Definition 4.7. Assume $d \in P_\lambda(\mathfrak{D})$.

(1) Assume $T \subseteq OB(d)^{<\xi}$ ($1 < \xi \leq \omega$) and let $n < \omega$. Then

- $Lev_n(T) = T \cap OB(d)^{n+1}$,
- $\text{Suc}_T(\langle \rangle) = Lev_0(T)$,
- $\text{Suc}_T(\langle \nu_0, \dots, \nu_{n-1} \rangle) = \{\mu \in OB(d) : \langle \nu_0, \dots, \nu_{n-1}, \mu \rangle \in T\}$.

(2) For $\langle \nu \rangle \in T$, let

$$T_{\langle \nu \rangle} = \{\langle \nu_0, \dots, \nu_{k-1} \rangle : k < \omega, \langle \nu, \nu_0, \dots, \nu_{k-1} \rangle \in T\}$$

and define by recursion for $\langle \nu_0, \dots, \nu_{n-1} \rangle \in T$,

$$T_{\langle \nu_0, \dots, \nu_{n-1} \rangle} = (T_{\langle \nu_0, \dots, \nu_{n-2} \rangle})_{\langle \nu_{n-1} \rangle}.$$

(3) The measures $E_\xi(d)$ ($\xi < o(\bar{E})$) on $OB(d)$ are defined as follows:

$$\forall X \subseteq OB(d) \quad (X \in E_\xi(d) \Leftrightarrow mc_\xi(d) \in j_{E_\xi}(X)),$$

where

$$mc_\xi(d) = \{\langle j_{E_\xi}(\bar{\alpha}), R_\xi(\bar{\alpha}) \rangle : \bar{\alpha} \in d, \alpha < j_{E_\xi}(\lambda)\},$$

and R_ξ is defined for each $\kappa \leq \alpha < \epsilon$ by

$$R_\xi(\alpha) = \langle \alpha \rangle \frown \langle E_{\xi'} : \xi' < \xi, \alpha < j_{E_{\xi'}}(\lambda) \rangle.$$

Also set

$$E(d) = \bigcap \{E_\xi(d) : \xi < o(\bar{E})\}.$$

- (4) A tree $T \subseteq OB(d)^{<\omega}$ is called a d -tree, if
- For each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in T$, we have $\nu_0 < \dots < \nu_{n-1}$,
 - $\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in T, \text{Suc}_T(\langle \nu_0, \dots, \nu_{n-1} \rangle) \in E(d)$.
- (5) Assume $c \in P_{\kappa^+}(\mathfrak{D}), c \subseteq d$, and T is a tree with elements from $OB(d)$. Then the projection of T to a tree with elements from $OB(c)$ is

$$T \upharpoonright c = \{ \langle \nu_0 \upharpoonright c, \dots, \nu_{n-1} \upharpoonright c \rangle : n < \omega, \langle \nu_0, \dots, \nu_{n-1} \rangle \in T \}.$$

Definition 4.8. $\mathbb{P}_{\bar{E}, \epsilon}^*$ consists of all functions $f : d \rightarrow^{<\omega} \mathfrak{R}$ such that

- (1) $\bar{\kappa} \in d \in P_\lambda(\mathfrak{D})$,
- (2) For each $\bar{\alpha} \in d, f(\bar{\alpha}) = \langle f_0(\bar{\alpha}), \dots, f_{k-1}(\bar{\alpha}) \rangle$ is an increasing sequence in \mathfrak{R} ,
- (3) For each $\bar{\alpha} \in d$ and $i < |f(\bar{\alpha})|$, $(o(f_i(\bar{\alpha})) < o(\bar{\alpha}))$,
- (4) For each $\bar{\alpha} \in d$, the sequence $\langle o(f_i(\bar{\alpha})) : i < |f(\bar{\alpha})| \rangle$ is non-increasing.

Definition 4.9. For $f, g \in \mathbb{P}_{\bar{E}, \epsilon}^*$, we say f is an extension of g ($f \leq_{\mathbb{P}_{\bar{E}, \epsilon}^*}^* g$) if $f \supseteq g$.

Remark 4.10. Clearly $\mathbb{P}_{\bar{E}, \epsilon}^* \simeq \text{Add}(\lambda, \epsilon)$.

We write $OB(f), E_\xi(f), E(f), mc_\xi(f)$ and f -tree, for $OB(\text{dom}(f)), E_\xi(\text{dom}(f)), E(\text{dom}(f)), mc_\xi(\text{dom}(f))$ and $\text{dom}(f)$ -tree respectively, where $f \in \mathbb{P}_{\bar{E}, \epsilon}^*$. The following lemma is clear.

Lemma 4.11. $\langle \mathbb{P}_{\bar{E}, \epsilon}^*, \leq_{\mathbb{P}_{\bar{E}, \epsilon}^*}^* \rangle$ satisfies the λ^+ -c.c.

Definition 4.12. Assume $f \in \mathbb{P}_{\bar{E}, \epsilon}^*$ and $\nu \in OB(f)$. Define $g = f_{\langle \nu \rangle}$ to be of the form

$g = g_{\leftarrow} \hat{\ } g_{\rightarrow}$ (the case $g_{\leftarrow} = \emptyset$ is allowed) where

- (1) $\text{dom}(g_{\rightarrow}) = \text{dom}(f)$,
- (2) For each $\bar{\alpha} \in \text{dom}(g_{\rightarrow})$

$$g_{\rightarrow}(\bar{\alpha}) = \begin{cases} f(\bar{\alpha}) \upharpoonright k \hat{\ } \langle \nu(\bar{\alpha}) \rangle & \text{if } \bar{\alpha} \in \text{dom}(\nu), \nu(\bar{\alpha}) > f_{|f(\bar{\alpha})|-1}(\bar{\alpha}), \\ f(\bar{\alpha}) & \text{Otherwise.} \end{cases}$$

where

$$k = \min \{ l \leq |f(\bar{\alpha})| : \forall l \leq i < |f(\bar{\alpha})|, o(f_i(\bar{\alpha})) < o(\nu(\bar{\alpha})) \}.$$

The above value of k is defined so as to ensure that $\langle o(f_i(\bar{\alpha})) : i < k \rangle \hat{\ } \langle o(\nu(\bar{\alpha})) \rangle$ is non-increasing.

- (3) $\text{dom}(g_{\leftarrow}) = \{ \nu(\bar{\alpha}) : \bar{\alpha} \in \text{dom}(\nu), o(\nu(\bar{\alpha})) > 0 \}$,

(4) For each $\bar{\alpha} \in \text{dom}(\nu)$ with $o(\nu(\bar{\alpha})) > 0$ we have

$$g_{\leftarrow}(\nu(\bar{\alpha})) = f(\bar{\alpha}) \upharpoonright (|f(\bar{\alpha})| \setminus k),$$

where k is defined as above.

Definition 4.13. $\mathbb{P}_{\bar{E}, \epsilon, \rightarrow}$ consists of pairs $p = \langle f, A \rangle$ where

- (1) $f \in \mathbb{P}_{\bar{E}, \epsilon}^*$,
- (2) A is an f -tree such that for each $\langle \nu \rangle \in A$ and each $\bar{\alpha} \in \text{dom}(\nu)$

$$f_{|f(\bar{\alpha})|-1}(\bar{\alpha}) < \nu(\bar{\alpha})$$

We write f^p, A^p and $mc_{\xi}(p)$ for f, A and $mc_{\xi}(f)$, respectively.

Definition 4.14. Let $p, q \in \mathbb{P}_{\bar{E}, \epsilon, \rightarrow}$. We say $p \leq_{\mathbb{P}_{\bar{E}, \epsilon, \rightarrow}}^* q$ (p is a Prikry extension of q) if

- (1) $f^p \leq_{\mathbb{P}_{\bar{E}, \epsilon}^*}^* f^q$,
- (2) $A^p \upharpoonright \text{dom}(f^q) \subseteq A^q$.

Definition 4.15. Assume $\langle e_i : i < n \rangle$ ($n < \omega$) is a sequence of extenders such that $e_i \in V_{\text{crit}(e_{i+1})}$. The product forcing notion $\mathbb{P} = \prod_{i < n} \mathbb{P}_{e_i}$ is defined by applying the definitions of the Prikry with extenders forcing notions coordinatewise. That is, for each $\langle p_i : i < n \rangle, \langle q_i : i < n \rangle \in \mathbb{P}$,

$$\langle p_i : i < n \rangle \leq_{\mathbb{P}} \langle q_i : i < n \rangle \Leftrightarrow \forall i < n, p_i \leq_{\mathbb{P}_{e_1}} q_i,$$

and

$$\langle p_i : i < n \rangle \leq_{\mathbb{P}}^* \langle q_i : i < n \rangle \Leftrightarrow \forall i < n, p_i \leq_{\mathbb{P}_{e_1}}^* q_i.$$

For $p = \langle p_i : i < n \rangle \in \mathbb{P}$, we use the notation $p_{\leftarrow} = p_0 \widehat{\ } \dots \widehat{\ } p_{n-2}$ and $p_{\rightarrow} = p_{n-1}$. Assume $\langle \nu \rangle \in A^{p_{\leftarrow}}$. Define the condition $p_{\langle \nu \rangle}$ recursively as follows:

$$p_{\langle \nu \rangle} = p_{\leftarrow} \widehat{\ } p_{\rightarrow \langle \nu \rangle}.$$

Note that with p_{\leftarrow} and p_{\rightarrow} defined we have, for each $p, q \in \mathbb{P}$,

$$p \leq q \Leftrightarrow (p_{\leftarrow} \leq q_{\leftarrow} \text{ and } p_{\rightarrow} \leq q_{\rightarrow} \leq),$$

and

$$p \leq^* q \Leftrightarrow (p_{\leftarrow} \leq^* q_{\leftarrow} \text{ and } p_{\rightarrow} \leq^* q_{\rightarrow} \leq).$$

It is a standard fact that $(\mathbb{P}, \leq_{\mathbb{P}}, \leq_{\mathbb{P}}^*)$ is a Prikry type forcing notion, provided that all forcing notions $(\mathbb{P}_{e_i}, \leq_{\mathbb{P}_{e_i}}, \leq_{\mathbb{P}_{e_i}}^*)$ are of Prikry type. Note that as the extenders e_i are disjoint,

factoring of \mathbb{P} is easily achieved, thus a generic extension by \mathbb{P} can be analyzed by inspecting generic extensions by each factor $\mathbb{P}_{\bar{e}_i}$. We are now ready to define the forcing notion $\mathbb{P}_{\bar{E},\epsilon}$.

Definition 4.16. *A condition p in the forcing notion $\mathbb{P}_{\bar{E},\epsilon}$ is of the form $p_{\leftarrow} \frown p_{\rightarrow}$ where*

- (1) $p_{\rightarrow} \in \mathbb{P}_{\bar{E},\epsilon,\rightarrow}$,
- (2) $p_{\leftarrow} \in \prod_{i < n} \mathbb{P}_{\bar{e}_i}$ ($n < \omega$), where \bar{e}_i are extender sequences such that
 - (2-1) $o(\bar{e}_i) \leq o(\bar{E})$,
 - (2-2) $\bar{e}_i \in V_{\text{crit}(\bar{e}_{i+1})}$,
 - (2-3) $\langle \nu \rangle \in A^{p_{\leftarrow}} \Rightarrow \nu(\bar{\kappa})_0 > \text{crit}(\bar{e}_{n-1})$.

Conditions in $\mathbb{P}_{\bar{E},\epsilon}$ have lower parts $\mathbb{P}_{\bar{E},\epsilon,\leftarrow}$ defined by

$$\mathbb{P}_{\bar{E},\epsilon,\leftarrow} = \{p_{\leftarrow} : p \in \mathbb{P}_{\bar{E},\epsilon}\}.$$

Also for $p \in \mathbb{P}_{\bar{E},\epsilon}$, we define f^p recursively to be $f^{p_{\leftarrow}} \frown f^{p_{\rightarrow}}$, and we write f_{\leftarrow}^p and f_{\rightarrow}^p for $f^{p_{\leftarrow}}$ and $f^{p_{\rightarrow}}$, respectively.

Definition 4.17. *Let $p, q \in \mathbb{P}_{\bar{E},\epsilon}$. Then $p \leq_{\mathbb{P}_{\bar{E},\epsilon}}^* q$ (p is a Prikry extension of q) if:*

- (1) $p_{\rightarrow} \leq^* q_{\rightarrow}$,
- (2) $p_{\leftarrow} \leq^* q_{\leftarrow}$.

Definition 4.18. (1) *Assume $\mu, \nu \in OB(f)$ are such that $\mu < \nu$, and for each $\bar{\alpha} \in \text{dom}(\mu)$, $o(\mu(\bar{\alpha})) < o(\nu(\bar{\alpha}))$. Then $\mu \downarrow \nu$, the reflection of μ by ν is defined by*

$$\text{dom}(\mu \downarrow \nu) = \{\nu(\bar{\alpha}) : \bar{\alpha} \in \text{dom}(\mu)\}$$

and for each $\bar{\alpha} \in \text{dom}(\mu)$

$$(\mu \downarrow \nu)(\nu(\bar{\alpha})) = \mu(\bar{\alpha}).$$

If $\mu_0, \dots, \mu_n, \nu \in OB(f)$ are such that $\mu_i < \nu$ and for each $\bar{\alpha} \in \text{dom}(\mu_i)$, $o(\mu_i(\bar{\alpha})) < o(\nu(\bar{\alpha}))$, then $\langle \mu_0, \dots, \mu_n \rangle \downarrow \nu$, the reflection of $\langle \mu_0, \dots, \mu_n \rangle$ by ν is defined to be $\langle \mu_0 \downarrow \nu, \dots, \mu_n \downarrow \nu \rangle$.

(2) *Assume A is an f -tree and $\langle \nu \rangle \in A$. The tree $A \downarrow \nu$ is defined as follows: $A \downarrow \nu$ consists of all $\langle \mu_0, \dots, \mu_n \rangle \downarrow \nu$ where:*

- (a) $n < \omega$,
- (b) $\langle \mu_0, \dots, \mu_n \rangle \in A$,

(c) $\forall i \leq n$ ($\mu_i < \nu$, $\text{dom}(\mu_i) \subseteq \text{dom}(\nu)$ and $\forall \bar{\alpha} \in \text{dom}(\mu_i)$, $o(\mu_i(\bar{\alpha})) < o(\nu(\bar{\alpha}))$).

It is easily seen that

$$\{\langle \nu \rangle \in A : A \downarrow \nu \text{ is an } f_{\langle \nu \rangle \leftarrow} \text{-tree}\} \in E_1(f),$$

and if we consider \emptyset to be an \emptyset -tree, then

$$\{\langle \nu \rangle \in A : A \downarrow \nu \text{ is an } f_{\langle \nu \rangle \leftarrow} \text{-tree}\} \in E(f).$$

Definition 4.19. Assume $q \in \mathbb{P}_{\bar{E}, \epsilon, \rightarrow}$ and $\langle \nu \rangle \in A^q$. The condition $p \in \mathbb{P}_{\bar{E}, \epsilon}$ is the one point extension of q by $\langle \nu \rangle$ ($p = q_{\langle \nu \rangle}$) if it is of the form $p_{\leftarrow} \hat{\cap} p_{\rightarrow}$ where $p_{\leftarrow} \in \mathbb{P}_{\bar{E}, \rightarrow}$ and $p_{\rightarrow} \in \mathbb{P}_{\bar{E}, \rightarrow}$ are defined as follows:

- (1) $f^p = f_{\langle \nu \rangle}^q$,
- (2) $A^{p_{\rightarrow}} = A_{\langle \nu \rangle}^q$,
- (3) $A^{p_{\leftarrow}} = A^q \downarrow \nu$.

Define $q_{\langle \nu_0, \dots, \nu_n \rangle}$ recursively by

$$q_{\langle \nu_0, \dots, \nu_n \rangle} = (q_{\langle \nu_0, \dots, \nu_{n-1} \rangle})_{\langle \nu_n \rangle},$$

where $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^q$.

Definition 4.20. Assume $p \in \mathbb{P}_{\bar{E}, \epsilon}$ and $\langle \nu \rangle \in A^{p_{\rightarrow}}$. Then

$$p_{\langle \nu \rangle} = p_{\leftarrow} \hat{\cap} p_{\rightarrow} \langle \nu \rangle.$$

Define $p_{\langle \nu_0, \dots, \nu_n \rangle}$ recursively by

$$p_{\langle \nu_0, \dots, \nu_n \rangle} = (p_{\langle \nu_0, \dots, \nu_{n-1} \rangle})_{\langle \nu_n \rangle}$$

Definition 4.21. Let $p, q \in \mathbb{P}_{\bar{E}, \epsilon}$. Then $p \leq_{\mathbb{P}_{\bar{E}, \epsilon}} q$ (p is stronger than q) if $p = r \hat{\cap} s$ and there is $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{q_{\rightarrow}}$ such that

- (1) $s \leq_{\mathbb{P}_{\bar{E}, \epsilon}}^* q_{\rightarrow \langle \nu_0, \dots, \nu_{n-1} \rangle}$,
- (2) $r \leq q_{\leftarrow}$.

Definition 4.22. Assume $p \in \mathbb{P}_{\bar{E}, \epsilon}$. Then $p_{\rightarrow} \in \mathbb{P}_{\bar{E}, \epsilon}$, and we define

$$\mathbb{P}_{\bar{E}, \epsilon} \downarrow p_{\rightarrow} = \{q \in \mathbb{P}_{\bar{E}, \epsilon} : q \leq p_{\rightarrow}\}$$

and

$$\mathbb{P}_{\bar{E}, \epsilon} \downarrow p_{\leftarrow} = \{r : r \leq p_{\leftarrow}\}.$$

Let's state the main properties of our forcing notion.

Lemma 4.23. $\mathbb{P}_{\bar{E},\epsilon}$ satisfies the λ^+ -c.c.

Proof. Assume not, and let $A \subseteq \mathbb{P}_{\bar{E},\epsilon}$ be an antichain of size λ^+ . We may assume without loss of generality that all p_{\leftarrow} , for $p \in A$ are the same (as there are only λ -many such p_{\leftarrow}). Note that for any $p, q \in \mathbb{P}_{\bar{E},\epsilon}$, if f^p is compatible with f^q in $\mathbb{P}_{\bar{E},\epsilon}^*$, then p and q are compatible in $\mathbb{P}_{\bar{E},\epsilon}$. It follows that $\{f^p : p \in A\} \subseteq \mathbb{P}_{\bar{E},\epsilon}^*$ is an antichain of size λ^+ , which contradicts Lemma 4.17. \square

The following factorization property is clear.

Lemma 4.24. (*Factorization lemma*) For any $p \in \mathbb{P}_{\bar{E},\epsilon}$,

$$\mathbb{P}_{\bar{E},\epsilon} \downarrow p \simeq \mathbb{P}_{\bar{E},\epsilon} \downarrow p_{\leftarrow} \times \mathbb{P}_{\bar{E},\epsilon} \downarrow p_{\rightarrow}.$$

The next lemmas are proved in [12].

Lemma 4.25. $\langle \mathbb{P}_{\bar{E},\epsilon}, \leq, \leq^* \rangle$ satisfies the Prikry property.

Lemma 4.26. In a $\mathbb{P}_{\bar{E},\epsilon}$ -generic extension, λ is preserved.

Let G be $\mathbb{P}_{\bar{E},\epsilon}$ -generic over V , and for $\kappa \leq \alpha < \epsilon$ let

$$G^{\bar{\alpha}} = \bigcup \{f_{\rightarrow}^p(\bar{\alpha}) : p \in G, \bar{\alpha} \in \text{dom}(f_{\rightarrow}^p)\}$$

and

$$C^{\bar{\alpha}} = \bigcup \{\bar{v}_0 : \bar{v} \in G^{\bar{\alpha}}\}.$$

Lemma 4.27. (1) $C^{\bar{\kappa}}$ is a club of κ ,

$$(2) \alpha \neq \beta \Rightarrow C^{\bar{\alpha}} \neq C^{\bar{\beta}},$$

(3) Forcing with $\mathbb{P}_{\bar{E},\epsilon}$ collapses all cardinal in (κ, λ) onto κ .

It follows from our results that

$$\lambda = (\kappa^+)^{V[G]}.$$

Also by Lemma 4.27, $2^\kappa \geq |\epsilon|$, and using the λ -chain condition of the forcing, we can conclude that $(2^\kappa)^{V[G]} \leq (|\mathbb{P}_{\bar{E},\epsilon}|^{<\lambda})^\kappa = \epsilon$, and hence

$$V[G] \models "2^\kappa = |\epsilon|".$$

Lemma 4.28. *If $cf(o(\bar{E})) > |\epsilon|$, then κ remains measurable in $V[G]$.*

The next theorem follows from the above results, the factorization property of the forcing notion $\mathbb{P}_{\bar{E},\epsilon}$ and using some reflection arguments.

Lemma 4.29. *Assume G is $\mathbb{P}_{\bar{E},\epsilon}$ -generic over V . Let $\langle \kappa_\xi : \xi < \mu \rangle$ be an increasing enumeration of $C^{\bar{\kappa}}$, and for each $\xi < \mu$, let λ_ξ be the least inaccessible above κ_ξ . Then*

- (1) *A cardinal $\eta < \kappa$ is collapsed in $V[G]$ iff there exists a limit ordinal $\xi < \mu$, such that $\eta \in (\kappa_\xi, \lambda_\xi)$, and then η is collapsed to κ_ξ ,*
- (2) *For each limit $\xi < \mu$, $(\kappa_\xi^+)^{V[G]} = \lambda_\xi$.*

4.2. Projected supercompact extender based Radin forcing. Here we define our projected forcing $\mathbb{P}_{\bar{E},\epsilon}^\pi$. The forcing construction is very similar to that of $\mathbb{P}_{\bar{E},\epsilon}$, and so we just list the main changes which are required to define $\mathbb{P}_{\bar{E},\epsilon}^\pi$. We add the superscript π to denote it is related to the projected forcing.

Definition 4.30. $\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$ consists of all functions $f : d \rightarrow {}^{<\omega}\mathfrak{R}$ such that

- (1) $\bar{\kappa} \in d \in P_\lambda(\mathfrak{D})$,
- (2) $f(\bar{\kappa}) = \langle f_0(\bar{\kappa}), \dots, f_{k-1}(\bar{\kappa}) \rangle$ is an increasing sequence in \mathfrak{R} ,
- (3) $\forall \bar{\kappa} \neq \bar{\alpha} \in d, f(\bar{\alpha}) = \langle \rangle$ (here is the place we use the extra assumption $\langle \rangle \in \mathfrak{R}$),
- (4) For each $i < |f(\bar{\kappa})|$, $(o(f_i(\bar{\kappa})) < o(\bar{\kappa}))$,
- (5) The sequence $\langle o(f_i(\bar{\kappa})) : i < |f(\bar{\kappa})| \rangle$ is non-increasing.

Definition 4.31. For $f, g \in \mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$, we say f is an extension of g ($f \leq_{\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}}^* g$) if $f \supseteq g$.

Remark 4.32. $\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$ is forcing isomorphic to the trivial forcing.

Lemma 4.33. $\langle \mathbb{P}_{\bar{E},\epsilon}^{*,\pi}, \leq_{\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}}^* \rangle$ is κ^+ -c.c.

Proof. The result follows from the fact that for $f, g \in \mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$, if $f(\bar{\kappa}) = g(\bar{\kappa})$, then f and g are compatible, and that there are only κ possible choices for $f(\bar{\kappa})$, as each $f(\bar{\kappa}) \in V_\kappa^{<\omega}$. \square

Definition 4.34. Assume $f \in \mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$ and $\nu \in OB(f)$. Define $g = f_{\langle \nu \rangle}$ to be of the form $g = g_{\leftarrow} \frown g_{\rightarrow}$ (the case $g_{\leftarrow} = \emptyset$ is allowed) where

- (1) $\text{dom}(g_{\rightarrow}) = \text{dom}(f)$,

(2) For each $\bar{\alpha} \in \text{dom}(g_{\rightarrow})$

$$g_{\rightarrow}(\bar{\alpha}) = \begin{cases} f(\bar{\alpha}) \upharpoonright k \frown \langle \nu(\bar{\alpha}) \rangle & \text{if } \bar{\alpha} = \bar{\kappa}, \nu(\bar{\alpha}) > f_{|f(\bar{\alpha})|-1}(\bar{\alpha}), \\ \langle \rangle & \text{Otherwise.} \end{cases}$$

where

$$k = \min\{l \leq |f(\bar{\kappa})| : \forall l \leq i < |f(\bar{\kappa})|, o(f_i(\bar{\kappa})) < o(\nu(\bar{\kappa}))\}.$$

The above value of k is defined so as to ensure that $\langle o(f_i(\bar{\kappa})) : i < k \rangle \frown \langle o(\nu(\bar{\kappa})) \rangle$ is non-increasing.

(3) $\text{dom}(g_{\leftarrow}) = \{\nu(\bar{\alpha}) : \bar{\alpha} \in \text{dom}(\nu), o(\nu(\bar{\alpha})) > 0\}$,

(4) For each $\bar{\alpha} \in \text{dom}(\nu)$ with $o(\nu(\bar{\alpha})) > 0$ we have

$$g_{\leftarrow}(\nu(\bar{\alpha})) = f(\bar{\alpha}) \upharpoonright (|f(\bar{\alpha})| \setminus k),$$

where k is defined as above.

Definition 4.35. $\mathbb{P}_{\bar{E}, \epsilon, \rightarrow}^{\pi}$ consists of pairs $p = \langle f, A \rangle$ where

- (1) $f \in \mathbb{P}_{\bar{E}, \epsilon}^{*, \pi}$,
- (2) A is an f -tree such that for each $\langle \nu \rangle \in A$

$$f_{|f(\bar{\kappa})|-1}(\bar{\kappa}) < \nu(\bar{\kappa})$$

Definition 4.36. Let $p, q \in \mathbb{P}_{\bar{E}, \epsilon, \rightarrow}^{\pi}$. We say $p \leq_{\mathbb{P}_{\bar{E}, \epsilon, \rightarrow}^{*, \pi}} q$ (p is a Prikry extension of q) if

- (1) $f^p \leq_{\mathbb{P}_{\bar{E}, \epsilon}^{*, \pi}} f^q$,
- (2) $A^p \upharpoonright \text{dom}(f^q) \subseteq A^q$.

We are now ready to define the forcing notion $\mathbb{P}_{\bar{E}, \epsilon}^{\pi}$.

Definition 4.37. A condition p in the forcing notion $\mathbb{P}_{\bar{E}, \epsilon}^{\pi}$ is of the form $p_{\leftarrow} \frown p_{\rightarrow}$ where

- (1) $p_{\rightarrow} \in \mathbb{P}_{\bar{E}, \epsilon, \rightarrow}^{\pi}$,
- (2) $p_{\leftarrow} \in \prod_{i < n} \mathbb{P}_{\bar{e}_i}^{\pi}$ ($n < \omega$), where \bar{e}_i are extender sequences such that
 - (2-1) $o(\bar{e}_i) \leq o(\bar{E})$,
 - (2-2) $\bar{e}_i \in V_{\text{crit}(\bar{e}_{i+1})}$,
 - (2-3) $\langle \nu \rangle \in A^{p_{\leftarrow}} \Rightarrow \nu(\bar{\kappa})_0 > \text{crit}(\bar{e}_{n-1})$.

Definition 4.38. Let $p, q \in \mathbb{P}_{\bar{E}, \epsilon}^{\pi}$. Then $p \leq_{\mathbb{P}_{\bar{E}, \epsilon}^{*, \pi}} q$ (p is a Prikry extension of q) if:

- (1) $p_{\rightarrow} \leq_{\mathbb{P}_{\bar{E}, \epsilon, \rightarrow}^{*, \pi}} q_{\rightarrow}$,

$$(2) p \leftarrow \leq^{*,\pi} q \leftarrow.$$

Definition 4.39. Let $p, q \in \mathbb{P}_{\bar{E},\epsilon}^\pi$. Then $p \leq_{\mathbb{P}_{\bar{E},\epsilon}^\pi} q$ (p is stronger than q) if $p = r \cap s$ and there is $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{q \leftarrow}$ such that

- (1) $s \leq_{\mathbb{P}_{\bar{E},\epsilon}^\pi}^{*,\pi} q \rightarrow \langle \nu_0, \dots, \nu_{n-1} \rangle$,
- (2) $r \leq q \leftarrow$.

In the next subsection, we produce a projection $\pi : \mathbb{P}_{\bar{E},\epsilon} \rightarrow \mathbb{P}_{\bar{E},\epsilon}^\pi$ which is of Prikry type, and we will use it in subsection 4.4, to prove the main properties of the projected forcing.

4.3. Projecting $\mathbb{P}_{\bar{E},\epsilon}$ to $\mathbb{P}_{\bar{E},\epsilon}^\pi$. We now produce a Prikry type projection $\pi : \mathbb{P}_{\bar{E},\epsilon} \rightarrow \mathbb{P}_{\bar{E},\epsilon}^\pi$ from $\mathbb{P}_{\bar{E},\epsilon}$ into $\mathbb{P}_{\bar{E},\epsilon}^\pi$. We do it in a few steps, by producing projection between earlier forcing notions and their corresponding projected versions.

We start with defining a projection from $\mathbb{P}_{\bar{E},\epsilon}^*$ to $\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$. Let $f \in \mathbb{P}_{\bar{E},\epsilon}^*$, $f : d \rightarrow \mathfrak{A}$. Let $\pi^*(f)$ be a function with the same domain as f , such that

$$\pi^*(f)(\bar{\alpha}) = \begin{cases} f(\bar{\alpha}) & \text{if } \bar{\alpha} = \bar{\kappa}, \\ \langle \rangle & \text{Otherwise.} \end{cases}$$

It is clear that $\pi^*(f) \in \mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$, and hence

$$\pi^* : \mathbb{P}_{\bar{E},\epsilon}^* \rightarrow \mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$$

is well-defined.

Lemma 4.40. π^* is a projection from $\mathbb{P}_{\bar{E},\epsilon}^*$ to $\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$.

Proof. Let $f \in \mathbb{P}_{\bar{E},\epsilon}^*$, $g \in \mathbb{P}_{\bar{E},\epsilon}^{*,\pi}$ and $g \leq_{\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}}^{*,\pi} \pi^*(f)$, which means $g \supseteq \pi^*(f)$. Let $f^* \in \mathbb{P}_{\bar{E},\epsilon}^*$ have the same domain as $\text{dom}(g)$, so that:

$$f^*(\bar{\alpha}) = \begin{cases} g(\bar{\alpha}) & \text{if } \bar{\alpha} = \bar{\kappa}, \\ f(\bar{\alpha}) & \text{if } \bar{\alpha} \neq \bar{\kappa}, \bar{\alpha} \in \text{dom}(f), \\ \langle \rangle & \text{Otherwise.} \end{cases}$$

Then $f^* \in \mathbb{P}_{\bar{E},\epsilon}^*$, $f^* \leq_{\mathbb{P}_{\bar{E},\epsilon}^*}^* f$ and $\pi^*(f^*) \leq_{\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}}^{*,\pi} g$. \square

We now produce a projection from $\mathbb{P}_{\bar{E},\epsilon,\rightarrow}$ into $\mathbb{P}_{\bar{E},\epsilon,\rightarrow}^\pi$. Let $p = \langle f, A \rangle \in \mathbb{P}_{\bar{E},\epsilon,\rightarrow}$, and define $\pi_{\rightarrow}(\langle f, A \rangle)$ to be the pair $\langle \pi^*(f), A \rangle$. Clearly $\pi_{\rightarrow}(\langle f, A \rangle) \in \mathbb{P}_{\bar{E},\epsilon,\rightarrow}^\pi$, and so

$$\pi_{\rightarrow} : \mathbb{P}_{\bar{E},\epsilon,\rightarrow} \rightarrow \mathbb{P}_{\bar{E},\epsilon,\rightarrow}^\pi$$

is well-defined.

Lemma 4.41. π_{\rightarrow} is a projection from $\langle \mathbb{P}_{\bar{E},\epsilon,\rightarrow}, \leq_{\mathbb{P}_{\bar{E},\epsilon,\rightarrow}}^* \rangle$ into $\langle \mathbb{P}_{\bar{E},\epsilon,\rightarrow}^{\pi}, \leq_{\mathbb{P}_{\bar{E},\epsilon,\rightarrow}^{\pi}}^{*,\pi} \rangle$.

Proof. Let $p = \langle f, A \rangle \in \mathbb{P}_{\bar{E},\epsilon,\rightarrow}, q = \langle g, B \rangle \in \mathbb{P}_{\bar{E},\epsilon,\rightarrow}^{\pi}$ and $q \leq_{\mathbb{P}_{\bar{E},\epsilon,\rightarrow}^{\pi}}^{*,\pi} \pi_{\rightarrow}(p) = \langle \pi^*(f), A \rangle$. This means

- (1) $g \leq_{\mathbb{P}_{\bar{E},\epsilon}^{*,\pi}}^{*,\pi} \pi^*(f)$,
- (2) $B \upharpoonright \text{dom}(\pi^*(f)) \subseteq A$.

By Lemma 4.45, there exists $f^* \in \mathbb{P}_{\bar{E},\epsilon}^*$ with $f^* \leq_{\mathbb{P}_{\bar{E},\epsilon}^*}^*$ and $\pi(f^*) \leq_{\mathbb{P}_{\bar{E},\epsilon}^{\pi}}^{*,\pi} g$. Let $p^* = \langle f^*, B \rangle \in \mathbb{P}_{\bar{E},\epsilon,\rightarrow}$ and $\pi_{\rightarrow}(p^*) \leq_{\mathbb{P}_{\bar{E},\epsilon,\rightarrow}^{\pi}}^{*,\pi} q$. \square

Finally we present a projection between our main forcing notions, namely from the forcing notion $\mathbb{P}_{\bar{E},\epsilon}$ into $\mathbb{P}_{\bar{E},\epsilon}^{\pi}$, further we show that it is in fact of Prikry type.

Theorem 4.42. *There is $\pi : \mathbb{P}_{\bar{E},\epsilon} \rightarrow \mathbb{P}_{\bar{E},\epsilon}^{\pi}$ which is a Prikry type projection.*

Proof. We define the projection by recursion on the critical point of the extenders. The base case was dealt in section 3. Now assume that for each extender sequence $\bar{e} \in V_{\kappa}$, we have a projection $\pi_{\bar{e}} : \mathbb{P}_{\bar{e}} \rightarrow \mathbb{P}_{\bar{e}}^{\pi}$, and we define a projection from $\mathbb{P}_{\bar{E},\epsilon}$ into $\mathbb{P}_{\bar{E},\epsilon}^{\pi}$ (where κ is the critical point of extenders in \bar{E}).

So let $p \in \mathbb{P}_{\bar{E},\epsilon}$. Then p has the form $p_{\leftarrow} \frown p_{\rightarrow}$ where

- (1) $p_{\rightarrow} \in \mathbb{P}_{\bar{E},\epsilon,\rightarrow}$,
- (2) $p_{\leftarrow} \in \prod_{i < n} \mathbb{P}_{\bar{e}_i}$ ($n < \omega$), where \bar{e}_i are extender sequences such that
 - (2-1) $o(\bar{e}_i) \leq o(\bar{E})$,
 - (2-2) $\bar{e}_i \in V_{\text{crit}(\bar{e}_{i+1})}$,
 - (2-3) $\langle \nu \rangle \in A^{p_{\rightarrow}} \Rightarrow \nu(\bar{\kappa})_0 > \text{crit}(\bar{e}_{n-1})$.

Let $p_{\leftarrow} = \langle p_{\bar{e}_i} : i < n \rangle \in \prod_{i < n} \mathbb{P}_{\bar{e}_i}$, and set

$$\pi(p) = \langle \pi_{\bar{e}_i}(p_{\bar{e}_i}) : i < n \rangle \frown \pi_{\rightarrow}(p_{\rightarrow}).$$

It is easily seen that π is as required. \square

4.4. More on $\mathbb{P}_{\bar{E},\epsilon}^{\pi}$. We use the projection π above to prove some properties of the forcing notion $\mathbb{P}_{\bar{E},\epsilon}^{\pi}$. The next lemma can be proved as in Lemma 4.23, using Lemma 4.33 (instead of Lemma 4.11).

Lemma 4.43. $\mathbb{P}_{\bar{E},\epsilon}^\pi$ satisfies the κ^+ - c.c.

Using the projection π and the results of Section 2, we can conclude that

Theorem 4.44. $\mathbb{P}_{\bar{E},\epsilon}^\pi$ satisfies the Prikry property.

We also have the following analogue of Lemma 4.24

Lemma 4.45. (Factorization lemma for projected forcing) For any $p \in \mathbb{P}_{\bar{E},\epsilon}^\pi$,

$$\mathbb{P}_{\bar{E},\epsilon}^\pi \downarrow p \simeq \mathbb{P}_{\bar{E},\epsilon}^\pi \downarrow p_{\leftarrow} \times \mathbb{P}_{\bar{E},\epsilon}^\pi \downarrow p_{\rightarrow}.$$

Let H be $\mathbb{P}_{\bar{E},\epsilon}^\pi$ -generic over V , and let

$$H^{\bar{\kappa}} = \bigcup \{f_{\bar{\kappa}}^p : p \in H, \bar{\kappa} \in \text{dom}(f_{\bar{\kappa}}^p)\}$$

and

$$D^{\bar{\kappa}} = \bigcup \{\bar{v}_0 : \bar{v} \in G^{\bar{\kappa}}\}.$$

Then as in Lemma 4.27, $D^{\bar{\kappa}}$ can be proved to be a club of κ . Also by the same arguments as in the last subsection, we can prove the following.

Theorem 4.46. $\mathbb{P}_{\bar{E},\epsilon}^\pi$ preserves all cardinals.

4.5. Homogeneity properties. Let G be $\mathbb{P}_{\bar{E},\epsilon}$ -generic over V , and let G^π be the filter generated by $\pi[G]$. It follows that G^π is $\mathbb{P}_{\bar{E},\epsilon}^\pi$ -generic over V and that $V \subseteq V[G^\pi] \subseteq V[G]$. By standard forcing theorems, $V[G]$ is itself a forcing extension of $V[G^\pi]$. We show that this forcing has enough homogeneity properties, which guarantees that $HOD^{V[G]} \subseteq V[G^\pi]$. The homogeneity of our quotient forcing follows from the next theorem.

Lemma 4.47. (Homogeneity lemma) For all $p, q \in \mathbb{P}_{\bar{E},\epsilon}$, if $\pi(p) = \pi(q)$, then

$$\mathbb{P}_{\bar{E},\epsilon} \downarrow p \simeq \mathbb{P}_{\bar{E},\epsilon} \downarrow q.$$

In particular, if $p \Vdash \text{“}\phi(\alpha, \vec{\gamma})\text{”}$, where $\alpha, \vec{\gamma}$ are ordinals, then it is not the case that $q \Vdash \text{“}\neg\phi(\alpha, \vec{\gamma})\text{”}$.

Proof. To prove the lemma, we first state and prove an analogous result for forcing notions $\mathbb{P}_{\bar{E},\epsilon}^*$ and $\mathbb{P}_{\bar{E},\epsilon,\rightarrow}^*$, and then use these results to prove the lemma by recursion.

- **(Homogeneity lemma for $\mathbb{P}_{\bar{E},\epsilon}^*$):** For all $f, g \in \mathbb{P}_{\bar{E},\epsilon}^*$, if $\pi^*(f) = \pi^*(g)$, then

$$\mathbb{P}_{\bar{E},\epsilon}^* \downarrow f \simeq \mathbb{P}_{\bar{E},\epsilon}^* \downarrow g.$$

Proof. As $\pi^*(f) = \pi^*(g)$, we have $\text{dom}(f) = \text{dom}(g) = \Delta$, and $f(\bar{\kappa}) = g(\bar{\kappa})$. Define

$$\Phi^* : \mathbb{P}_{\bar{E}, \epsilon}^* \downarrow f \rightarrow \mathbb{P}_{\bar{E}, \epsilon}^* \downarrow g$$

as follows: let $f^* \leq_{\mathbb{P}_{\bar{E}, \epsilon}^*}^* f$. Let $\Phi^*(f^*) = g \cup f^* \upharpoonright (\text{dom}(f^*) \setminus \Delta)$. Clearly $\Phi^*(f^*) \leq_{\mathbb{P}_{\bar{E}, \epsilon}^*}^* g$, and so Φ^* is well-defined. It is clearly an isomorphism. \square

- **(Homogeneity lemma for $\mathbb{P}_{\bar{E}, \epsilon, \rightarrow}$):** For all $p, q \in \mathbb{P}_{\bar{E}, \epsilon, \rightarrow}$, if $\pi_{\rightarrow}(p) = \pi_{\rightarrow}(q)$, then

$$\mathbb{P}_{\bar{E}, \epsilon, \rightarrow} \downarrow p \simeq \mathbb{P}_{\bar{E}, \epsilon, \rightarrow} \downarrow q.$$

Proof. Let $p = \langle f, A \rangle$ and $q = \langle g, B \rangle$, and assume that $\pi_{\rightarrow}(p) = \langle \pi^*(f), A \rangle = \langle \pi^*(g), B \rangle = \pi_{\rightarrow}(q)$. Define

$$\Phi_{\rightarrow} : \mathbb{P}_{\bar{E}, \epsilon, \rightarrow} \downarrow p \rightarrow \mathbb{P}_{\bar{E}, \epsilon, \rightarrow} \downarrow q$$

as follows: $\Phi_{\rightarrow}(\langle f^*, A^* \rangle) = \langle \Phi^*(f^*), A^* \rangle$. Φ_{\rightarrow} is easily seen to be well-defined and an isomorphism. \square

We are now ready to complete the proof of the homogeneity lemma 4.47. We produce the isomorphism by recursion on the critical point of the extenders. The base case was dealt in section 3. Now assume that for each extender sequence $\bar{e} \in V_{\kappa}$, and each $p, q \in \mathbb{P}_{\bar{e}}$, if $\pi_{\bar{e}}(p) = \pi_{\bar{e}}(q)$, then we have an isomorphism

$$\Phi_{\bar{e}} : \mathbb{P}_{\bar{e}} \downarrow p \simeq \mathbb{P}_{\bar{e}} \downarrow q,$$

and we define an isomorphism

$$\Phi : \mathbb{P}_{\bar{E}, \epsilon} \downarrow p \simeq \mathbb{P}_{\bar{E}, \epsilon} \downarrow q,$$

where $p, q \in \mathbb{P}_{\bar{E}, \epsilon}$ are such that $\pi(p) = \pi(q)$.

So let $p^* \in \mathbb{P}_{\bar{E}, \epsilon}, p^* \leq_{\mathbb{P}_{\bar{E}, \epsilon}}^* p$. Then p^* has the form $p_{\leftarrow}^* \hat{\smallfrown} p_{\rightarrow}^*$, where

- (1) $p_{\rightarrow}^* \in \mathbb{P}_{\bar{E}, \epsilon, \rightarrow}$,
- (2) $p_{\leftarrow}^* \in \prod_{i < n} \mathbb{P}_{\bar{e}_i}$ ($n < \omega$), where \bar{e}_i are extender sequences such that
 - (2-1) $o(\bar{e}_i) \leq o(\bar{E})$,
 - (2-2) $\bar{e}_i \in V_{\text{crit}(\bar{e}_{i+1})}$,
 - (2-3) $\langle \nu \rangle \in A^{p_{\leftarrow}^*} \Rightarrow \nu(\bar{\kappa})_0 > \text{crit}(\bar{e}_{n-1})$.

Let $p_{\leftarrow}^* = \langle p_{\bar{e}_i}^* : i < n \rangle \in \prod_{i < n} \mathbb{P}_{\bar{e}_i}$, and define

$$\Phi(p^*) = \langle \Phi_{\bar{e}_i}(p_{\bar{e}_i}^*) : i < n \rangle \frown \Phi_{\rightarrow}(p_{\rightarrow}^*).$$

Φ is easily seen to be as required. \square

4.6. Completing the proof of Theorem 1.3. We are finally ready to complete the proof of Theorem 1.3. As it was stated at the beginning of this section, it suffices to prove Theorem 4.1. Thus fix a Mitchell increasing sequence $\bar{E} = \langle E_\xi : \xi < o(\bar{E}) \rangle$ of extenders on κ as in Subsection 4.1, where $cf(o(\bar{E})) > \lambda$, and consider the forcing notions $\mathbb{P}_{\bar{E}} = \mathbb{P}_{\bar{E},\lambda}$ and $\mathbb{P}_{\bar{E}}^\pi = \mathbb{P}_{\bar{E},\lambda}^\pi$ (thus we are assuming $\epsilon = \lambda$). By Theorem 4.42, there exists a projection $\pi : \mathbb{P}_{\bar{E}} \rightarrow \mathbb{P}_{\bar{E}}^\pi$. Let G be $\mathbb{P}_{\bar{E}}$ -generic over V and let G^π be the filter generated by $\pi[G]$. We know that G^π is $\mathbb{P}_{\bar{E}}^\pi$ -generic over V and $V \subseteq V[G^\pi] \subseteq V[G]$. Note that the clubs $C^{\bar{\kappa}}$ (added by G) and $D^{\bar{\kappa}}$ (added by G^π) are the same. Let $\langle \kappa_\xi : \xi < \kappa \rangle$ be an increasing enumeration of them. For each limit ordinal $\xi < \kappa$

- (1) In $V[G]$, $(\kappa_\xi)^+ = \lambda_\xi$, where λ_ξ is the least inaccessible above κ_ξ ,
- (2) In $V[G^\pi]$, all cardinals are preserved and each λ_ξ is an inaccessible cardinal,
- (3) $HOG^{V[G]} \subseteq V[G^\pi]$.

It follows that for each limit ordinal $\xi < \kappa$, $(\kappa_\xi^+)^{V[G]}$ is an inaccessible cardinal in $HOG^{V[G]}$.

The result follows immediately.

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