ALMOST SOUSLIN KUREPA TREES

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Abstract. We show that the existence of an almost Souslin Kurepa tree is consistent with ZFC. We also prove their existence in L. These results answer two questions from [16].

1. Introduction

The theory of trees forms a significant and highly interesting part of set theory. In this paper we study ω_1-trees and prove some consistency results concerning them. Let T be a normal ω_1- tree. Let's recall that:

- T is an Aronszajn tree if it has no branches,
- T is a Kurepa tree if it has at least ω_2-many branches,
- T is a Souslin tree if it has no uncountable antichains (and hence no branches),
- T is an almost Souslin tree if for any antichain $X \subseteq T$, the set $S_X = \{ht(x) : x \in X\}$ is not stationary (see [1], [16]),
- T is regressive if for any limit ordinal $\alpha < \omega_1$, there is a function $f : T_\alpha \to T_{<\alpha}$ such that for any $x \in T_\alpha, f(x) <_T x$, and for any $x \neq y$ in T_α, at least one of $f(x)$ or $f(y)$ is above the meet of x and y (see [8]).

Intuitively a Kurepa tree is very thick. On the other hand a Souslin tree is very thin, and obviously no Kurepa tree is a Souslin tree. We can think of an almost Souslin tree as a fairly thin tree. The following are well known:

- There is an Aronszajn tree (Aronszajn, see [3] for proof),
- It is consistent with ZFC that a Souslin tree exists (Jech [2], Tennenbaum [13]),
- $V = L$ implies the existence of a Souslin tree (Jensen [5], see also [6]),

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- It is consistent with $ZFC + \neg CH$ to assume there is no Souslin tree (Solovay and Tennenbaum [11]),
- It is consistent with $ZFC + GCH$ to assume there is no Souslin tree (Jensen [7], see [9] for proof),
- It is consistent with ZFC that a Kurepa tree exists (Stewart [12], see [3] for proof),
- $V = L$ implies the existence of a Kurepa tree (Solovay, see [3] for proof),
- It is consistent, relative to the existence of an inaccessible cardinal, that there is no Kurepa tree (Silver [10]).

For more details on trees, we refer the readers to the articles [3] and [14]. The following example shows that almost Souslin trees exist in ZFC.

Example 1.1. *Let $T = \{t \in {}^{<\omega_1}2 : Supp(t) \text{ is finite}\}$. Then it is easily seen that T is an almost Souslin tree with ω_1-many branches and with no Aronszajn subtrees (See also [14] Theorem 4.1).*

Now, in [16], Zakrzewski asked the following questions:

Question 1.2. *Is the existence of an almost Souslin Kurepa tree consistent with ZFC?*

Question 1.3. *Does the axiom of constructibility guarantee the existence of an almost Souslin Kurepa tree?*

In this paper, we answer both of these questions positively. In section 2 we answer question 1.2 by building a model of ZFC in which an almost Souslin Kurepa tree exists, and in section 3 we answer question 1.3 by showing that the existence of an $(\omega_1, 1)-$morass implies the existence of an almost Souslin Kurepa tree.

2. Forcing an almost Souslin Kurepa tree

In this section we answer question 1.2. In fact we will prove something stronger, which also extends some results from [8].

Theorem 2.1. *Assume GCH. Then there exists a cardinal preserving generic extension of V in which an almost Souslin regressive Kurepa tree exists.*

Proof. Let $\kappa \geq \omega_2$. We produce a cardinal preserving generic extension of V which contains an almost Souslin regressive Kurepa tree with $\kappa-$many branches. First we define a forcing notion $\mathbb{P}$ which adds a regressive Kurepa tree with $\kappa-$many branches. This forcing is essentially the forcing notion of [8]. Conditions in $\mathbb{P}$ are of the form $p = \langle T_p, \leq_p, g_p, f_p \rangle$ where:

(1) $T_p \subseteq \omega_1$ is countable,

(2) $\langle T_p, \leq_p \rangle$ is a normal $\alpha_p + 1-$tree, where α_p is an ordinal less than ω_1,

(3) g_p is a bijection from a subset of κ onto $(T_p)_{\alpha_p}$, the α_p-th level of T_p,

(4) $f_p : T_{p,lim} \to T_p$, where $T_{p,lim} = \{x \in T_p : ht(x)$ is a limit ordinal $\}$,

(5) For all $x \in T_{p,lim}, f_p(x) <_p x$,

(6) For each $x \neq y$ in $T_{p,lim}$, if $ht(x) = ht(y)$, then at least one of $f_p(x)$ or $f_p(y)$ is above the meet of x and y,

The order relation on $\mathbb{P}$ is defined by $p \leq q$ (p is an extension of q) iff:

(1) $\langle T_p, \leq_p \rangle$ end extends $\langle T_q, \leq_q \rangle$,

(2) $dom g_p \supseteq dom g_q$,

(3) for all $\alpha \in dom g_q, g_p(\alpha) \geq_p g_q(\alpha)$,

(4) $f_p \supseteq f_q$.

The following lemma can be proved easily (see also [8] Theorem 5).

Lemma 2.2. (a) *Let $p \in \mathbb{P}$ and $\alpha < \kappa$. Then there exists $q \leq p$ such that $\alpha \in dom g_q$. Furthermore q can be chosen so that $\alpha_q = \alpha_p + 1$, and $dom g_q = dom g_p \cup \{\alpha\}$.*

(b) *Let $\langle p_n : n < \omega \rangle$ be a descending sequence of conditions in $\mathbb{P}$. Then there exists $q \in \mathbb{P}$ which extends all of the p_n's. Furthermore q can be chosen so that $\alpha_q = sup_{n\in\omega}\alpha_{p_n}$, and $dom(g_q) = \bigcup_{n<\omega} dom(g_{p_n})$.*

(c) *$\mathbb{P}$ satisfies the $\omega_2 - c.c.$.* □

It follows from the above lemma that $\mathbb{P}$ is a cardinal preserving forcing notion. Let G be $\mathbb{P}-$generic over V. Let

- $T = \bigcup_{p\in G} T_p$,
- $\leq_T = \bigcup_{p\in G} \leq_p$,
- $f = \bigcup_{p\in G} f_p : T_{lim} \to T$, where $T_{lim} = \{x \in T : ht(x)$ is a limit ordinal$\}$.

It is easy to show that $\langle T, \leq_T\rangle$ is a normal regressive ω_1-tree.

Lemma 2.3. *$\langle T, \leq_T\rangle$ has $\kappa-$many branches, in particular it is a Kurepa tree.*

Proof. The lemma follows easily from the following facts:

(1) For each $\xi < \kappa, \{g_p(\xi) : p \in G, \xi \in dom(g_p)\}$ determines a branch b_ξ of T.

(2) For $\xi \neq \zeta$ in $\kappa, b_\xi \neq b_\zeta$. □

Let $S = \{\alpha_p : p \in G, domg_p = \bigcup\{domg_q : q \in G, \alpha_q < \alpha_p\}\}$.

Lemma 2.4. *S is a stationary subset of ω_1.*

Proof. Let $\dot{S}$ be a $\mathbb{P}-$name for S. Let $p \in \mathbb{P}$ and $\dot{C}$ be a $\mathbb{P}-$name such that

$$p \Vdash \ulcorner \dot{C} \text{ is a club subset of } \omega_1 \urcorner.$$

We find $q \leq p$ which forces $\dot{S} \cap \dot{C} \neq \emptyset$. Define by induction two sequences $\langle p_n : n < \omega\rangle$ of conditions in $\mathbb{P}$ and $\langle \beta_n : n < \omega\rangle$ of countable ordinals such that:

- $p_0 = p$,
- $p_{n+1} \leq p_n$,
- $\alpha_{p_n} < \beta_n < \alpha_{p_{n+1}}$,
- $p_{n+1} \Vdash \ulcorner \beta_n \in \dot{C} \urcorner$

By Lemma 2.2(b), there is $q \in \mathbb{P}$ such that q extends p_n's, $n < \omega$, and such that $\alpha_q = sup_{n \in \omega} \alpha_{p_n}$, and $dom(g_q) = \bigcup_{n<\omega} dom(g_{p_n})$. Then it is easily seen that $q \Vdash \ulcorner \alpha_q \in \dot{S} \cap \dot{C} \urcorner$.□

Working in $V[G]$, let $\mathbb{Q}$ be the usual forcing notion for adding a club subset to S. Thus conditions in $\mathbb{Q}$ are closed bounded subsets of S ordered by end extension. Let H be $\mathbb{Q}-$generic over $V[G]$. The following is well-known (see [4] Theorem 23.8).

Lemma 2.5. *(a) $\mathbb{Q}$ is ω_1-distributive,*

(b) $\mathbb{Q}$ satisfies the $\omega_2 -$ c.c.,

(c) $C = \bigcup H \subseteq S$ is a club subset of ω_1. □

It follows that $\mathbb{Q}$ is a cardinal preserving forcing notion and hence $\langle T, \leq_T\rangle$ remains a regressive Kurepa tree with $\kappa-$many branches in $V[G][H]$. We show that in $V[G][H], \langle T, \leq_T\rangle$ is also almost Souslin.

Lemma 2.6. *In* $V[G][H]$, $\langle T, \leq_T\rangle$ *is almost Souslin.*

Proof. Suppose not. Let $Z \subseteq T$ be an antichain of T such that $S_Z = \{ht(x) : x \in Z\}$ is stationary in ω_1. We may further suppose that for $x \neq y$ in $Z, ht(x) \neq ht(y)$, and that $S_Z \subseteq C$.

First we define a map h on $T \restriction C = \{x \in T : ht(x) \in C\}$ as follows: Let $\alpha \in C$ and $x \in T_\alpha$. Pick $p \in G$ such that $\alpha = \alpha_p$. Then $x = g_p(\xi)$, for some $\xi \in domg_p$. Let $h(x) = g_q(\xi)$, where $q \in G$ is such that α_q is minimal with $\xi \in domg_q$. Note that $\alpha_q < \alpha_p$, and $h(x) <_T x$ (as $C \subseteq S$).

The map $ht(x) \mapsto ht(h(x))$ is well-defined and regressive on S_Z, hence by Fodor's lemma there is $Y \subseteq Z$ and an ordinal $\gamma < \omega_1$ such that S_Y is stationary and for all $x \in Y, ht(h(x)) = \gamma$. Since T_γ is countable, we can find $X \subseteq Y$, and $t \in T$ such that S_X is stationary, and for all $x \in X, h(x) = t$. Now

$$\forall x \in X, \exists p_x \in G, \exists \xi_x \in dom(g_{p_x})(x = g_{p_x}(\xi_x)).$$

and then for all $x \in X, h(x) = g_{q_x}(\xi_x)$, where $q_x \in G$ is such that α_{q_x} is minimal with $\xi_x \in domg_{q_x}$. The map $\alpha_{p_x} \mapsto \alpha_{q_x}$ is regressive on S_X, and hence we can find $W \subseteq X$, and $\eta < \omega_1$ such that S_W is stationary and for all $x \in W, \alpha_{q_x} = \eta$. Let $q \in G$ be such that $\alpha_q = \eta$ Then for all $x \in W, h(x) = g_q(\xi_x)$. As $domg_q$ is countable, there are $V \subseteq W$ and $\xi \in domg_q$ such that S_V is stationary and for all $x \in V, \xi_x = \xi$. Then for all $x \in V, x = g_{p_x}(\xi)$. Choose $x \neq y$ in V, and let $p \in G$ be such that $p \leq p_x, p_y$. Then $g_p(\xi) \geq_P x = g_{p_x}(\xi), y = g_{p_y}(\xi)$. It follows that x and y are compatible and we get a contradiction. The lemma follows. $\square$

Thus in $V[G][H]$, $\langle T, \leq_T\rangle$ is an almost Souslin regressive Kurepa tree with $\kappa-$many branches. This completes the proof of Theorem 2.1. $\square$

Remark 2.7. *As we will see in the next section, just working in* $V[G]$, *it is possible to define a subtree* T^* *of* T *which is an almost Souslin regressive Kurepa tree with* $\kappa-$*many branches. We gave the above argument for Theorem 2.1, since it was our original motivation for defining* T^*.

3. Almost Souslin Kurepa trees in L

In this section, answering question 2.2, we show that an almost Souslin Kurepa tree exists in L. Again as in section 2, we prove something stronger.

Theorem 3.1. *If there exists an $(\omega_1, 1)-$morass, then there is an almost Souslin regressive Kurepa tree.*

To prove the above theorem, we need some definitions and facts from [15]. Let $(\mathbb{P}, \leq)$ be a partial order and $\mathbb{D} = \{D_\alpha : \alpha < \omega_2\}$ be a family of open dense subsets of $\mathbb{P}$. For $p \in \mathbb{P}$, let $rlm(p) = \{\alpha < \omega_2 : p \in D_\alpha\}$, and for $\alpha < \omega_2$ let $\mathbb{P}_\alpha = \{p \in \mathbb{P} : rlm(p) \subseteq \alpha\}$. Also let $\mathbb{P}^* = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$.

Definition 3.2. *$\mathbb{D}$ is an ω_1-indiscernible family if the following conditions are satisfied:*

(1) $\mathbb{P}^* \neq \emptyset$, and for all $\alpha < \omega_1, \mathbb{P}^* \cap D_\alpha$ is open dense in $\mathbb{P}^*$,

(2) For all $\alpha < \omega_1, (\mathbb{P}_\alpha, \leq)$ is ω_1-closed,

Also for each order preserving function $f : \alpha \to \gamma, \alpha < \omega_1, \gamma < \omega_2$ there is a function $\sigma_f : \mathbb{P}_\alpha \to \mathbb{P}_\gamma$ such that

(3) σ_f is order preserving,

(4) For all $p \in \mathbb{P}_\alpha, rlm(\sigma_f(p)) = f[rlm(p)]$,

(5) If $\beta < \omega_1, f \restriction \beta = id \restriction \beta, f(\beta) \geq \alpha, \gamma < \omega_1$ and $p \in \mathbb{P}_\alpha$, then p and $\sigma_f(p)$ are compatible in $\mathbb{P}^*$,

(6) If $f_1 : \alpha_1 \to \alpha_2, f_2 : \alpha_2 \to \gamma$ are order preserving, $\alpha_1, \alpha_2 < \omega_1, \gamma < \omega_2$, then $\sigma_{f_2 \circ f_1} = \sigma_{f_2} \circ \sigma_{f_1}$.

We also need the following theorem (See [15] Theorem 1.1.3).

Theorem 3.3. *The following are equivalent:*

(a) There exists an $(\omega_1, 1)-$morass,

(b) Whenever $\mathbb{P}$ is a partial order and $\mathbb{D}$ is an ω_1-indiscernible family of open dense subsets of $\mathbb{P}$, then there is a set G which is $\mathbb{P}-$generic over $\mathbb{D}$. Furthermore G can be chosen to be ω_1-complete. □

We are now ready to give the proof of Theorem 3.1.

Assume an $(\omega_1, 1)-$morass exists. Let $(\mathbb{P}, \leq)$ be the forcing notion of section 2, when $\kappa = \omega_2$, for adding a regressive Kurepa tree with ω_2-many branches. For each $\alpha < \omega_2$ let $D_\alpha = \{p \in \mathbb{P} : \alpha \in domg_p\}$, and let $\mathbb{D} = \{D_\alpha : \alpha < \omega_2\}$. Then it is easy to see that for each $p \in \mathbb{P}, rlm(p) = domg_p$, for each $\alpha < \omega_2, \mathbb{P}_\alpha = \{p \in \mathbb{P} : domg_p \subseteq \alpha\}$, and $\mathbb{P}^* = \mathbb{P}_{\omega_1}$.

By Theorem 1.2.1 of [15], $\mathbb{D}$ is an ω_1-indiscernible family of open dense subsets of $\mathbb{P}$. Thus using Theorem 3.3, there exists $G \subseteq \mathbb{P}$ which is $\mathbb{P}-$generic over $\mathbb{D}$ and is ω_1-complete. Define $T, \leq_T$ and f exactly as in the proof of Theorem 2.1. By Theorem 1.2.2 of [15], $\langle T, \leq_T\rangle$ is a normal Kurepa tree, and using f it is regressive.

We now define a subtree of T which is an almost Souslin regressive Kurepa tree. Let $S = \{\alpha_p : p \in G\}$ and let C be the set of limit points of S. Then C is a club subset of ω_1. We first define by induction on $\alpha < \omega_1$ a sequence $\langle T^\alpha : \alpha < \omega_1\rangle$ of subtrees of T as follows:

- $\alpha = 0$: Let $T^0 = T$,
- $\alpha = \beta + 1$: Let $T^\alpha = T^\beta$,
- α is a limit ordinal, $\alpha \notin C$: Let $T^\alpha = \bigcap_{\beta<\alpha} T^\beta$,
- $\alpha \in C$: First let $T^* = \bigcap_{\beta<\alpha} T^\beta$. Now we define T^α as follows:

– $(T^\alpha)_{<\alpha} = (T^*)_{<\alpha}$,

– $(T^\alpha)_\alpha = \{x \in (T^*)_\alpha : \exists\xi < \kappa, \forall p \in G(\alpha_p < \alpha \wedge \xi \in domg_p \Rightarrow g_p(\xi) \leq_T x)\}$,

– for $\gamma > \alpha, (T^\alpha)_\gamma = \{x \in (T^*)_\gamma : \exists y \in (T^\alpha)_\alpha, x \geq_T y\}$.

Remark 3.4. *For $x \in (T^\alpha)_\alpha$, the required ξ is unique. Furthermore If $y \in T^\alpha$, and $x \leq_T y$, then $x \in T^\alpha$.*

Finally let $T^* = \bigcap_{\alpha<\omega_1} T^\alpha$. Clearly T^* is a subtree of T with ω_2-many branches, and hence it is a regressive Kurepa tree. We show that it is almost Souslin.

For $\alpha \in C$ we define $g_\alpha : \bigcup\{domg_p : p \in G, \alpha_p < \alpha\} \to (T^*)_\alpha$ as follows: Let $\xi \in domg_p$, where $p \in G, \alpha_p < \alpha$. Then there is a unique $x \in (T^*)_\alpha$ such that:

$$\forall q \in G(\alpha_q < \alpha \wedge \xi \in domg_q \Rightarrow x \geq_T g_q(\xi)).$$

Let $g_\alpha(\xi) = x$. Let us note that g_α is a bijection and for $\alpha \in C \cap S, g_\alpha = g_p$ where $p \in G$ is such that $\alpha_p = \alpha$. Next we define a function $h : T^* \restriction C \to T^*$ as follows: Let $\alpha \in C$ and $x \in (T^*)_\alpha$. Then $x = g_\alpha(\xi)$ for some $\xi \in domg_\alpha$. Let $h(x) = g_\beta(\xi)$, where β is the least ordinal such that for some $p \in G, \beta = \alpha_p$, and $\xi \in domg_p$. Note that $\beta < \alpha$ and $h(x) <_T x$.

Now as in the proof of Lemma 2.6, we can show that T^* is almost Souslin. Hence T^* is an almost Souslin regressive Kurepa tree. This completes the proof of Theorem 3.1. □

Remark 3.5. *The methods of this paper can be used to get more consistency results about trees. For example we can show that the existence of an almost Souslin Kurepa tree with no Aronszajn subtrees is consistent with ZFC, and that such a tree exists in L.*

The following question from [16] remained open.

Question 3.6. *Does there exist a Souslin tree T such that for each G which is $T-$generic over V, T is an almost Souslin Kurepa tree in $V[G]$?*

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