

COLLAPSING THE CARDINALS OF HOD

JAMES CUMMINGS; SY DAVID FRIEDMAN AND MOHAMMAD GOLSHANI

ABSTRACT. Assuming that GCH holds and κ is κ^{+3} -supercompact, we construct a generic extension W of V in which κ remains strongly inaccessible and $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal $\alpha < \kappa$. In particular the rank-initial segment W_κ is a model of ZFC in which $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal α .

1. INTRODUCTION

The study of *covering lemmas* has played a central role in set theory. The original Jensen covering lemma for L [4] states that if 0^\sharp does not exist then every uncountable set of ordinals in V is covered by a set of ordinals in L with the same V -cardinality. The restriction to uncountable sets is necessary because of the example of Namba forcing, which preserves \aleph_1^V but changes the cofinality of \aleph_2^V to ω . In combination with the “internal” theory of L coming from fine structure [3], the Jensen covering lemma is a powerful tool for proving lower bounds in consistency strength.

If V_0 is an inner model of V_1 , then *Jensen covering holds between V_0 and V_1* if every uncountable set of ordinals in V_1 is covered by a set of ordinals in V_0 with the same V_1 -cardinality. The example of Prikry forcing shows that Jensen’s covering lemma does not directly generalise to Kunen’s model $L[\mu]$. If $V = L[\mu]$ and G is Prikry generic over V then

Cummings was partially supported by NSF grant DMS-1101156, and thanks NSF for their support. He also thanks the University of Vienna for inviting him as a Guest Professor in January 2013.

Friedman would like to thank the FWF (Austrian Science Fund) for its support through Project P23316-N13.

Golshani’s research was in part supported by a grant from IPM (No. 91030417). He also thanks the European Science Foundation for their support through the grant 5317 within the INFITY program.

The authors thank Hugh Woodin for a very helpful discussion of this problem, including the suggestion to use supercompact Radin forcing. In an email exchange with the first author [12] Woodin suggested an alternative to using our projected forcing $\mathbb{R}_w^{\text{proj}}$, but we have opted to retain our original formulation.

cardinals are preserved in $V[G]$, the Prikry generic set cannot be covered by any set in $L[\mu]$ of cardinality less than κ , and 0^\dagger does not exist in $V[G]$.

Weak covering holds between V_0 and V_1 if $(\lambda^+)^{V_0} = (\lambda^+)^{V_1}$ for every singular cardinal λ in V_1 . It is not hard to see that Jensen covering implies weak covering, the key point is that if $(\lambda^+)^{V_0} < (\lambda^+)^{V_1}$ then $\text{cf}^{V_1}((\lambda^+)^{V_0}) < \lambda$. Note that for example weak covering holds between $L[\mu]$ and a Prikry extension of it. Jensen and Steel [5] have shown that if there is no inner model with a Woodin cardinal then weak covering holds over the core model for a Woodin cardinal.

The Jensen-Steel covering result implies that we will require the strength of a Woodin cardinal to obtain any failure of weak covering between a model V and a generic extension $V[G]$. To see this suppose there is no inner model with a Woodin cardinal and λ is singular in $V[G]$. Then by Jensen-Steel $(\lambda^+)^{V[G]} = (\lambda^+)^{K^{V[G]}}$ where K is the core model for one Woodin cardinal. The core model K is generically absolute so $K^{V[G]} = K^V \subseteq V$, and hence $(\lambda^+)^{V[G]} = (\lambda^+)^{K^{V[G]}} = (\lambda^+)^{K^V} = (\lambda^+)^V$.

It is possible to force a failure of weak covering between V and some generic extension from a Woodin cardinal, using the stationary tower forcing [7]. Let δ be Woodin, let $S = \{\alpha < \aleph_{\omega+1} : \text{cf}(\alpha) = \omega\}$, and let G be generic for the stationary tower forcing $\mathbb{P}_{<\delta}$ with $S \in G$. Then $\bigcup S = \aleph_{\omega+1}^V$, and if $j : V \rightarrow M \subseteq V[G]$ is the generic embedding associated with G then by a standard fact $j[\aleph_{\omega+1}] \in j(S)$. Since $j(S)$ is a set of ordinals this implies that $\text{crit}(j) \geq \aleph_{\omega+1}^V$. Since $V[G] \models {}^{<\delta}M \subseteq M$, the cardinals of V agree with those of $V[G]$ up to \aleph_ω . Moreover $j[\aleph_{\omega+1}^V] = \aleph_{\omega+1}^V$ and hence $\text{cf}^M(\aleph_{\omega+1}^V) = \omega$, so that in particular $\aleph_{\omega+1}^V < (\aleph_{\omega+1}^V)^{V[G]}$. So $\aleph_{\omega+1}^V$ is a singular cardinal of $V[G]$ whose successor is not computed correctly by V , a failure of weak covering.

This argument is not the last word, because a key point in the theory of covering lemmas is that weak covering should hold over a “reasonably definable” inner model. Of course the term “reasonably definable” is quite vague, so let us stipulate that a reasonably definable model should at least be ordinal definable and have an ordinal definable well-ordering. If M is such a model then $M \subseteq HOD$, because every element of M can be defined in V as “the α^{th} element of M ” for some ordinal α ; what is more, HOD itself is such a model. It follows that a failure of weak covering over HOD will also be a failure of weak covering over every

reasonably definable inner model. The main result of this paper produces a model where $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal α , which is in a certain sense the ultimate failure of weak covering.

The problem of arranging that $(\alpha^+)^{HOD} < \alpha^+$ has a very easy solution for a single regular cardinal α . If we force with the Levy collapse $Coll(\alpha, \alpha^+)$ then we obtain a generic extension $V[G]$ such that α is still regular, $(\alpha^+)^V < (\alpha^+)^{V[G]}$ and (by the homogeneity of the collapse) $HOD^{V[G]} \subseteq V$. Work of Dobrinen and Friedman [1] shows that it is consistent that $(\alpha^+)^{HOD} < \alpha^+$ for every regular α , and that this can hold in the presence of very large cardinals (for example superstrong cardinals). This is in contrast to a covering result by Steel [10] stating that if there is no inner model with a Woodin cardinal and κ is measurable with some normal measure μ , then $\{\alpha : (\alpha^+)^K = \alpha^+\} \in \mu$.

The stationary tower argument given above does not provide a failure of weak covering over HOD , because the stationary tower forcing is very inhomogeneous and there is no reason to believe that $HOD^{V[G]} \subseteq V$. In fact the only arguments known to the authors for obtaining a failure of weak covering over HOD involve some use of supercompactness: we will outline one such argument below in Section 2.

To prove our main result we will start with a suitable large cardinal κ , which will still be inaccessible in our final model. We will first build generic extensions V_0^* and V^* with $V \subseteq V_0^* \subseteq V^*$, such that $(\alpha^+)^V = (\alpha^+)^{V_0^*} < (\alpha^+)^{V^*}$ for almost all (in the sense of the club filter) infinite V^* -cardinals $\alpha < \kappa$. We will also arrange that $HOD^{V^*} \subseteq V_0^*$, so that in V^* we have $(\alpha^+)^{HOD} < \alpha^+$ for almost all infinite cardinals $\alpha < \kappa$. The model V^* will be obtained by supercompact Radin forcing, and V_0^* will be a submodel generated by a projected version of supercompact Radin forcing; then V^* is a generic extension of V_0^* by a quotient forcing, and the projected forcing is constructed so that we have enough homogeneity for the quotient forcing to argue that $HOD^{V^*} \subseteq V_0^*$. To finish we will force over V^* with a sufficiently homogeneous iteration of Levy collapsing posets, and produce a model W such that $HOD^W \subseteq HOD^{V^*}$ and $(\alpha^+)^V = (\alpha^+)^{V_0^*} < (\alpha^+)^W$ for every W -cardinal $\alpha < \kappa$. Since κ is inaccessible in W , the initial segment W_κ will be a model of ZFC in which $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal α .

We do not know the exact consistency strength of the assertion “ $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal α ”. Since it implies that weak covering fails over every reasonably definable inner model at every singular cardinal, it is certainly very strong, and presumably it implies the consistency of many Woodin cardinals and of strong forms of determinacy. Our work in this paper gives the upper bound “GCH and κ is κ^{+3} -supercompact”, which seems quite reasonable since in the current state of knowledge we need “GCH and κ is κ^+ -supercompact” to get the consistency of “ $(\alpha^+)^{HOD} < \alpha^+$ for some singular cardinal α ”.

Our notation is quite standard. We will write the pointwise image of a function f on a set $A \subseteq \text{dom}(f)$ as $f[A]$ rather than $f^{\smallfrown}A$. If \mathbb{P} is a forcing poset with $p \in \mathbb{P}$ then we write $\mathbb{P} \downarrow p$ for $\{q \in \mathbb{P} : q \leq p\}$. If s and t are sequences then $s \smallfrown t$ is the concatenation of s and t .

2. PRELIMINARIES ON SUPERCOMPACT PRIKRY FORCING AND RADIN FORCING

To motivate our use of supercompact Radin forcing, in this section we give a discussion of supercompact Prikry forcing and the original (non-supercompact) form of Radin forcing. Supercompact Prikry forcing will give us a way of arranging that $(\kappa^+)^{HOD} < \kappa^+$ for a single singular cardinal κ .

Recall that Prikry forcing is defined from a normal measure U_0 on a measurable cardinal κ . It is a κ^+ -cc forcing poset which adds an ω -sequence cofinal in κ without adding bounded subsets of κ . Supercompact Prikry forcing and Radin forcing represent generalisations of Prikry forcing in different directions: supercompact Prikry forcing adds an increasing ω -sequence of elements of $P_\kappa \lambda$ whose union is λ for some $\lambda \geq \kappa$, while Radin forcing adds closed unbounded subsets of κ of order types greater than or equal to ω .

The supercompact Radin forcing which we define in Section 3 is a common generalisation of supercompact Prikry forcing and Radin forcing. Proofs of the various assertions we make about supercompact Prikry forcing and Radin forcing can mostly be found (in the more general setting of supercompact Radin forcing) in that section.

2.1. Supercompact Prikry forcing. Supercompact Prikry forcing was introduced by Magidor [8] in his work on the Singular Cardinals Hypothesis.

Let κ be λ -supercompact for some regular cardinal $\lambda \geq \kappa$, and let U be a supercompactness measure on $P_\kappa \lambda$. It will be convenient to identify a subset on which U concentrates:

Definition 2.1. Let $A(\kappa, \lambda)$ be the set of those $x \in P_\kappa \lambda$ such that $x \cap \kappa$ is an inaccessible cardinal and $\text{ot}(x)$ is a regular cardinal with $\text{ot}(x) \geq x \cap \kappa$.

We claim that U concentrates on $A(\kappa, \lambda)$. The argument will be the prototype for several later arguments, and will use the following standard fact: if $j : V \rightarrow M = \text{Ult}(V, U)$ is the ultrapower map associated with U and $B \subseteq P_\kappa \lambda$, then $B \in U$ if and only if $j[\lambda] \in j(B)$. Since $j[\lambda] \cap j(\kappa) = \kappa$ and $\text{ot}(j[\lambda]) = \lambda$, it is clear that $j[\lambda] \in j(A(\kappa, \lambda))$.

Let $A = A(\kappa, \lambda)$ and define $\kappa_x = x \cap \kappa$, $\lambda_x = \text{ot}(x)$ for $x \in A$; the functions $x \mapsto \kappa_x$ and $x \mapsto \lambda_x$ represent κ and λ respectively in the ultrapower of V by U , and it is often useful to view κ_x and λ_x as “local” versions of κ and λ . We equip A with a strict partial ordering \prec by defining $x \prec y$ if and only if $x \subseteq y$ and $\lambda_x < \kappa_y$.

Remark 2.2. If $x \in A(\kappa, \lambda)$ then $\kappa_x = \min(ON \setminus x)$ and $\lambda_x = \text{ot}(x)$, so that both κ_x and λ_x can be computed without knowing the values of κ and λ .

Definition 2.3. The supercompact Prikry forcing \mathbb{P}_U defined from U has conditions (s, B) where $B \in U$ with $B \subseteq A$, and s is a finite \prec -increasing sequence drawn from A . The sequence s is the stem of the condition, and the measure one set B is the upper part. The ordering is just like Prikry forcing, that is to say the condition (s, B) is extended by prolonging s using elements from B and shrinking B : a direct extension of (s, B) is an extension of the form (s, C) for some $C \subseteq B$.

Remark 2.4. By a classical theorem of Solovay we have $\lambda^{<\kappa} = \lambda$, so that there are only λ possible stems. If $\lambda = \kappa$ then $A(\kappa, \lambda) = \kappa$, and \mathbb{P}_U is just the standard Prikry forcing defined from a normal measure on κ .

The generic object for the forcing poset \mathbb{P}_U is a \prec -increasing ω -sequence $\langle x_i : i < \omega \rangle$ such that $\bigcup_i x_i = \lambda$. The poset satisfies the *Prikry lemma*, that is to say that any question about the generic extension can be decided by a direct extension. This implies that \mathbb{P}_U adds no bounded subsets of κ , in particular κ remains a cardinal. By contrast with the standard Prikry forcing at κ , which is κ^+ -cc and preserves all cardinals, forcing with \mathbb{P}_U will collapse λ to have cardinality κ . Since any two conditions with the same stem are compatible, and

there are only λ stems, we see that \mathbb{P}_U is λ^+ -c.c. and so it is only cardinals μ with $\kappa < \mu \leq \lambda$ that are collapsed.

The main point in the proof of the Prikry lemma (see Lemma 3.32) is that U enjoys a form of normality. If s is a stem and $y \in A$ then we write “ $s \prec y$ ” for the assertion that either s is empty or the last entry in s is below y in the \prec ordering: normality states that if I is a set of stems and $\langle B_s : s \in I \rangle$ is such that $B_s \in U$ for all s , and we define the diagonal intersection $\Delta_s B_s = \{y \in A : \forall s \prec y \ y \in B_s\}$, then $\Delta_s B_s \in U$.

We can use supercompact Prikry forcing to obtain a model where $(\alpha^+)^{HOD} < \alpha^+$ for a single singular cardinal α . Let us assume for simplicity that $\lambda = \kappa^+$. Let $\langle x_i : i < \omega \rangle$ be a \mathbb{P}_U -generic sequence and let U_0 be the projection of U to a measure on κ via the map $x \mapsto \kappa_x$. Using the well-known criterion for Prikry genericity, one can show that $\langle \kappa_{x_i} : i < \omega \rangle$ is generic for \mathbb{P}_{U_0} , the Prikry forcing at κ defined from the measure U_0 .

So starting with G which is \mathbb{P}_U -generic, we obtain a chain of models $V \subseteq V[G_0] \subseteq V[G]$ where G_0 is \mathbb{P}_{U_0} -generic, and $(\kappa^+)^V = (\kappa^+)^{V[G_0]} < (\kappa^+)^{V[G]}$. Once we have shown that $HOD^{V[G]} \subseteq V[G_0]$, it will follow that in the model $V[G]$ weak covering fails for HOD at κ . This can be proved (see Lemma 4.18) using permutations of κ^+ that fix all points below κ ; the key idea is (roughly speaking) that such permutations induce many automorphisms of \mathbb{P}_U which commute with the operation of projecting a \mathbb{P}_U -generic filter down to a \mathbb{P}_{U_0} -generic filter.

2.2. Radin forcing. Radin forcing [9] is a generalisation of Prikry forcing in which closed unbounded sets of order types greater than or equal to ω are added to a large cardinal κ . We will outline the basic theory of Radin forcing.

Definition 2.5. A κ -measure sequence is a sequence w such that $w(0) = \kappa$, and $w(\alpha)$ is a κ -complete measure on V_κ for $0 < \alpha < \text{lh}(w)$.

Definition 2.6. Given an elementary embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$, we derive a κ -measure sequence u_j by setting $u_j(0) = \kappa$ and then $u_j(\alpha) = \{X \subseteq V_\kappa : u_j \upharpoonright \alpha \in j(X)\}$ for $\alpha > 0$, continuing for as long as $\alpha < j(\kappa)$ and $u_j \upharpoonright \alpha \in M$.

Each $u_j(\alpha)$ for $\alpha > 0$ is a κ -complete non-principal ultrafilter on V_κ , concentrating on objects which “resemble” $u_j \upharpoonright \alpha$. Note that $u_j(1)$ is essentially the usual normal measure

on κ derived from j ; it concentrates on sequences of length one rather than ordinals because it is generated by the sequence $u_j \upharpoonright 1 = \langle \kappa \rangle$ rather than the ordinal κ . In this informal discussion, we will sometimes ignore the distinction between the ordinal α and the sequence $\langle \alpha \rangle$.

Definition 2.7. *If w is an initial segment of u_j then we say that j is a constructing embedding for w .*

Definition 2.8. *We define a class \mathcal{U}_∞ of good measure sequences as follows:*

- \mathcal{U}_0 is the class of w such that for some κ , w is a κ -measure sequence which has a constructing embedding.
- $\mathcal{U}_{n+1} = \{w \in \mathcal{U}_n : \text{for all nonzero } \alpha < \text{lh}(w), w(\alpha) \text{ concentrates on } \mathcal{U}_n\}$.
- $\mathcal{U}_\infty = \bigcap_{n < \omega} \mathcal{U}_n$.

Note that by countable completeness, if w is a good measure sequence then every measure in w concentrates on good measure sequences. By choosing j to witness some modest degree of strength for κ , we may arrange that long initial segments of u_j are good measure sequences. The key point here is that if $j : V \rightarrow M$ then there are extenders in M which approximate j sufficiently to serve as constructing embeddings (in M) for initial segments of j .

Given a measure sequence u with $\text{lh}(u) > 1$ we define the Radin forcing \mathbb{R}_u as follows.

Definition 2.9. *Conditions in \mathbb{R}_u are sequences $p = \langle (u^i, A^i) : i \leq n \rangle$ where each u^i is a good measure sequence in $V_{u^{i+1}(0)}$, $u^n = u$, $u^i(0)$ increases with i , and either $\text{lh}(u^i) = 1$ and $A^i = \emptyset$ or $\text{lh}(u^i) > 1$ and $A^i \subseteq V_{u^i(0)}$ with A^i a set of good measure sequences which is measure one for every measure in u^i . A condition is extended by shrinking some of the sets A^i , and interpolating new pairs (w, B) such that $w \in A^i$ and $B \subseteq A^i \cap V_{w(0)}$ between (u^{i-1}, A^{i-1}) and (u^i, A^i) for some i .*

The generic object can be viewed as a sequence of good measure sequences $\langle v_\alpha : \alpha < \mu \rangle$ where $\langle v_\alpha(0) : \alpha < \mu \rangle$ increases continuously with α and is cofinal in $u(0)$: the condition p above carries the information that each v_α either appears among the u^i or is the first entry in a pair which can legally be added to p . The uniform definition of the extension relation readily implies that if $\text{lh}(v_\alpha) > 1$ then the sequence $\langle v_\beta : \beta < \alpha \rangle$ is generic over V for \mathbb{R}_{v_α} .

It is easy to see that if p is a condition as above and $i < n$ with $\text{lh}(u^i) > 1$, then $\mathbb{R}_u \downarrow p$ is isomorphic to a product of the form $R_{u^i} \downarrow q \times \mathbb{R}_u \downarrow r$ for suitable conditions $q \in \mathbb{R}_{u^i}$ and $r \in \mathbb{R}_u$. The forcing \mathbb{R}_u is also $u(0)^+$ -cc and satisfies a version of the Prikry lemma, stating as usual that any question about the generic extension can be decided just by shrinking measure one sets. It follows from these facts that forcing with \mathbb{R}_u preserves all cardinals.

As motivation, we outline a version of Radin forcing intended to add a cofinal and continuous sequence of type ω^2 in a large cardinal κ . Consider the Radin forcing defined from a good measure sequence u of length three, so u has two measures $u(1)$ and $u(2)$. If we force below the condition (u, A) where A consists of good measure sequences in $V_{u(0)}$ of length one or two, then we will obtain a generic sequence $\langle v_\alpha : \alpha < \omega^2 \rangle$. where v_α has length one (so is morally just an ordinal) for successor α but has length two for limit α .

It can be shown that:

- For each $m < \omega$, the ω -sequence $\langle v_{\omega \cdot m + n}(0) : n < \omega \rangle$ is Prikry generic for the Prikry forcing defined from $v_{\omega \cdot (m+1)}(1)$.
- The ω -sequence $\langle v_{\omega \cdot m} : m < \omega \rangle$ is generic for the version of Prikry forcing defined from the measure $u(2)$ (stems are finite with each entry a measure sequence of length two and critical points increasing, upper parts are $u(2)$ -large sets).
- For any sequence $\langle \beta_i : i < \omega \rangle$ which consists of successor ordinals and is cofinal in ω^2 , the ω -sequence $\langle v_{\beta_i}(0) : i < \omega \rangle$ is Prikry generic for the Prikry forcing defined from $u(1)$.

3. SUPERCOMPACT RADIN FORCING

Supercompact Radin forcing was introduced by Foreman and Woodin [2] in their consistency proof for “GCH fails everywhere”. In particular they proved that supercompact Radin forcing satisfies a version of the Prikry lemma, and can preserve large cardinals. The main forcing of [2] is rather complicated as it aims to interleave generic objects for Cohen posets along the generic sequence, and conditions must contain machinery for constraining these generic objects. Other accounts of supercompact Radin forcing appear in the literature, for example Krueger [6] has described a version constructed from a coherent sequence of supercompactness measures.

In this section we define a version of supercompact Radin forcing. To make the paper self-contained, we will prove all the properties of this forcing which we will use. To motivate some technical aspects of the general definition, we will first define a special case of the forcing which adds a continuous and \prec -increasing ω^2 -sequence in $P_\kappa\lambda$.

Definition 3.1. *Let $\kappa \leq \lambda \leq \mu$, where λ and μ are regular and κ is μ -supercompact. Let $j : V \rightarrow M$ witness μ -supercompactness of κ , and then define a sequence u_j by the recursion $u_j(0) = j[\lambda]$, $u_j(\alpha) = \{X : u_j \upharpoonright \alpha \in j(X)\}$ for as long as $\alpha < j(\kappa)$ and $u_j \upharpoonright \alpha \in M$.*

Remark 3.2. If $\lambda = \kappa$ then we are just defining the kind of measure sequences constructed in Section 2.2.

Recall from Section 2.1 that we defined $A(\kappa, \lambda)$ as the set of $x \in P_\kappa\lambda$ such that $x \cap \kappa$ is an inaccessible cardinal and $\text{ot}(x)$ is a regular cardinal with $\text{ot}(x) \geq x \cap \kappa$. We also defined $\kappa_x = x \cap \kappa$ and $\lambda_x = \text{ot}(x)$ for such x .

Definition 3.3. *Let $S(\kappa, \lambda)$ be the set of non-empty sequences w such that $\text{lh}(w) < \kappa$, $w(0) \in A(\kappa, \lambda)$ and $w(\alpha) \in V_\kappa$ for all α with $0 < \alpha < \text{lh}(w)$.*

It is easy to see that for every $\alpha < j(\kappa)$ such that the measure $u_j(\alpha)$ is defined, it concentrates on $S(\kappa, \lambda)$. When $\text{lh}(w) = 1$, we will sometimes be careless about the distinction between the sequence $w = \langle w(0) \rangle$ and the set $w(0)$.

Remark 3.4. We can refine the definition of the set $S(\kappa, \lambda)$ to find a smaller subset on which the measures on u_j will concentrate, reflecting more of the properties of initial segments of u_j . We do this in generality in Definition 3.8 below, for the purposes of the following example we just note that $u_j(1)$ is (essentially) the λ -supercompactness measure on $P_\kappa\lambda$ derived from j in the standard way, and so $u_j(2)$ concentrates on pairs (x, w) where $x \in P_\kappa\lambda$ and w is (essentially) a supercompactness measure on $P_{\kappa_x}\lambda_x$.

Suppose that $u_j(1)$ and $u_j(2)$ are defined and let $u = u_j \upharpoonright 3 = (j[\lambda], u_j(1), u_j(2))$. Conditions in the supercompact Radin forcing to add an ω^2 -sequence in $P_\kappa\lambda$ will be finite sequences $\langle (u^i, A^i) : 0 \leq i \leq n \rangle$ where

- $u^i \in S(\kappa, \lambda)$ for $i < n$, and $\langle x^i : i < n \rangle$ is \prec -increasing where $x^i = u^i(0)$.

- For $i < n$, either $\text{lh}(u^i) = 1$ and $A^i = \emptyset$, or $\text{lh}(u^i) = 2$ and $u^i(1)$ is a supercompactness measure on $A(\kappa_{x^i}, \lambda_{x^i})$ with $A^i \in u^i(1)$.
- $u^n = u$, $A^n \subseteq S(\kappa, \lambda)$, $A^n \in u^n(1) \cap u^n(2)$ and it consists of sequences of length at most 2.

The ordering is basically as in the Radin forcing described in Section 2.2, with one complication. If $i < n$ and $\text{lh}(u^i) = 2$, then the elements of A^i are in $A(\kappa_{x^i}, \lambda_{x^i})$ but the objects we would like to interpolate between (u^{i-1}, A^{i-1}) and (u^i, A^i) are elements $y \in A(\kappa, \lambda)$ such that $x^{i-1} \prec y \prec x^i$. The solution will be to use the order isomorphism $\pi : x^i \simeq \lambda_{x^i}$, so that y can be legally be interpolated if $x^{i-1} \prec y \prec x^i$ and $\pi[y] \in A^i$.

This gives some insight into the way that supercompact Radin forcing will work in general. An entry w on the generic sequence will have $x = w(0) \in A(\kappa, \lambda)$, and the remainder of w will consist of measures on $S(\kappa_x, \lambda_x)$. The measures appearing on w define a “local” supercompact Radin forcing for $P_{\kappa_x} \lambda_x$, and the role of x will be to integrate the generic sequence for this “local” forcing into the “global” sequence. On a related point, in the definition of the supercompact Radin forcing from a sequence u the value of $u(0)$ is actually irrelevant.

3.1. Supercompact measure sequences. Before defining supercompact Radin forcing, we need to define the “good measure sequences” which will form the building blocks for this forcing. The reader should note that the terms *constructing embedding*, *(good) measure sequence*, will be used in a more general sense than in Section 2.2.

Definition 3.5. *A sequence u is a (κ, λ) -measure sequence if $u(0)$ is a set of ordinals with $\kappa = \min(ON \setminus u(0))$ and $\lambda = \text{ot}(u(0))$, and $u(\alpha)$ is a κ -complete measure on $S(\kappa, \lambda)$ for all α with $0 < \alpha < \text{lh}(u)$.*

The sequence u_j constructed in definition 3.1 is an example of such a sequence.

Definition 3.6. *If u is a (κ, λ) -measure sequence and $A \subseteq S(\kappa, \lambda)$, then A is u -large if and only if $A \in u(\beta)$ for all β with $0 < \beta < \text{lh}(u)$.*

Definition 3.7. *Given a (κ, λ) -measure sequence u , j is a constructing embedding for u if and only if j witnesses that κ is λ -supercompact, and for all α with $0 < \alpha < \text{lh}(u)$ we have that $u_j(\alpha)$ is defined with $u_j(\alpha) = u(\alpha)$.*

Note that possibly $u(0) \neq u_j(0)$ in the last definition, which may seem surprising. We will discuss this point further after Lemma 3.12. The basic issue is that if $j : V \rightarrow M$ witness μ -supercompactness for some large $\mu > \lambda$, and we build a (κ, λ) -measure sequence u_j then we should like initial segments of u_j to have a constructing embedding in M ; this will be true with our definition.

Definition 3.8. *We define a class $\mathcal{U}_\infty^{\text{sup}}$ of good measure sequences as follows:*

- $\mathcal{U}_0^{\text{sup}}$ is the class of sequences u such that for some regular cardinals κ and λ with $\kappa \leq \lambda$, u is a (κ, λ) -measure sequence which has a constructing embedding.
- $\mathcal{U}_{n+1}^{\text{sup}} = \{u \in \mathcal{U}_n : \text{for all nonzero } \alpha < \text{lh}(u), u(\alpha) \text{ concentrates on } \mathcal{U}_n\}$.
- $\mathcal{U}_\infty^{\text{sup}} = \bigcap_{n < \omega} \mathcal{U}_n$.

As in Section 2.2, it follows from the countable completeness of the measures in u that if $u \in \mathcal{U}_\infty^{\text{sup}}$ every measure in u concentrates on $\mathcal{U}_\infty^{\text{sup}}$.

Definition 3.9. *Given a measure sequence $u \in \mathcal{U}_\infty^{\text{sup}}$, we let $\kappa_u = \min(\text{ON} \setminus u(0))$ and $\lambda_u = \text{ot}(u(0))$.*

Definition 3.10. *Given a (κ, λ) -measure sequence u , a non-zero ordinal $\alpha < \text{lh}(u)$ is a weak repeat point for u if and only if for all $X \in u(\alpha)$ there exists a non-zero $\beta < \alpha$ such that $X \in u(\beta)$.*

Failure to be a weak repeat point is witnessed by a “novel” subset of $S(\kappa, \lambda)$, and $|S(\kappa, \lambda)| = \lambda^{<\kappa}$, so we have the following easy result.

Lemma 3.11. *A (κ, λ) -measure sequence of length $(2^{\lambda^{<\kappa}})^+$ contains a weak repeat point.*

For the purposes of our main result, we need a good (κ, κ^+) -measure sequence with a weak repeat point.

Lemma 3.12. *Let GCH hold and let $j : V \rightarrow M$ witness that κ is κ^{+3} -supercompact. Then there exists a (κ, κ^+) -measure sequence u such that $u \in \mathcal{U}_\infty^{\text{sup}}$ and u has a weak repeat point.*

Proof. Evidently M contains every (κ, κ^+) -measure sequence of length less than κ^{+3} , so that the construction of u_j runs for at least κ^{+3} steps. By Lemma 3.11 a weak repeat point α appears before stage κ^{+3} , so it suffices to check that $u \in \mathcal{U}_\infty^{\text{sup}}$, where $u = u_j \upharpoonright (\alpha + 1)$. Clearly j is a constructing embedding for u . To verify that u is good, we will first use a reflection argument to show that u has a constructing embedding in M .

Let W be the supercompactness measure on $P_\kappa(\kappa^{+2})$ induced by the embedding j and let $j_0 : V \rightarrow M_0 = \text{Ult}(V, W)$ be the usual ultrapower map. The following claims are all standard and easy to verify:

- (1) $W \in M$.
- (2) If we define a map k from M_0 to M by setting $k : [F]_W \mapsto j(F)(j[\kappa^{+2}])$ then k is elementary, and $k \circ j_0 = j$.
- (3) $\kappa^{+3} = (\kappa^{+3})^{M_0} < \text{crit}(k) = (\kappa^{+4})^{M_0} < \kappa^{+4} = (\kappa^{+4})^M$.
- (4) The sets $j_0[\kappa^+]$ and $j_0[\kappa^{+2}]$ are in M_0 , $k(j_0[\kappa^+]) = j[\kappa^+]$, and $k(j_0[\kappa^{+2}]) = j[\kappa^{+2}]$.
- (5) Since $\kappa^{+3} \subseteq \text{rge}(k)$, it follows easily from *GCH* that $H_{\kappa^{+3}} \subseteq \text{rge}(k)$, and so that $k \upharpoonright H_{\kappa^{+3}} = \text{id}$.
- (6) Let $j_0^* : M \rightarrow M^* = \text{Ult}(M, W)$ be the ultrapower map defined from W in M . For every $a \in M$, $P_\kappa(\kappa^{+2})TC(a) \subseteq M$ and so $j_0(M) = j_0^*(M)$, that is $j_0^* = j_0 \upharpoonright M$.

Now let $u^* = u_{j_0^*}^M$, the measure sequence constructed by the embedding j_0^* in the model M .

We claim that the construction of u^* proceeds for at least $\alpha + 1$

steps and that $u(\beta) = u^*(\beta)$ for all β with $0 < \beta \leq \alpha$. At the start, $u^*(0) = j_0^*[\kappa^+] = j_0[\kappa^+]$, and we now proceed by induction on β with $0 < \beta \leq \alpha$. Note that all the models V , M , M_0 and M^* agree on the computation of $P(S(\kappa, \kappa^+))$.

Suppose that $u^*(\eta) = u(\eta)$ when $0 < \eta < \beta$. Since M_0 is closed under κ^{+2} -sequences, $u^* \upharpoonright \beta \in M_0$ and the properties of k listed above imply that $k(u^* \upharpoonright \beta) = u \upharpoonright \beta$. Now for every $X \subseteq S(\kappa, \kappa^+)$ we have that

$$\begin{aligned} X \in u^*(\beta) &\iff u^* \upharpoonright \beta \in j_0^*(X) \\ &\iff u^* \upharpoonright \beta \in j_0(X) \\ &\iff u \upharpoonright \beta \in j(X) \\ &\iff X \in u(\beta), \end{aligned}$$

where the equivalences follow respectively from the definition of u^* , the agreement between j_0 and j_0^* , the properties of k , and the definition of u .

We can now verify that $u \in \mathcal{U}_1^{\text{sup}}$. This holds because for each $\beta \leq \alpha$ the sequence $u \upharpoonright \beta$ has a constructing embedding in M , and so by the recursive definition of u the measure $u(\beta)$ concentrates on the class of sequences with a constructing embedding.

The rest of the argument is straightforward. We start by observing that since $\kappa^{+3} = (\kappa^{+3})^M$, every measure in u concentrates on the set of measure sequences x such that $x(0) \in P_\kappa(\kappa^+)$ and $\text{lh}(x) < \kappa_x^{+3}$. By the agreement between V and M , a routine induction shows that for all n and for all such measure sequences x we have $x \in \mathcal{U}_n \iff x \in \mathcal{U}_n^M$.

We now establish by induction on $n \geq 1$ that $u \in \mathcal{U}_n^{\text{sup}}$. We just did the base case $n = 1$, so suppose that we established $u \in \mathcal{U}_n^{\text{sup}}$. By definition, $u \in \mathcal{U}_{n+1}^{\text{sup}}$ if and only if $u(\beta)$ concentrates on $\mathcal{U}_n^{\text{sup}}$ for $0 < \beta \leq \alpha$, that is if $u \upharpoonright \beta \in (\mathcal{U}_n^{\text{sup}})^M$ for all such β . By definition $u \upharpoonright \beta \in (\mathcal{U}_n^{\text{sup}})^M$ if and only if $u(\gamma)$ concentrates on $(\mathcal{U}_{n-1}^{\text{sup}})^M$ for all γ with $0 < \gamma < \beta$, and by the remarks in the previous paragraph this amounts to verifying that $u(\gamma)$ concentrates on $\mathcal{U}_{n-1}^{\text{sup}}$ which is true since $u \in \mathcal{U}_n$. \square

Recall that in Definition 3.7, we permitted that $w(0) \neq j[\lambda]$ in the definition of “ j is a constructing embedding for w ”. In light of the proof of Lemma 3.12, this may seem less surprising: the set $j[\lambda]$ is necessarily an element of M , but may not be of the form $i[\lambda]$ for a suitable supercompactness embedding *defined in M* .

Before defining supercompact Radin forcing, we need to define a family of “type changing maps”. These are needed because if $x \in S(\kappa, \lambda) \cap \mathcal{U}_\infty^{\text{sup}}$, then the measures in x concentrate on $S(\kappa_x, \lambda_x)$ rather than $S(\kappa, \lambda)$.

Remark 3.13. The type changing maps are functions from ordinals to ordinals, whose role is to change the type of a measure sequence x via pointwise application to $x(0)$. Accordingly we will systematically abuse notation in the following way: whenever ν is one of the type changing maps π_ν , ρ_ν or $\sigma_{\nu w}$ from the forthcoming Definitions 3.15 and 3.16, and x is a measure sequence then

$$\nu(x) = \langle \nu[x(0)] \rangle \frown \langle x(\beta) : 0 < \beta < \text{lh}(x) \rangle.$$

Recall that we defined an ordering \prec on $A(\kappa, \lambda)$ by stipulating that $x \prec y$ if and only if $x \subseteq y$ and $\lambda_x < \kappa_y$.

Definition 3.14. *Let $v, w \in S(\kappa, \lambda)$. Then $w \prec v$ if and only if $w(0) \prec v(0)$, $\text{lh}(w) < \kappa_v$ and $w(\beta) \in V_{\kappa_v}$ for all β such that $0 < \beta < \text{lh}(w)$.*

In the sequel there will be situations where several spaces of the general form $S(\kappa, \lambda)$ appear at once. We will only compare sequences v and w when they lie in the same such space, and the values of κ and λ should always be clear from the context.

Definition 3.15. *Let $v \in S(\kappa, \lambda)$. Then we define $\pi_v : v(0) \simeq \lambda_v$ to be the unique order isomorphism between $v(0)$ and λ_v , and also define $\rho_v : \lambda_v \simeq v(0)$ by $\rho_v = \pi_v^{-1}$.*

Definition 3.16. *Let $v, w \in S(\kappa, \lambda)$ with $w \prec v$. Then we define $\sigma_{wv} : \lambda_w \rightarrow \lambda_v$ by $\sigma_{wv} = \pi_v \circ \rho_w$.*

Informally σ_{wv} is a ‘‘collapsed’’ version of the inclusion map from $w(0)$ to $v(0)$. The following Lemmas are straightforward.

Lemma 3.17. *Let $v \in S(\kappa, \lambda)$. Then $\rho_v \upharpoonright \kappa_v$ is the identity, $\rho_v(\kappa_v) \geq \kappa$, and for every $w \in S(\kappa_v, \lambda_v)$ we have $\rho_v(w) \in S(\kappa, \lambda)$ with $\rho_v(w) \prec v$.*

Lemma 3.18. *Let $v, w \in S(\kappa, \lambda)$ with $w \prec v$. Then $\sigma_{wv} \upharpoonright \kappa_w$ is the identity, $\pi_v(w) \in S(\kappa_v, \lambda_v)$, and for every $x \in S(\kappa_w, \lambda_w)$ we have $\sigma_{wv}(x) \in S(\kappa_v, \lambda_v)$.*

Definition 3.19. *A good pair is a pair (u, A) where $u \in \mathcal{U}_\infty^{\text{sup}}$, and either $\text{lh}(u) = 1$ and $A = \emptyset$ or $\text{lh}(u) > 1$, $A \subseteq \mathcal{U}_\infty^{\text{sup}} \cap S(\kappa_u, \lambda_u)$ and A is u -large.*

We define, for each $u \in \mathcal{U}_\infty^{\text{sup}}$ with length greater than 1, the corresponding *supercompact Radin forcing* $\mathbb{R}_u^{\text{sup}}$.

Definition 3.20. *Let $u \in \mathcal{U}_\infty^{\text{sup}}$ with length greater than 1. A condition in $\mathbb{R}_u^{\text{sup}}$ is a finite sequence*

$$p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^n, A^n) \rangle$$

where:

- (1) $u^n = u$.
- (2) Each (u^i, A^i) is a good pair.
- (3) $u^i \in S(\kappa_{u^n}, \lambda_{u^n})$ for all $i < n$.
- (4) $u^i \prec u^{i+1}$ for all $i < n - 1$.

The sequence $\langle (u^0, A^0), \dots, (u^{n-1}, A^{n-1}) \rangle$ is the stem of the condition, and A^n is the upper part.

Definition 3.21. *Let*

$$p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^m = u, A^m) \rangle$$

and

$$q = \langle (v^0, B^0), \dots, (v^j, B^j), \dots, (v^n = u, B^n) \rangle$$

be in $\mathbb{R}_u^{\text{sup}}$. Then $q \leq p$ (q is an extension of p) iff:

- (1) There exist natural numbers $j_0 < \dots < j_m = n$ such that $v^{j_k} = u^k$ and $B^{j_k} \subseteq A^k$.
- (2) If j is such that $0 \leq j \leq n$ and $j \notin \{j_0, \dots, j_m\}$, and i is least such that $\kappa_{v^j} < \kappa_{u^i}$, then:

- If $i = m$, then $v^j \in A^m$ and $\rho_{v^j}[B^j] \subseteq A^m$.
- If $i < m$, then $\pi_{u^i}(v^j) \in A^i$ and $\sigma_{v^j u^i}[B^j] \subseteq A^i$.

We also define $q \leq^* p$ (q is a direct extension of p) iff $q \leq p$ and $n = m$.

Remark 3.22. Any two conditions with the same stem are compatible. Just as for supercompact Prikry forcing, that there are only λ possible stems. Therefore $\mathbb{R}_u^{\text{sup}}$ satisfies the $\lambda^+ - c.c.$.

The following Lemma shows that conditions can be extended in many ways by adding in new good pairs immediately below the top entry.

Lemma 3.23. *Let $u \in \mathcal{U}_\infty^{\text{sup}}$ with $\text{lh}(u) > 1$, and let*

$$p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^m = u, A^m) \rangle$$

be a condition in $\mathbb{R}_u^{\text{sup}}$. Let X be the set of sequences v with the following property: there exists a set B such that (v, B) is a good pair, and if we set

$$q = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^{m-1}, A^m), (v, B), (u^m = u, A^m) \rangle$$

then q is a condition extending p . Then X is u -large.

Proof. Let j be a constructing embedding for u . To show X is u -large we must show that $u_j \upharpoonright \beta \in j(X)$ for all β with $0 < \beta < \text{lh}(u)$. It is easy to see that $v \in X$ if and only if it satisfies the conditions:

- (1) $v \in A^m$.
- (2) $u^{m-1} \prec v$ (in case $m > 0$).
- (3) $\{w \in S(\kappa_v, \lambda_v) : \rho_v(w) \in A^m\}$ is v -large.

Since A_m is u -large, $A_m \in u(\beta)$ and so $u_j \upharpoonright \beta \in j(A^m)$. It is easy to check that for any $x \in S(\kappa_u, \lambda_u)$, $j(x) \prec u_j \upharpoonright \beta$; the main points are that $j(x(0)) = j[x(0)]$ and $u_j(0) = j[\lambda_u]$. Since $u_j(0) = j[\lambda_u]$, $\rho_{u_j \upharpoonright \beta} = j \upharpoonright \lambda_u$, and so easily $\rho_{u_j \upharpoonright \beta}(w) = j(w)$ for all $w \in S(\kappa_u, \lambda_u)$; so $A^m = \{w \in S(\kappa_u, \lambda_u) : \rho_{u_j \upharpoonright \beta}(w) \in j(A^m)\}$, and since A^m is clearly $u_j \upharpoonright \beta$ -large we have that $u_j \upharpoonright \beta \in j(X)$. \square

Lemma 3.23 states that u -many of the elements of A^m can be interpolated into the condition p . We can iterate this argument ω times, by setting $B^0 = A^m$ and then defining a decreasing sequence $\langle B^i : i < \omega \rangle$ such that each B^i is a u -large set and every element of B^{i+1} can be interpolated between (u^{m-1}, A^{m-1}) and (u, B^i) . Now let $A' = \bigcap_{i < \omega} B^i$. Since the measures in u are countably complete, A' is u -large and we obtain the following corollary of Lemma 3.23.

Corollary 3.24. *Let $u \in \mathcal{U}_\infty^{\text{sup}}$ with $\text{lh}(u) > 1$, and let*

$$p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^m = u, A^m) \rangle$$

be a condition in $\mathbb{R}_u^{\text{sup}}$. Then there is a u -large set $A' \subseteq A^m$ such that if we set

$$p' = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^m = u, A') \rangle,$$

then for every $v \in A'$ there exists $q \leq p'$ of the form

$$q = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^{m-1}, A^m), (v, B), (u^m = u, A') \rangle.$$

Similarly, whenever $\text{lh}(u^i) > 1$ there are many candidates for interpolation between (u^{i-1}, A^{i-1}) and (u^i, A^i) in an extension of p . Before we prove this, we state a very easy but useful factoring lemma.

Lemma 3.25. *Suppose that*

$$p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^m = u, A^m) \rangle \in \mathbb{R}_u^{\text{sup}}$$

and $i < m$ with $\text{lh}(u^i) > 1$. Let

$$p^{>i} = \langle (u^{i+1}, A_*^{i+1}), \dots, (u^m = u, A_*^m) \rangle,$$

where $A_*^j = A^j$ for all $j > i + 1$ while $A_*^{i+1} = \{w \in A^{i+1} : u^i \prec \rho_{u^{i+1}}(w)\}$. Let

$$p^{\leq i} = \langle (u_*^0, A^0), \dots, (u_*^{i-1}, A^{i-1}), (u^i, A^i) \rangle,$$

where $u_*^j = \pi_{u^i}(u^j)$ for each $j < i$.

Then $p^{\leq i} \in \mathbb{R}_{u^i}^{\text{sup}}$, $p^{>i} \in \mathbb{R}_u^{\text{sup}}$ and there exists

$$i : \mathbb{R}_{u^i}^{\text{sup}} \downarrow p^{\leq i} \times \mathbb{R}_u^{\text{sup}} \downarrow p^{>i} \rightarrow \mathbb{R}_u^{\text{sup}}$$

which is an isomorphism between its domain and a dense subset of $\mathbb{R}_u^{\text{sup}} \downarrow p$.

Proof. Let $q_0 \leq p^{\leq i}$ and $q_1 \leq p^{>i}$, say $q_0 = \langle (v^0, B^0), \dots, (v^j = u^i, B^j) \rangle$ and $q_1 = \langle (v^{j+1}, B^{j+1}), \dots, (v^n = u, B^n) \rangle$. Define $i(q_0, q_1) = \langle (v_*^j, B^j) : 0 \leq j \leq n \rangle$ where $v_*^k = \rho_{u^i}(v^k)$ for all $k < j$, $v_*^k = v^k$ for all $k \geq j$. \square

Lemma 3.26. *Let $u \in \mathcal{U}_\infty^{\text{sup}}$ with $\text{lh}(u) > 1$, and let*

$$p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^m = u, A^m) \rangle$$

be a condition in $\mathbb{R}_u^{\text{sup}}$. Let $i < m$ with $\text{lh}(u^i) > 1$, and let Y be the set of sequences v with the following property: there exists a set B such that (v, B) is a good pair, and if we set

$$q = \langle (u^0, A^0), \dots, (u^{i-1}, A^{i-1}), (v, B), (u^i, A^i), \dots, (u^{m-1}, A^{m-1}), (u^m = u, A^m) \rangle$$

then q is a condition extending p . Then $\pi_{u^i}[Y]$ is u^i -large.

Proof. As we should expect from Lemma 3.25, we prove this by considering addability in $\mathbb{R}_u^{\text{sup}}$. To be more precise, by the argument of Lemma 3.23 there are u^i -many $\bar{v} \in A^i$ such that some (\bar{v}, B) can be interpolated between $(\pi_{u^i}(u^{i-1}), A^{i-1})$ and (u^i, A^i) in $p^{\leq i}$. For each such \bar{v} and B , if we set $v = \rho_{u^i}(\bar{v})$ then (v, B) can be interpolated between (u^{i-1}, A^{i-1}) and (u^i, A^i) in p . \square

Definition 3.27. *Given $p \in \mathbb{R}_u^{\text{sup}}$, $p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^n = u, A^n) \rangle$ and $w \in \mathcal{U}_\infty^{\text{sup}}$, we say w appears in p if and only if $w = u^i$ for some $i < n$.*

Very much as for the Radin forcing \mathbb{R}_u discussed in Section 2.2, the generic object for the supercompact Radin forcing $\mathbb{R}_u^{\text{sup}}$ can be viewed as a sequence $\langle v_\alpha : \alpha < \mu \rangle$ where $\langle v_\alpha(0) : \alpha < \mu \rangle$ is a continuous \prec -increasing sequence with union λ . As before, the condition p above carries the information that each v_α either appears among the u^i or is the first entry in a pair which can legally be added to p .

The main technical fact that we will need is the Prikry lemma for supercompact Radin forcing, which states (as usual) that every question about the generic extension can be decided by a direct extension. Before giving the proof of the Prikry lemma, we need a suitable version of normality for good measure sequences.

Definition 3.28. *Let s be a stem for $\mathbb{R}_u^{\text{sup}}$, say $s = \langle (u^0, A^0), \dots, (u^{m-1}, A^{m-1}) \rangle$, and let $w \in S(\kappa_u, \lambda_u)$. Then we define $s \prec w$ if and only if either s is empty or $u^{m-1} \prec w$.*

Definition 3.29. *Let L be a set of stems for $\mathbb{R}_u^{\text{sup}}$, and let $\langle A_s : s \in L \rangle$ be an L -indexed family of subsets of $S(\kappa_u, \lambda_u)$. Then the diagonal intersection of the family $\Delta_s A_s$ is defined to be $\{w : \forall s \prec w \ w \in A_s\}$.*

The following lemma is a form of normality for the measures on a good measure sequence.

Lemma 3.30. *Let $u \in \mathcal{U}_\infty^{\text{sup}}$ with $\text{lh}(u) > 1$ and let $0 < \beta < \text{lh}(u)$. Let L be a set of lower parts for $\mathbb{R}_u^{\text{sup}}$ and let $\langle A_s : s \in L \rangle$ be a sequence such that $A_s \in u(\beta)$ for all s . Then $\Delta_s B_s = \{w : \forall s \prec w \ w \in B_s\} \in u(\beta)$.*

Proof. As we remarked in the course of proving Lemma 3.23, if j is constructing for u then $j(w) \prec u_j \upharpoonright \beta$ for every $w \in S(\kappa_u, \lambda_u)$. It is easy to check that the converse also holds: if

$v \in j(S(\kappa_u, \lambda_u))$ and $v \prec u_j \upharpoonright \beta$ then $v = j(w)$ for some $w \in S(\kappa_u, \lambda_u)$. It is now routine to verify that $u_j \upharpoonright \beta \in j(\Delta_s B_s)$, so that $\Delta_s B_s \in u(\beta)$. \square

Corollary 3.31. *The diagonal intersection of a family of u -large sets is u -large.*

The proof of the Prikry lemma follows the usual template for proving such lemmas. We include it to make this paper more self-contained and to confirm that our definition of *good measure sequence* causes no problems.

Lemma 3.32. *Let ϕ be a sentence of the forcing language and let $p \in \mathbb{R}_u^{\text{sup}}$. Then there is $q \leq^* p$ such that q decides ϕ .*

Proof. We start by reducing to the case when the stem of the condition p is empty. To do this assume that we have the Prikry lemma for conditions with empty stems, let $p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^n = u, A^n) \rangle$ with $n > 0$, and use Lemma 3.25 with $m = n - 1$ to view the truth value of ϕ (which is an $\mathbb{R}_u^{\text{sup}} \downarrow p$ -name for an element of 2) as a $\mathbb{R}_u^{\text{sup}} \downarrow p^{>n-1}$ -name for a $\mathbb{R}_{u^{n-1}}^{\text{sup}} \downarrow p^{\leq n-1}$ -name for an element of 2. Since $\lambda_{u^{n-1}} < \kappa_u$ and $p^{>n-1}$ has an empty stem, we may shrink the measure one set in $p^{>n-1}$ to determine the value of this $\mathbb{R}_{u^{n-1}}^{\text{sup}}/p^{\leq n-1}$ -name; working downwards and repeating the argument a further n times, we end with the conclusion of the Prikry lemma.

So we let $p = \langle (u, A) \rangle$. Let I be the set of stems s such that there is $B \subseteq A$ with $s \frown \langle (u, B) \rangle$ deciding ϕ , and for each $s \in I$ let A_s be such a set B . Let $A_1 = \Delta_{s \in I} A_s$, then easily A_1 has the following property: for every stem s , if there exists $B \subseteq A$ such that $s \frown \langle (u, B) \rangle$ decides ϕ , then $s \frown (u, A_1)$ decides ϕ .

Now for each stem s , we partition A_1 into three parts: $A_{1,0}^s$ is the set of $w \in A_1$ such that for some w -large $B \subseteq V_{\kappa_w}$

$$s \frown \langle (w, B) \rangle \frown \langle (u, A_1) \rangle \Vdash \phi,$$

$A_{1,1}^s$ is the set of $w \in A_1$ such that for some w -large B the condition

$$s \frown \langle (w, B) \rangle \frown \langle (u, A_1) \rangle \Vdash \neg \phi,$$

and $A_{1,2}^s$ is the remainder of A_1 . For each α with $0 < \alpha < \text{lh}(u)$, let B_s^α be $A_{1,n}^s$ for the unique n such that $A_{1,n}^s \in u(\alpha)$, let $B^\alpha = \Delta_s B_s^\alpha$, and let $A_2 = \bigcup_\alpha B^\alpha$. By construction A_2

has the following property: if

$$s \frown \langle (w, B) \rangle \frown \langle (u, C) \rangle \leq s \frown \langle (u, A_2) \rangle$$

is a condition which forces ϕ , then there exists α such that

$$\{w' : \exists B' s \frown \langle (w', B') \rangle \frown \langle (u, A_1) \rangle \text{ forces } \phi\}$$

is $u(\alpha)$ -large, and similarly for $\neg\phi$.

To finish the argument, we fix a condition $t \frown \langle (u, C) \rangle \leq \langle (u, A_2) \rangle$ which decides ϕ with $\text{lh}(t)$ minimal, and argue that t must be empty. If not let $t = s \frown \langle (w, B) \rangle$, and assume (without loss of generality) that $s \frown \langle (w, B) \rangle \frown \langle (u, C) \rangle$ forces ϕ . By construction we find α and a function f with $\text{dom}(f) \in u(\alpha)$ such that $s \frown \langle (v, f(v)) \rangle \frown \langle (u, A_2) \rangle$ forces ϕ for every $v \in \text{dom}(f)$.

We will now construct a set $A_3 \subseteq A_2$ such that every extension of $s \frown \langle (u, A_3) \rangle$ is compatible with some condition of the form $s \frown \langle (v, f(v)) \rangle \frown \langle (u, A_2) \rangle$. This property implies that $s \frown \langle (u, A_3) \rangle$ forces ϕ , contradicting the minimal choice of $\text{lh}(t)$. We note that by the definition of extension in the forcing $\mathbb{R}_u^{\text{sup}}$, we may assume from this point on that s is empty.

We define various subsets of A_2 :

- For $v' \in A_2$, $Y(v')$ is the set of $v \in \text{dom}(f) \cap A_2$ such that $v' \prec v$, $\pi_v(v') \in f(v)$, and $\{x \in S(\kappa_{v'}, \lambda_{v'}) : \sigma_{v'v}(x) \in f(v)\}$ is v' -large.
- X is the set of $v' \in A_2$ such that $Y(v')$ is $u(\alpha)$ -large.
- Y is the diagonal intersection $\Delta_{v' \in X} Y(v')$.
- Z is the set of $w \in A_2$ such that $\{x \in S(\kappa_w, \lambda_w) : \rho_w(x) \in Y\}$ is $w(\beta)$ -large for some β with $0 < \beta < \text{lh}(w)$.

Let $A_3 = X \cup Y \cup Z$. We will verify that A_3 is u -large.

Claim 1. X is $u \upharpoonright \alpha$ -large.

Proof. Let $j : V \rightarrow M$ be constructing for u , let $B = j(f)(u_j \upharpoonright \alpha)$. Applying Lemma 3.26 in M to the condition $\langle (u_j \upharpoonright \alpha, B), (j(u), j(A_2)) \rangle$ we obtain exactly the conclusion that there are $u \upharpoonright \alpha$ -many v' such that $u_j \upharpoonright \alpha \in j(Y(v'))$. \square

Claim 2. Y is $u(\alpha)$ -large.

Proof. Immediate by Lemma 3.30. \square

Claim 3. Z is $u(\beta)$ -large for all β with $\alpha < \beta < \text{lh}(w)$.

Proof. : We verify that $u_j \upharpoonright \beta \in j(Z)$. As we saw in the proof of Lemma 3.23, $\rho_{u_j \upharpoonright \beta} = j \upharpoonright \lambda_u$, and $\rho_{u_j \upharpoonright \beta}(x) = j(x)$ for $x \in S(\kappa_u, \lambda_u)$, so that $\{x \in S(\kappa_u, \lambda_u) : \rho_{u_j \upharpoonright \beta}(x) \in j(Y)\} = Y$. Since $\alpha < \beta$ and Y is $u(\alpha)$ -large, we are done. \square

Now consider an arbitrary extension

$$q = \langle (u^0, A^0), \dots, (u^{m-1}, A^{m-1}), (u^m = u, A^m) \rangle$$

of $\langle (u, A_3) \rangle$. There are various cases: the first case is the most important one, in that we handle the other cases by making a further extension to get into the first case.

- (1) There exists j such that $u^j \in Y$ and $u^i \in X$ for all $i < j$. In this case we can readily verify that the condition $\langle (u^j, f(u^j)), (u, A_2) \rangle$ is compatible with q : the main point to check is that each pair $(u^i, A^i \cap f(u^j))$ for $i < j$ can legally be added below $(u^j, f(u^j))$, which is immediate from the definitions of X and Y .
- (2) For the least j such that $u^j \notin X$, $u^j \in Z$. In this case by Lemma 3.26 we can interpolate some sequence in Y between u^{j-1} and u^j , and reduce to the first case.
- (3) $u^j \in X$ for all j . In this case we can interpolate some sequence in Y between u^{m-1} and u , and again reduce to the first case.

\square

The next result collects some useful information about the extension by $\mathbb{R}_u^{\text{sup}}$ where u is a good (κ, λ) -measure sequence.

Lemma 3.33. *Let $u \in \mathcal{U}_\infty^{\text{sup}}$ be a (κ, λ) -measure sequence with $\text{lh}(u) > 1$, and let G be $\mathbb{R}_u^{\text{sup}}$ -generic over V . The following hold in $V[G]$:*

- (1) *Let $\vec{w} = \langle w_\alpha : \alpha < \mu \rangle$ enumerate $\{w : w \text{ appears in some } p \in G\}$, so that we have $w_\alpha \prec w_\beta$ for all α, β with $\alpha < \beta < \mu$. Then μ is a limit ordinal with $\mu \leq \kappa_u$, and $V[G] = V[\vec{w}]$.*
- (2) *Let $\vec{C} = \langle w_\alpha(0) : \alpha < \mu \rangle$. Then \vec{C} is a \prec -increasing and continuous sequence in $P_{\kappa_u}(\lambda_u)$. Furthermore if $\text{lh}(u) \geq \kappa_u$, then $\mu = \kappa_u$.*

- (3) For each $\alpha < \mu$, $\kappa_{w_\alpha} < \lambda_{w_\alpha} < \kappa_{w_{\alpha+1}}$.
- (4) $\lambda = \bigcup_{\alpha < \mu} w_\alpha(0)$, in particular if $\lambda > \kappa$ then λ is collapsed.
- (5) If we let $D = \{\kappa_{w_\alpha(0)} : \alpha < \mu\}$ then D is a club subset of κ .
- (6) For every limit ordinal $\beta < \mu$, $\langle w_\alpha^* : \alpha < \beta \rangle$ is $\mathbb{R}_{w_\beta}^{\text{sup}}$ -generic over V , where $w_\alpha^* = \pi_{w_\beta}(w_\alpha)$, and $\vec{w} \upharpoonright [\beta, \mu)$ is $\mathbb{R}_u^{\text{sup}}$ -generic over $V[\langle w_\alpha^* : \alpha < \beta \rangle]$.
- (7) For every $\gamma < \kappa$ and $A \subseteq \gamma$ with $A \in V[\vec{w}]$, $A \in V[\langle w_\alpha^* : \alpha < \beta \rangle]$. where $\beta < \mu$ is the largest ordinal such that $\kappa_{w_\beta} \leq \gamma$.
- (8) If γ is a cardinal in V such that $\kappa_{w_\beta} < \gamma \leq \lambda_{w_\beta}$ for some limit $\beta < \mu$, then γ is collapsed and in fact $V[G] \models |\gamma| = \kappa_{w_\beta}$. Other cardinals are preserved.

Proof. We will prove each claim in turn.

- (1) As we noted above, G is the set of conditions such that each w_α appears either in p or in some extension of p . It follows that $V[G] = V[\vec{w}]$. By Lemma 3.23 and an easy density argument, μ must be a limit ordinal.
- (2) It follows from the definition of the forcing that \vec{C} is \prec -increasing, and from Lemmas 3.23 and 3.26 that \vec{C} is continuous at every limit ordinal. An easy induction argument shows that $\mu \geq \omega^{\text{lh}(u)-1}$, in particular if $\text{lh}(u) \geq \kappa_u$ then necessarily $\mu = \kappa_u$.
- (3) We have that w_α and $w_{\alpha+1}$ are good measure sequences with $w_\alpha \prec w_{\alpha+1}$, and the conclusion is immediate.
- (4) It follows from Lemma 3.23 that \vec{C} is cofinal in $(A(\kappa_u, \lambda_u), \prec)$, so in particular its union is λ_u .
- (5) This follows from Lemmas 3.23 and 3.26.
- (6) This is immediate from Lemma 3.25.
- (7) This follows from Lemma 3.25 and Lemma 3.32, together with the remark that for each α the measures in w_α are κ_{w_α} -complete.
- (8) Since $\langle w_\alpha^* : \alpha < \beta \rangle$ is $\mathbb{R}_{w_\beta}^{\text{sup}}$ -generic over V , the cardinal γ is collapsed as claimed. Preservation of other cardinals follows from the analysis of bounded subsets of κ in $V[G]$, and the fact that $\mathbb{R}_{w_\beta}^{\text{sup}}$ is $\lambda_{w_\beta}^+$ -c.c. for all limit β .

□

Lemma 3.34. *Let $v \in \mathcal{U}_\infty^{\text{sup}}$ be a (κ, λ) -measure sequence constructed from a suitable embedding j . Assume that v has a weak repeat point α and j witnesses that κ is μ -supercompact for some $\mu > \lambda$. Let $u = v \upharpoonright \alpha$ and let G be $\mathbb{R}_u^{\text{sup}}$ -generic over V . Then in $V[G]$, κ remains μ -supercompact.*

Proof. We prove the lemma in a sequence of claims.

Claim 1. *If $A \in u(\beta)$ for all β with $0 < \beta < \alpha$, then $u \in j(A)$.*

Proof. It suffices to show that A is $v(\alpha)$ -large. Suppose not. Then $A^c = S(\kappa, \lambda) \setminus A \in v(\alpha)$, so for some β with $0 < \beta < \alpha$, $A^c \in v(\beta) = u(\beta)$ which is in contradiction with $A \in u(\beta)$. \square

Note that any condition $p \in \mathbb{R}_u^{\text{sup}}$ is of the form $p_d \hat{\wedge} \langle (u, A) \rangle$, for some unique p_d and A .

By Claim 1 $u \in j(A)$, so we can form the condition

$$p^* = q_d \hat{\wedge} \langle (u, A) \rangle \hat{\wedge} \langle (j(u), j(A)) \rangle \in j(\mathbb{R}_u^{\text{sup}}),$$

where q_d is obtained from p_d by type changing to make the above condition well-defined, which is to say (arguing as in Lemma 3.23) that $q_d = j(p_d)$.

The following can be proved easily.

Claim 2. *$p^* \leq j(p)$ in $j(\mathbb{R}_u^{\text{sup}})$, and if $p \leq q$ then $p^* \leq q^*$.*

Given any condition $p = p_d \hat{\wedge} \langle (u, A) \rangle \in \mathbb{R}_u$ and any $j(u)$ -large set E , set

$$p^* \upharpoonright E = q_d \hat{\wedge} \langle (u, A) \rangle \hat{\wedge} \langle (j(u), j(A) \cap E) \rangle.$$

We now define U on $P_\kappa \mu$ as follows: $X \in U$ if and only if there exist $p \in G$ and E which is $j(u)$ -large such that $p^* \upharpoonright E \Vdash j[\mu] \in j(\dot{X})$, where \dot{X} is a name for X .

Claim 3. *The above definition of U does not depend on the choice of the name \dot{X} .*

Proof. Suppose that $i_G(\dot{X}_1) = i_G(\dot{X}_2)$ and that $p^* \upharpoonright E \Vdash j[\mu] \in j(\dot{X}_1)$. Strengthening p if necessary we may assume that $p \Vdash \dot{X}_1 = \dot{X}_2$, so that by elementarity $j(p) \Vdash j(\dot{X}_1) = j(\dot{X}_2)$. Now $p^* \upharpoonright E \leq p^* \leq j(p)$, so that $p^* \upharpoonright E \Vdash j[\mu] \in j(\dot{X}_2)$. \square

We show that U is a normal measure on $P_\kappa \mu$. U is easily seen to be a filter.

Claim 4. U is an ultrafilter.

Proof. Let \dot{X} name a subset X of $P_\kappa\mu$. Appealing to Lemmas 3.32 and Lemma 3.25, we may find E such that the condition $\langle (u, S(\kappa, \lambda)), (j(u), E) \rangle$ forces that the truth value of $j[\mu] \in j(\dot{X})$ is equal to a truth value for $\mathbb{R}_u^{\text{sup}}$. Hence we may find $p \in G$ such that $p^* \upharpoonright E$ decides $j[\mu] \in j(\dot{X})$, from which it follows that either $X \in U$ or $X^c \in U$. \square

Claim 5. U is fine.

Proof. Suppose that $\alpha < \mu$, $X = \{x \in P_\kappa(\mu) : \alpha \in x\}$ and that $p \in \mathbb{R}_u^{\text{sup}}$. It is clear that $p^* \Vdash j[\mu] \in j(\dot{X})$, so p forces $\dot{X} \in \dot{U}$. \square

Claim 6. U is normal.

Proof. Let $F : P_\kappa\mu \rightarrow \lambda$ be regressive, that is $F(x) \in x$ for all non-empty $x \in P_\kappa\mu$, and let \dot{F} name F . Appealing to Lemmas 3.32 and Lemma 3.25, together with the facts that $\mu < j(\kappa)$ and the measures in U are $j(\kappa)$ -complete, we may find E such that the condition $\langle (u, S(\kappa, \lambda)), (j(u), E) \rangle$ forces that for every $\alpha \in \mu$ the truth value of $j(\dot{F})(j[\mu]) = j(\alpha)$ is equal to a truth value for $\mathbb{R}_u^{\text{sup}}$. Hence we may find $p \in G$ and $\alpha < \mu$ such that $p^* \upharpoonright E \Vdash j(F)(j[\mu]) = j(\alpha)$, from which we see that $\{x : F(x) = \alpha\} \in U$. \square

Claim 7. U is κ -complete.

Proof. This follows by a similar argument to the one we gave for normality in Claim 6. \square

It follows that U is a normal measure on $P_\kappa\mu$, and the lemma follows. \square

4. PROJECTED FORCING

As we saw in the last section, if $u \in \mathcal{U}_\infty^{\text{sup}}$ is a (κ, λ) -measure sequence and G is generic for $\mathbb{R}_u^{\text{sup}}$, then $(\alpha^+)^V < (\alpha^+)^{V[G]}$ for a closed unbounded set of cardinals $\alpha < \kappa$. Using a sequence with a repeat point, we may also arrange that κ is a large cardinal in $V[G]$. We wish to find a submodel $V[G^\phi]$ of $V[G]$ such that $V[G^\phi]$ is a cardinal-preserving extension of V , and $HOD^{V[G]} \subseteq V[G^\phi]$.

It will be technically convenient, and sufficient for the intended application, to assume from this point on that $\lambda = \kappa^+$. The measures in a good (κ, κ^+) -measure sequence will

concentrate on a certain subset of $S(\kappa, \kappa^+)$, namely $\{w \in S(\kappa, \kappa^+) : \lambda_w = \kappa_w^+\}$. It follows that by working below a suitable condition in $\mathbb{R}_u^{\text{sup}}$, we may assume that $\lambda_v = \kappa_v^+$ for every v appearing on the generic sequence.

In the interests of notational simplicity, we prefer to make a slight modification in certain definitions. From this point on we let $A(\kappa, \kappa^+)$ be the set of $x \in P_\kappa \kappa^+$ such that (as before) $\kappa_x = x \cap \kappa \in \kappa$ and κ_x is inaccessible, and (modified) $\lambda_x = \text{ot}(x) = \kappa_x^+$. We then modify the definitions of $S(\kappa, \kappa^+)$ (Definition 3.3), (κ, κ^+) measure sequence (Definition 3.5) good (κ, κ^+) -measure sequence (Definition 3.8), and $\mathbb{R}_u^{\text{sup}}$ in case u is a good (κ, κ^+) measure sequence (Definition 3.20) accordingly.

We will obtain $V[G^\phi]$ by defining a *projected sequence* $\phi(u)$, a *projected forcing* $\mathbb{R}_{\phi(u)}^{\text{proj}}$, and an order-preserving map ϕ from $\mathbb{R}_u^{\text{sup}}$ to $\mathbb{R}_{\phi(u)}^{\text{proj}}$, then arguing that $\phi[G]$ generates a $\mathbb{R}_{\phi(u)}^{\text{proj}}$ -generic filter G^ϕ . We note that our projected forcing is rather different from the parallel construction of Foreman and Woodin [2]. The reason is that we need our projected forcing to be as close to the supercompact Radin forcing as possible, so that the quotient forcing is sufficiently homogeneous.

Given any $u \in \mathcal{U}_\infty^{\text{sup}}$ we first define $\phi(u)$, and then we will define the projected forcing $\mathbb{R}_{\phi(u)}^{\text{proj}}$.

Definition 4.1. *Suppose (u, A) is a good pair.*

- $\phi(u) = \kappa_u \widehat{\langle} u(\zeta) : 0 < \zeta < \text{lh}(u) \rangle$.
- $\phi(A) = \{\phi(v) : v \in A\}$.

Also let $\mathcal{U}_\infty^{\text{proj}} = \{\phi(u) : u \in \mathcal{U}_\infty^{\text{sup}}\}$, and for $w \in \mathcal{U}_\infty^{\text{proj}}$ let $\kappa_w = w(0)$. For $u \in \mathcal{U}_\infty^{\text{sup}}$ and $0 < \alpha < \text{lh}(u)$, let $\phi(u(\alpha))$ be the Rudin-Keisler projection of $u(\alpha)$ via the map ϕ ; similarly if $w \in \mathcal{U}_\infty^{\text{proj}}$ and $0 < \alpha < \text{lh}(w)$, let $\phi(w(\alpha))$ be the Rudin-Keisler projection of $w(\alpha)$ via the map ϕ ;

Note that $\phi(u(\alpha)) \neq \phi(u)(\alpha) = u(\alpha)$, because the former is a measure on V_κ and the latter is a measure on $S(\kappa, \kappa^+)$. In fact $\phi(u(\alpha)) = \phi(\phi(u)(\alpha))$.

Definition 4.2. *Let $w, w' \in \mathcal{U}_\infty^{\text{proj}}$. Then $w \prec w'$ if and only if $w \in V_{\kappa_{w'}}$.*

It is routine to check that if $u, v \in \mathcal{U}_\infty^{\text{sup}}$ with $u \prec v$, then $\phi(u) \prec \phi(v)$.

Definition 4.3. A good pair for projected forcing is a pair (w, B) where $w \in \mathcal{U}_\infty^{\text{proj}}$, $B \subseteq \mathcal{U}_\infty^{\text{proj}} \cap V_{\kappa_w}$ and $B \in \phi(w(\alpha))$ for all α with $0 < \alpha < \text{lh}(w)$.

Remark 4.4. If (u, A) is a good pair, then $(\phi(u), \phi(A))$ is a good pair for projected forcing.

Given $w \in \mathcal{U}_\infty^{\text{proj}}$, we define the projected forcing $\mathbb{R}_w^{\text{proj}}$.

Definition 4.5. Let $w \in \mathcal{U}_\infty^{\text{proj}}$. A condition in $\mathbb{R}_w^{\text{proj}}$ is a finite sequence

$$p = \langle (w^0, B^0), \dots, (w^i, B^i), \dots, (w^m, B^m) \rangle$$

where:

- (1) $w^m = w$.
- (2) Each (w^i, B^i) is a good pair for projected forcing.
- (3) $w_i \prec w_{i+1}$, for all $i < m$.

We now define the extension relation.

Definition 4.6. Let

$$p = \langle (w^0, B^0), \dots, (w^i, B^i), \dots, (w^m = w, B^m) \rangle$$

and

$$q = \langle (x^0, C^0), \dots, (x^j, C^j), \dots, (x^n = w, C^n) \rangle$$

be in $\mathbb{R}_w^{\text{proj}}$. Then $q \leq p$ (q is an extension of p) if and only if:

- (1) There exists an increasing sequence of natural numbers $j_0 < \dots < j_m = n$ such that $x^{j_k} = w^k$ and $C^{j_k} \subseteq B^k$.
- (2) If j is such that $0 \leq j \leq n$ and $j \notin \{j_0, \dots, j_m\}$, and if i is least such that $\kappa_{x^j} < \kappa_{w^i}$, then $x^j \in B^i$ and $C^j \subseteq B^i$.

We also define $q \leq^* p$ (q is a direct extension of p) iff $q \leq p$ and $m = n$.

As motivation, we consider the special case when $w = \phi(u)$ and $\text{lh}(u) = 3$. Forcing with $\mathbb{R}_w^{\text{proj}}$ below a suitable condition we will obtain a generic object $\vec{w} = \langle w_\alpha : \alpha < \omega^2 \rangle$ where:

- (1) $w_\alpha = \langle \kappa_\alpha \rangle$ for $\alpha = 0$ or a successor ordinal, $w_\alpha = \langle \kappa_\alpha, U_\alpha \rangle$ for α a limit ordinal.
- (2) $\langle \kappa_\alpha : \alpha < \omega^2 \rangle$ is increasing and cofinal in κ_u .

(3) U_α is a κ_α^+ -supercompactness measure on $P_{\kappa_\alpha} \kappa_\alpha^+$.

As we see below, the generic extension $V[\bar{w}]$ preserves cardinals and can be viewed as a submodel of an extension $V[\bar{u}]$, where $\bar{u} = \langle u_\alpha : \alpha < \omega^2 \rangle$ and $w_\alpha = \phi(u_\alpha)$ for each α . The key idea is that the model $V[\bar{w}]$ “remembers” the definitions of the forcing posets $\mathbb{R}_{u_\alpha}^{\text{sup}}$ for each limit α , and retains enough information to singularise κ_α for each such α , but “forgot” the collapsing information that was present in the entries $u_\beta(0)$.

The theory of $\mathbb{R}_w^{\text{proj}}$ is very similar to that of $\mathbb{R}_u^{\text{sup}}$, but the statements and proofs are simpler because there is no need for the “type changing” maps. For example the following result is the analog of Lemmas 3.23 and 3.26 for $\mathbb{R}_w^{\text{proj}}$, and can be proved in exactly the same way.

Lemma 4.7. *Let $w \in \mathcal{U}_\infty^{\text{proj}}$ with $\text{lh}(w) > 1$, and let*

$$p = \langle (w^0, B^0), \dots, (w^i, B^i), \dots, (w^m = w, B^m) \rangle$$

be a condition in $\mathbb{R}_w^{\text{proj}}$. Let $i \leq m$ with $\text{lh}(w^i) > 1$, and let Y be the set of sequences v with the following property: there exists a set B such that (v, B) is a good pair, and if we set

$$q = \langle (w^0, B^0), \dots, (w^{i-1}, B^{i-1}), (v, B), (w^i, B^i), \dots, (w^{m-1}, B^{m-1}), (w^m = w, B^m) \rangle$$

then q is a condition extending p . Then Y is w^i -large.

Definition 4.8. *If $p \in \mathbb{R}_w^{\text{proj}}$ with $p = \langle (w^0, B^0), \dots, (w^i, B^i), \dots, (w^m = w, B^m) \rangle$, then x appears in p if and only if $x = w^i$ for some $i < m$.*

It is easy to see that $\mathbb{R}_w^{\text{proj}}$ satisfies the κ_w^+ -c.c.

Lemma 4.9. *Let G be $\mathbb{R}_w^{\text{proj}}$ -generic over V , and let*

$$D = \{\kappa_x : x \text{ appears in some } p \in G\}.$$

Then D is a club of κ_w . Furthermore if $\text{lh}(w) \geq \kappa_w$, then $\text{ot}(D) = \kappa_w$.

Proof. This is an easy consequence of Lemma 4.7. □

As before we have a factorization property for $\mathbb{R}_w^{\text{proj}}$.

Lemma 4.10. *Suppose that*

$$p = \langle (w^0, B^0), \dots, (w^i, B^i), \dots, (w^m = w, B^m) \rangle \in \mathbb{R}_w^{\text{proj}}$$

and $i < m$ with $\text{lh}(w^i) > 1$. Let

$$p^{>i} = \langle (w^{i+1}, B_*^{i+1}), \dots, (w^m = w, B_*^m) \rangle,$$

where $B_*^j = B^j$ for all $j > i + 1$ while $B_*^{i+1} = \{w \in B^{i+1} : w^i \prec w\}$. Let

$$p^{\leq i} = \langle (w^0, B^0), \dots, (w^{i-1}, B^{i-1}), (w^i, B^i) \rangle.$$

Then $p^{\leq i} \in \mathbb{R}_{w^i}^{\text{proj}}$, $p^{>i} \in \mathbb{R}_w^{\text{proj}}$ and there exists

$$i : \mathbb{R}_{w^i}^{\text{proj}} \downarrow p^{\leq i} \times \mathbb{R}_w^{\text{proj}} \downarrow p^{>i} \rightarrow \mathbb{R}_w^{\text{proj}}$$

which is an isomorphism between its domain and a dense subset of $\mathbb{R}_w^{\text{proj}}$.

Proof. Let $q_0 \leq p^{\leq i}$ and $q_1 \leq p^{>i}$, say $q_0 = \langle (v^0, B^0), \dots, (v^j = w^i, B^j) \rangle$ and $q_1 = \langle (v^{j+1}, B^{j+1}), \dots, (v^n = w, B^n) \rangle$. Define $i(q_0, q_1) = \langle (v^j, B^j) : 0 \leq j \leq n \rangle$. \square

Lemma 4.11. *Let $w \in \mathcal{U}_\infty^{\text{proj}}$ with $\text{lh}(w) > 1$ and let $0 < \beta < \text{lh}(w)$, let L be a set of lower parts for $\mathbb{R}_w^{\text{proj}}$ and let $\langle A_s : s \in L \rangle$ be a sequence such that $A_s \in \phi(w(\beta))$ for all s . Then $\Delta_s B_s = \{w : \forall s \prec w \ w \in B_s\} \in \phi(w(\beta))$.*

Proof. Exactly like the proof of Lemma 3.30. \square

Lemma 4.12. *$\mathbb{R}_w^{\text{proj}}$ satisfies the Prikry property, that is to say every sentence of the forcing language can be decided by a direct extension.*

Proof. The proof is like the proof of Lemma 3.32, using Lemma 4.11 and the fact that $w = \phi(u)$ for some good sequence u with a constructing embedding. \square

We sketch an alternative proof for Lemma 4.12 following the proof of Lemma 4.16 in the next section.

Lemma 4.13. *Let $w \in \mathcal{U}_\infty^{\text{proj}}$ with $\text{lh}(w) > 1$, and let G be $\mathbb{R}_w^{\text{proj}}$ -generic over V .*

- (1) *Let $\vec{x} = \langle x_\alpha : \alpha < \mu \rangle$ enumerate $\{x : x \text{ appears in some } p \in G\}$ so that $x_\alpha \prec x_\beta$ for all α, β with $\alpha < \beta < \mu$. Then μ is a limit ordinal with $\mu \leq \kappa_w$, and $V[G] = V[\vec{x}]$.*

- (2) Let $D = \{x_\alpha(0) : \alpha < \mu\}$. Then D is a club subset of κ_w .
- (3) For every limit ordinal $\beta < \mu$, $\vec{x} \upharpoonright \beta$ is $\mathbb{R}_{x_\beta}^{\text{proj}}$ -generic over V , and $\vec{x} \upharpoonright [\beta, \mu)$ is $\mathbb{R}_w^{\text{proj}}$ -generic over $V[\vec{x} \upharpoonright \beta]$.
- (4) Suppose $\gamma < \kappa$, $A \subseteq \gamma$, $A \in V[\vec{x}]$. Let $\beta < \mu$ be the largest ordinal such that $\kappa_{x_\beta} \leq \gamma$. Then $A \in V[\vec{x} \upharpoonright \beta]$.
- (5) The models V and $V[G]$ have the same cardinals.

Proof. The proof is like the proof of Lemma 3.33, using Lemma 4.12. \square

Lemma 4.14. *Let GCH hold and let $j : V \rightarrow M$ witness κ is μ -supercompact for some $\mu \geq \kappa^{+3}$. Let $v \in \mathcal{U}_\infty^{\text{sup}}$ be a (κ, κ^+) -measure sequence constructed from j which has a weak repeat point α . Let $u = v \upharpoonright \alpha$ and let G^ϕ be $\mathbb{R}_{\phi(u)}^{\text{proj}}$ -generic over V . Then in $V[G^\phi]$, κ remains μ -supercompact.*

Proof. The proof is similar to that of Lemma 3.34, using Lemmas 4.10 and 4.12. \square

4.1. Weak projection. Suppose that $u \in \mathcal{U}_\infty^{\text{sup}}$ and consider the forcing notions $\mathbb{R}_u^{\text{sup}}$ and $\mathbb{R}_{\phi(u)}^{\text{proj}}$. We define a map $\phi : \mathbb{R}_u^{\text{sup}} \rightarrow \mathbb{R}_{\phi(u)}^{\text{proj}}$ in the natural way, by

$$(1) \quad \phi : \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^n, A^n) \rangle \mapsto \langle (\phi(u^0), \phi(A^0)), \dots, (\phi(u^i), \phi(A^i)), \dots, (\phi(u^n), \phi(A^n)) \rangle$$

The map ϕ is possibly not a projection in the classical sense. The problem (in a representative special case) is that when we extend a condition $\phi(p)$ by drawing a \prec -increasing sequence $w^0 \prec \dots \prec w^n$ from $\phi(A)$ where (u, A) is the last pair in p , so that each sequence w^i is of form $\phi(v)$ for some $v \in A$, there may not exist a \prec -increasing sequence $v^0 \prec \dots \prec v^n$ from A with $\phi(v^i) = w^i$.

We will show that ϕ has a weaker property, introduced by Foreman and Woodin [2], which is sufficient for our purposes.

Definition 4.15. *Let \mathbb{P} and \mathbb{Q} be forcing posets. $\psi : \mathbb{P} \rightarrow \mathbb{Q}$ is a weak projection if and only if ψ is order preserving and for all $p \in \mathbb{P}$ there is $p^* \leq p$ such that for all $q \leq \psi(p^*)$ there exists $p' \leq p$ such that $\psi(p') \leq q$.*

It is easy to see that if $\psi : \mathbb{P} \rightarrow \mathbb{Q}$ is a weak projection and G is a \mathbb{P} -generic filter, then $\psi[G]$ generates a \mathbb{Q} -generic filter.

Lemma 4.16. $\phi : \mathbb{R}_u^{\text{sup}} \rightarrow \mathbb{R}_{\phi(u)}^{\text{proj}}$ is a weak projection, and in fact satisfies a stronger property: for all $p \in \mathbb{R}_u^{\text{sup}}$ there is $p^* \leq^* p$ such that for all $q \leq \phi(p^*)$ there is $p' \leq p$ such that $\phi(p') \leq^* q$.

Proof. It is easily seen that ϕ preserves both the ordering \leq and the direct extension ordering \leq^* .

Claim 1. Suppose that (u, A) is a good pair. Let A' be the set of $w \in A$ such that for all $x \in A$ with $\phi(x) \prec \phi(w)$ there is $\bar{x} \in A$ such that $\phi(\bar{x}) = \phi(x)$ and $\bar{x} \prec w$. Then A' is u -large.

Proof. Suppose that $j : V \rightarrow M$ constructs u and let $0 < \alpha < \text{lh}(u)$. We need to show that $A' \in u(\alpha)$ or equivalently $u_j \upharpoonright \alpha \in j(A')$. Thus we need to prove the following:

(*) If $x \in j(A)$ and $\phi(x) \prec \phi(u_j \upharpoonright \alpha)$, then there is $\bar{x} \in j(A)$ such that $\phi(\bar{x}) = \phi(x)$ and $\bar{x} \prec u_j \upharpoonright \alpha$.

Since $\phi(x) \prec \phi(u_j \upharpoonright \alpha)$, $\phi(x) \in V_{\kappa_u}$, and it follows that we may choose $\bar{x} \in \text{rge}(j)$ such that $\bar{x} \in j(A)$ and $\phi(\bar{x}) = \phi(x)$. Since $\bar{x} \in \text{rge}(j)$ we have $\bar{x} = j(z)$ for some $z \in A$, and then as in Lemma 3.23 we have that $\bar{x} \prec u_j \upharpoonright \alpha$.

□

We are now ready to complete the proof of Lemma 4.16 by showing that for all $p \in \mathbb{R}_u^{\text{sup}}$ there is $p^* \leq^* p$ such that for all $q \leq \phi(p^*)$ there is $p' \leq p$ such that $\phi(p') \leq^* q$. Using the factorization property from Lemma 3.25, it is sufficient to prove this for the special case where p has the form $\langle (u, A) \rangle$.

Let us say that a sequence w is *addable* to the good pair (u, A) if there exists B such that $\langle (w, B), (u, A) \rangle$ is a condition extending (u, A) in $\mathbb{R}_u^{\text{sup}}$. By Lemma 3.24 we may assume, without loss of generality, that every $v \in A$ is addable to (u, A) .

We will proceed by iterating the map $A \mapsto A'$ from Claim 1. Let $A^{(\omega)} = \bigcap_{n < \omega} A^{(n)}$ where $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})'$ for all $n < \omega$. We will show that $p^* = \langle (u, A^{(\omega)}) \rangle$ is

as required. Let

$$q = \langle (w^0, B^0), \dots, (w^i, B^i), \dots, (w^n, B^n) \rangle \in \mathbb{R}_{\phi(u)}^{\text{proj}},$$

with $q \leq \phi(p^*)$. We will find a sequence of good pairs (u^i, C^i) for $0 \leq i \leq n$, such that:

- (1) $u^n = u$.
- (2) $u^0 \prec u^1 \prec \dots \prec u^{n-1}$.
- (3) $u^i \in S(\kappa_{u^n}, \lambda_{u^n})$.
- (4) $\phi(u^i) = w^i$ and $\phi(C^i) \subseteq B^i$ for $0 \leq i \leq n$.
- (5) $u^i \in A^{(i)}$ for $i < n$.
- (6) $\langle (u^i, C^i), (u, A) \rangle \leq p = \langle (u, A) \rangle$

We choose (u^{n-i}, C^{n-i}) by induction on i for $0 \leq i \leq n$. For $i = 0$ let $u^n = u$ and let $C^n \subseteq B$ be such that $\phi(C^n) \subseteq B^n$. For $i = 1$ choose any $v \in A^{(\omega)}$ such that $\phi(v) = w^{n-1}$, and note that $v \in A^{(n-1)}$ and v is addable to (u, A) ; set $u^{n-1} = v$ and choose C^{n-1} such that $\langle (u^{n-1}, C^{n-1}), (u, A) \rangle \leq \langle (u, A) \rangle$ and $\phi(C^{n-1}) \subseteq B^{n-1}$.

Suppose now that $1 < i < n$ and we have chosen (u^{n-i}, C^{n-i}) . Choose $v \in A^{(\omega)}$ such that $\phi(v) = w^{n-i-1}$. Since $u^{n-i} \in A^{(n-i)} = (A^{(n-i-1)})'$ and $v \in A^{n-i-1}$, we can find $u^{n-i-1} \in A^{n-i-1}$ such that $\phi(u^{n-i-1}) = w^{n-i-1}$ and $u^{n-i-1} \prec u^{n-i}$. Since $u^{n-i-1} \in A$ it is addable to (u, A) , hence we may find C^{n-i-1} such that $\langle (u^{n-i-1}, C^{n-i-1}), (u, A) \rangle \leq \langle (u, A) \rangle$ and $\phi(C^{n-i-1}) \subseteq B^{n-i-1}$.

Let $p' = \langle (u^0, C^0), \dots, (u^i, C^i), \dots, (u^n, C^n) \rangle$. Then $p' \in \mathbb{R}_u^{\text{sup}}$, $p' \leq p$ and $\phi(p') \leq^* q$.

□

Lemma 4.16 allows us to give an alternative proof of Lemma 4.12, the Prikry lemma for $\mathbb{R}_w^{\text{proj}}$. We choose some u such that $\phi(u) = w$ and argue that $\phi[\mathbb{R}_u^{\text{sup}}]$ is dense in $\mathbb{R}_{\phi(u)}^{\text{proj}}$ under the direct extension relation \leq^* . Since ϕ preserves the direct extension relation, it is now fairly straightforward to argue that the Prikry lemma for $\mathbb{R}_u^{\text{sup}}$ implies the Prikry lemma for $\mathbb{R}_{\phi(u)}^{\text{proj}}$.

Remark 4.17. If $\langle u^i : i < \mu \rangle$ is the generic sequence for $\mathbb{R}_u^{\text{sup}}$ added by G , then it is easy to see that $\langle \phi(u^i) : i < \mu \rangle$ is the generic sequence for $\mathbb{R}_{\phi(u)}^{\text{proj}}$ added by G^ϕ .

4.2. Homogeneity property.

Lemma 4.18. *Let $u \in \mathcal{U}_\infty^{\text{sup}}$. For all $p, q \in \mathbb{R}_u^{\text{sup}}$, if $\phi(p)$ and $\phi(q)$ are compatible in the \leq^* ordering, then there exist $p^* \leq^* p$ and $q^* \leq^* q$ such that $\mathbb{R}_u^{\text{sup}} \downarrow p^* \simeq \mathbb{R}_u^{\text{sup}} \downarrow q^*$.*

Proof. We give the proof in a sequence of claims. We will ultimately induce an isomorphism between the cones $\mathbb{R}_u^{\text{sup}} \downarrow p^*$ and $\mathbb{R}_u^{\text{sup}} \downarrow q^*$ using a permutation of κ_u^+ , and so we begin with a general discussion of such permutations.

Let τ be a permutation of κ^+ with $\tau \upharpoonright \kappa = \text{id}$, then (in an abuse of notation) we define $\tau(y) = \tau[y]$ for $y \in P_\kappa \kappa^+$. Clearly $\tau(y) \in P_\kappa \kappa^+$. If $y \in A(\kappa, \kappa^+)$ then $y \cap \kappa = \tau(y) \cap \kappa$, but possibly $\text{ot}(y) \neq \text{ot}(\tau(y))$ and in this case $\tau(y) \notin A(\kappa, \kappa^+)$. However $\{y \in A(\kappa, \kappa^+) : \tau(y) = y\}$ is large, in a sense to be made precise in Claim 1 below.

If $y \in S(\kappa, \kappa^+)$, then we let

$$\tau(y) = \langle \tau(y(0)) \rangle \frown \langle y(\alpha) : 0 < \alpha < \text{lh}(y) \rangle.$$

Claim 1. *Suppose that v is a (κ, κ^+) -measure sequence which has a generating embedding j . Then $\{y \in S(\kappa, \kappa^+) : \tau(y) = y\}$ is v -large.*

Proof. We need to show that $j(\tau)(u_j \upharpoonright \alpha) = u_j \upharpoonright \alpha$, which is immediate because $u_j(0) = j[\kappa^+]$ and this set is closed under $j(\tau)$. □

We now complete the proof of Lemma 4.18. Thus let $p, q \in \mathbb{R}_u^{\text{sup}}$ be such that $\phi(p)$ and $\phi(q)$ are compatible in the \leq^* ordering.

Then p and q have the same length., say n . Let

$$p = \langle (u^0, A^0), \dots, (u^i, A^i), \dots, (u^n = u, A^n) \rangle$$

and

$$q = \langle (v^0, B^0), \dots, (v^i, B^i), \dots, (v^n = u, B^n) \rangle.$$

By our assumption $\phi(u^i) = \phi(v^i) = w^i$ say. Let $\kappa_{u^i} = \kappa_{v^i} = \kappa_i$ say, and note that $\text{ot}(u^i(0)) = \text{ot}(v^i(0)) = \kappa_i^+$, $\text{lh}(u^i) = \text{lh}(v^i) = \text{lh}(w^i)$, and $u^i(\alpha) = v^i(\alpha) = w^i(\alpha)$ for all $\alpha > 0$. Note also that $u^0(0) \subseteq u^1(0) \dots \subseteq u^{n-1}(0)$, and similarly $v^0(0) \subseteq v^1(0) \dots \subseteq v^{n-1}(0)$.

We may now easily build a permutation τ of κ_u^+ such that $\tau \upharpoonright \kappa_u = id$, and $\tau(u^i(0)) = v^i(0)$ for all i with $0 \leq i < n$. Note that $\tau(u^i) = v^i$ for each i .

Note that for each $i < n$, τ induces a permutation τ_i of κ_i^+ defined by $\tau_i = \pi_{v^i} \circ \tau \circ \rho_{u^i}$. By convention set $\tau_n = \tau$. For each i with $0 \leq i \leq n$, let

$$C^i = A^i \cap B^i \cap \{y \in S(\kappa_i, \kappa_i^+) : \tau_i(y) = y\}.$$

By Claim 1 and the remarks above, C^i is both u^i -large and v^i -large.

Now let

$$p^* = \langle (u^0, C^0), \dots, (u^i, C^i), \dots, (u^n = u, C^n) \rangle$$

and

$$q^* = \langle (v^0, C^0), \dots, (v^i, C^i), \dots, (v^n = u, C^n) \rangle.$$

We will define a function α with domain $\mathbb{R}_u^{\text{sup}} \downarrow p^*$ as follows: if

$$r = \langle (\bar{u}^0, D^0), \dots, (\bar{u}^j, D^j), \dots, (\bar{u}^t = u, C^t) \rangle \leq p^*,$$

then $\alpha(r)$ is the sequence obtained by replacing \bar{u}^j by $\tau(\bar{u}^j)$ for each j with $0 \leq j < t$.

We will verify that $\alpha(r) \in \mathbb{R}_u^{\text{sup}} \downarrow q^*$, and α is an isomorphism between $\mathbb{R}_u^{\text{sup}} \downarrow p^*$ and $\mathbb{R}_u^{\text{sup}} \downarrow q^*$.

Claim 2. $\alpha(r)$ is a condition.

Proof. Clearly each D^j is $\tau(\bar{u}^j)$ -large, because the measures of $\tau(\bar{u}^j)$ are the same as those of \bar{u}^j . We need to check that $\tau(\bar{u}^j) \in S(\kappa_{\bar{u}^j}, \kappa_{\bar{u}^j}^+)$ and that the sequences $\tau(\bar{u}^j)$ are increasing in the \prec -ordering.

Note that each of the sequences u^i appears as \bar{u}^j for some j , and in this case $\tau(\bar{u}^j) = \tau(u^i) = v^i$.

Subclaim 2.1. If $u^{n-1} \prec y$ and $y \in C^n$, then $v^{n-1} \prec \tau(y) = y$. Moreover if $u^{n-1} \prec y_0 \prec y_1$ and $y_0, y_1 \in C^n$, then $\tau(y_0) \prec \tau(y_1)$.

Proof. As $y \in C^n$, $\tau(y) = y$ and so $v^{n-1}(0) = \tau(u^{n-1}(0)) \subseteq \tau(y)(0) = y(0)$, hence easily $v^{n-1} \prec y$. The second part is immediate since $\tau(y_i) = y_i$. \square

Subclaim 2.2. *For all $i < n$, if $u^{i-1} \prec y \prec u^i$ and $\pi_{u^i}(y) \in C^i$, then $v^{i-1} \prec \tau(y) = \rho_{v^i}(\pi_{u^i}(y)) \prec v^i$ and in particular $\tau(y) \in S(\kappa_y, \kappa_y^+)$. Moreover if $u^{i-1} \prec y_0 \prec y_1 \prec u^i$ and $\pi_{u^i}(y_0), \pi_{u^i}(y_1) \in C^i$, then $\tau(y_0) \prec \tau(y_1)$.*

Proof. Since $\pi_{u^i}(y) \in C^i$ it is fixed by τ_i , hence $\pi_{v^i}(\tau(y)) = \pi_{u^i}(y)$. Also $u^{i-1} \prec y \prec u^i$ and so $v^{i-1}(0) = \tau(u^{i-1}(0)) \subseteq \tau(y)(0) \subseteq v^i(0) = \tau(u^i(0))$. Since the type-changing maps are order preserving, $\text{ot}(\tau(y)) = \text{ot}(y) = \kappa_y^+$ and hence $\tau(y) \in S(\kappa_y, \kappa_y^+)$. The final part is routine. \square

Combining the results of subclaims 2.1 and 2.2, we see that $\alpha(r)$ is a condition. In the course of proving the subclaims we obtained a description of the of the map α in terms of the type-changing maps, which we will use freely below. \square

Claim 3. $\alpha(r) \leq q^*$.

Proof. As we already mentioned each of the sequences v^i appears as $\tau(\bar{u}^j)$ for some j . By Definition 3.21, if \bar{u}^j is not among the sequences of the form u^i then either $\bar{u}^j \in C^n$, or $\pi_{u^i}(\bar{u}^j) \in C^i$ where $u^{i-1} \prec \bar{u}^j \prec u^i$. In the former case we have that $\tau(\bar{u}^j) = \bar{u}^j \in C^n$, and in the latter case we have $\pi_{v^i}(\tau(\bar{u}^j)) = \pi_{u^i}(\bar{u}^j) \in C^i$.

To finish the proof that $\alpha(r) \leq q^*$, we must check that the sets D^j behave correctly with respect to the sets C^i and the type-changing maps. If $\bar{u}^j = u^i$ then $D^j \subseteq C^i$ because $r \leq p^*$. Otherwise we distinguish as before the cases $u^{n-1} \prec \bar{u}^j \in C^n$ and $\pi_{u^i}(\bar{u}^j) \in C^i$ where $u^{i-1} \prec \bar{u}^j \prec u^i$.

In the former case $\rho_{\bar{u}^j}[D^j] \subseteq C^n$ by clause 2 of Definition 3.21 for $r \leq p^*$. The latter case is slightly more complicated because of the type changing. By clause 2 of Definition 3.21 for $r \leq p^*$, $\sigma_{\bar{u}^j u^i}[D^j] \subseteq C^i$. We have that $\tau(\bar{u}^j) = \rho_{v^i}(\pi_{u^i}(\bar{u}^j))$, and since $v^i = \rho_{v^i}(\pi_{u^i}(u^i))$ and the type changing maps are order-preserving we see that $\sigma_{\bar{u}^j u^i} = \sigma_{\tau(\bar{u}^j) v^i}$. It follows that $\sigma_{\tau(\bar{u}^j) v^i}[D^j] \subseteq C^i$ as required. \square

It is clear that α is bijective, with an inverse defined in the same way using the permutation τ^{-1} . To finish the proof we must check that α is order-preserving. Let $r \leq p^*$ be a condition as above, and let $s = \langle (\hat{u}^0, E^0), \dots, (\hat{u}^k, E^k), \dots, (\hat{u}^\ell = u, C^\ell) \rangle \leq r$,

Claim 4. $\alpha(s) \leq \alpha(r)$.

Proof. If \hat{u}^k appears among the sequences of form \bar{u}^j or if $\bar{u}^t \prec \hat{u}^k$, then there are no new technical points in checking Definition 3.21 at the pair $(\tau(\hat{u}^k), E^k)$. So we assume that neither of these cases holds, let j be least such that $\hat{u}^k \prec \bar{u}^j$, and observe that there is no new technical issue if \bar{u}^j is among the sequences of form u^i . This leaves us with the cases where $u^{n-1} \prec \hat{u}^k \prec \bar{u}^n$, and $u^{i-1} \prec \hat{u}^k \prec \bar{u}^j \prec u^i$ for some $i < n$.

In the first case $\tau(\hat{u}^k) = \hat{u}^k$ and $\tau(\bar{u}^j) = \bar{u}^j$, so it is easy to verify clause 2 at $\tau(\hat{u}^k)$ in Definition 3.21, using the same clause at \hat{u}^k from Definition 3.21 for $s \leq r$.

In the second case we have that $\pi_{\bar{u}^j}(\hat{u}^k) \in D^j$ and $\sigma_{\hat{u}^k \bar{u}^j}[E^k] \subseteq D^j$. We also have that $\tau(\hat{u}^k) = \rho_{v_i}(\pi_{u_i}(\hat{u}^k))$ and $\tau(\bar{u}^j) = \rho_{v_i}(\pi_{u_i}(\bar{u}^j))$, so that $\pi_{\tau(\bar{u}^j)}(\tau(\hat{u}^k)) = \pi_{\bar{u}^j}(\hat{u}^k)$ and $\sigma_{\hat{u}^k \bar{u}^j} = \sigma_{\tau(\hat{u}^k) \tau(\bar{u}^j)}$. Clause 2 at $\tau(\hat{u}^k)$ in Definition 3.21 now follows from the same clause at \hat{u}^k from Definition 3.21 for $s \leq r$. \square

This completes the proof of Lemma 4.18. \square

Corollary 4.19. (*Weak homogeneity*). *Suppose $p, q \in \mathbb{R}_u^{\text{sup}}$ and $\phi(p)$ is compatible with $\phi(q)$ in the \leq^* -ordering. If $p \Vdash \psi(\alpha, \vec{\gamma})$, where $\alpha, \vec{\gamma}$ are ordinals, then it is not the case that $q \Vdash \neg\psi(\alpha, \vec{\gamma})$.*

It follows that:

Corollary 4.20. *Suppose that G is $\mathbb{R}_u^{\text{sup}}$ -generic and let G^ϕ be the filter generated by $\phi[G]$.*

Then:

- (1) G^ϕ is $\mathbb{R}_{\phi(u)}^{\text{proj}}$ -generic over V .
- (2) $HOD^{V[G]} \subseteq V[G^\phi]$.

Proof. Part 1 is immediate because ϕ is a weak projection.

For part 2 suppose that a is a set of ordinals in $V[G]$ which is definable in $V[G]$ with ordinal parameters. We show that a belongs to $V[G^\phi]$. Write $a = \{\alpha : V[G] \models \psi(\alpha, \vec{\gamma})\}$. Then

$$a = \{\alpha : p \Vdash \psi(\alpha, \vec{\gamma}) \text{ for some } p \in G\}.$$

Let $\langle w_i : i < \mu \rangle$ be the generic sequence induced by G , so that $\langle \phi(w_i) : i < \mu \rangle$ is the generic sequence induced by G^ϕ . Let H be the set of conditions q in $\mathbb{R}_u^{\text{sup}}$ such that for

every finite set $b \subseteq \mu$ with $\{i < \mu : \phi(w_i) \text{ appears in } \phi(q)\} \subseteq b$, there is $r \leq q$ such that $\{i < \mu : \phi(w_i) \text{ appears in } \phi(r)\} = b$. Clearly $G \subseteq H$ and $H \in V[G^\phi]$.

We claim that

$$a = \{\alpha : p \Vdash \phi(\alpha, \vec{\gamma}) \text{ for some } p \in H \}.$$

Clearly if $\alpha \in a$ then it is a member of the set on the right hand side, so suppose for a contradiction that there exist p and q such that $p \in G$, $q \in H$, $q \Vdash \phi(\alpha, \vec{\gamma})$ and $p \Vdash \neg\phi(\alpha, \vec{\gamma})$. Let $b = \{i < \mu : w_i \text{ appears in } p \text{ or } \phi(w_i) \text{ appears in } q\}$. Since $q \in H$ we may find $q' \leq q$ such that $b = \{i < \mu : \phi(w_i) \text{ appears in } \phi(q')\}$, and since $p \in G$ we may find $p' \leq p$ such that $b = \{i < \mu : w_i \text{ appears in } p'\}$.

It follows that $\phi(p')$ and $\phi(q')$ are compatible in the \leq^* ordering, contradicting Corollary 4.20. \square

5. THE MAIN THEOREM

Theorem 1. *Let GCH hold and let κ be κ^{+3} -supercompact. Then there exists a generic extension W of V in which κ remains strongly inaccessible and $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal $\alpha < \kappa$. In particular the rank-initial segment W_κ is a model of ZFC in which $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal α .*

Proof. Let $j : V \rightarrow M$ witness that κ is a κ^{+3} -supercompact cardinal. Let $v \in \mathcal{U}_\infty^{\text{sup}}$ be a (κ, κ^+) -measure sequence constructed from j which has a weak repeat point α and let $u = v \upharpoonright \alpha$.

Consider the forcing notions $\mathbb{R}_u^{\text{sup}}$ and $\mathbb{R}_{\phi(u)}^{\text{proj}}$. Let G be $\mathbb{R}_u^{\text{sup}}$ -generic over V and let G^ϕ be the induced generic filter for $\mathbb{R}_{\phi(u)}^{\text{proj}}$. Let $\mu = \kappa^{+3}$. From previous results:

- κ remains μ -supercompact in $V[G]$ and in $V[G^\phi]$ (by Lemmas 3.34 and 4.14).
- There exists $D \in V[G^\phi]$ a club subset of κ such that for every limit point α of D , we have $(\alpha^+)^{V[G^\phi]} = (\alpha^+)^V < (\alpha^+)^{V[G]}$ (by Lemmas 3.33 and 4.13).

By part 2 of Corollary 4.20 we have $HOD^{V[G]} \subseteq V[G^\phi]$, in particular for every limit point α of D we have

$$(\alpha^+)^{HOD^{V[G]}} \leq (\alpha^+)^{V[G^\phi]} = (\alpha^+)^V < (\alpha^+)^{V[G]}.$$

Let $\langle \kappa_i : i < \kappa \rangle$ be an increasing enumeration of D . Working in $V[G]$, let \mathbb{Q} be the reverse Easton iteration for collapsing κ_{i+1} to have cardinality κ_i^+ for each $i < \kappa$, and let H be \mathbb{Q} -generic over $V[G]$. By standard arguments about iterated forcing:

- (1) $CARD^{V[G*H]} \cap (\kappa_0, \kappa) = \{\kappa_i^+ : i < \kappa\} \cup \{\kappa_i : i < \kappa, i \text{ is a limit ordinal}\}$.
- (2) κ remains inaccessible in $V[G * H]$.

It also follows from results of Dobrinen and Friedman [1] that \mathbb{Q} is *cone homogeneous*, that is for all $p, q \in \mathbb{Q}$ there are $p^* \leq p$, $q^* \leq q$ and an isomorphism $\alpha : \mathbb{Q}/p^* \rightarrow \mathbb{Q}/q^*$. Hence by [1] we have

$$HOD^{V[G*H]} \subseteq HOD^{V[G]}.$$

Finally let $\mathbb{P} = Col(\aleph_0, \kappa_0)^{V[G*H]}$ over $V[G * H]$, and let K be \mathbb{P} -generic over $V[G * H]$. It is now easily seen that κ remains inaccessible in $V[G * H * K]$, and by homogeneity of \mathbb{P}

$$HOD^{V[G*H*K]} \subseteq HOD^{V[G*H]}.$$

Hence

$$HOD^{V[G*H*K]} \subseteq HOD^{V[G*H]} \subseteq HOD^{V[G]} \subseteq V[G^\phi].$$

Thus for all infinite cardinals $\alpha < \kappa$ of $V[G * H * K]$ we have

$$(\alpha^+)^{HOD^{V[G*H*K]}} \leq (\alpha^+)^{V[G^\phi]} = (\alpha^+)^V < (\alpha^+)^{V[G*H*K]}.$$

Let $W = V[G * H * K]$. Then W is the required model and the theorem follows. \square

Remark 5.1. If we start with a cardinal κ which is supercompact, then we may find u such that κ remains supercompact in the generic extension by $\mathbb{R}_u^{\text{sup}}$. This gives a model where κ is supercompact and $(\alpha^+)^{HOD} < \alpha^+$ for club-many $\alpha < \kappa$. We note that Dobrinen and Friedman [1] gave a model where κ is measurable with some normal measure U , and $(\alpha^+)^{HOD} < \alpha^+$ for U -many $\alpha < \kappa$.

To preserve supercompactness we argue as follows. We choose for each $\mu \geq \kappa^{+3}$ an embedding witnessing that κ is μ -supercompact, appeal to Lemmas 3.12 and 3.34 to find a (κ, κ^+) -measure sequence u (depending on μ) such that $\text{lh}(u) < \kappa^{+++}$ and $\mathbb{R}_u^{\text{sup}}$ preserves the μ -supercompactness of κ , and finally use the Axiom of Replacement to find a sequence u such that $\mathbb{R}_u^{\text{sup}}$ preserves the μ -supercompactness of κ for unboundedly many values of μ .

Remark 5.2. We can show that κ is measurable in W in Theorem 1. Since K is generic for small forcing, it suffices to show that κ is measurable in $V[G * H]$.

To do this let $W \in V[G]$ be a normal measure on κ , and let $i : V[G] \rightarrow N$ be the associated ultrapower map. Clearly κ is a limit point of the club set $i(\mathbb{D})$. By standard facts about Easton iterations the poset \mathbb{Q} is κ -c.c. and we may write $i(\mathbb{Q}) = \mathbb{Q} * \dot{\mathbb{R}}$ where \mathbb{R} is the tail of the iteration. In $N[H]$ the first step of the iteration \mathbb{R} is a κ^+ -closed Levy collapse, and by standard arguments \mathbb{R} is κ^+ -closed.

Since GCH holds in $V[G]$, we have that $2^\kappa = \kappa^+ = |i(\kappa)|$ in $V[G]$. In $N[H]$ the iteration \mathbb{R} is a $i(\kappa)$ -c.c. poset of cardinality $i(\kappa)$ and so

$$N[H] \models \mathbb{R} \text{ has } i(\kappa) \text{ maximal antichains.}$$

Since $V[G] \models {}^\kappa N \subseteq N$ and H is generic for κ -c.c. forcing, $V[G * H] \models {}^\kappa N[H] \subseteq N[H]$. It follows that working in $V[G * H]$ we may enumerate the antichains of \mathbb{R} which lie in $N[H]$ in order type κ^+ , and build a decreasing κ^+ -chain of conditions in \mathbb{R} that meets each of these antichains. This allows us to construct a filter $H^* \in V[G * H]$ which is \mathbb{R} -generic over $N[H]$.

Since \mathbb{Q} is an iteration with supports bounded in κ , $i^{\ast}H \subseteq H * H^*$. It follows that we may lift i to obtain $i : V[G * H] \rightarrow N[H][H^*]$ definable in $V[G * H]$, and hence that κ is measurable in $V[G * H]$.

We conclude with some open questions:

- What is the exact consistency strength of the assertion “ $(\alpha^+)^{HOD} < \alpha^+$ for every infinite cardinal α ”?
- What is $HOD^{V[G]}$?
- Is it consistent that all uncountable regular cardinals are inaccessible cardinals of HOD ?
- Is it consistent that κ is supercompact and $(\alpha^+)^{HOD} < \alpha^+$ for all cardinals $\alpha < \kappa$?

As a consequence of his “HOD Conjecture” (see [11]), Woodin has conjectured a negative answer to the last of these questions.

REFERENCES

- [1] N. Dobrinen and S.-D. Friedman, Homogeneous iteration and measure one covering relative to HOD . *Arch. Math. Logic* 47 (2008), no. 7–8, 711–718.
- [2] M. Foreman and W. H. Woodin, The generalized continuum hypothesis can fail everywhere. *Ann. of Math. (2)* 133 (1991), no. 1, 1–35.
- [3] R. B. Jensen, The fine structure of the constructible hierarchy. *Ann. Math. Logic* 4 (1972), 229–308.
- [4] K. Devlin and R. B. Jensen, Marginalia to a theorem of Silver. *ISILC Logic Conference (Proc. Internat. Summer Inst. and Logic Colloq., Kiel, 1974)*, pp. 115–142. *Lecture Notes in Math. Vol. 499*, Springer, Berlin, 1975.
- [5] R. Jensen and J. Steel, K without the measurable. *J. Symbolic Logic* 78 (2013), no. 3, 708–734.
- [6] J. Krueger, Radin forcing and its iterations. *Arch. Math. Logic* 46 (2007), no. 3–4, 223–252.
- [7] P. B. Larson, The stationary tower. *University lecture series vol. 32*, American Mathematical Society, Providence RI, 2004.
- [8] M. Magidor, On the singular cardinals problem. I. *Israel J. Math.* 28 (1977), no. 1–2, 1–31.
- [9] L. B. Radin, Adding closed cofinal sequences to large cardinals. *Ann. Math. Logic* 22 (1982), no. 3, 243–261.
- [10] J. R. Steel, The Core Model Iterability Problem, volume 8 of *Lecture Notes in Logic*. Springer, Berlin, 1996
- [11] W. H. Woodin, Suitable extender models I. *Journal of Mathematical Logic*, Vol. 10, Nos. 1&2 (2010) 101–339.
- [12] W. H. Woodin, Private communication.

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213-3890, USA.

E-mail address: jcumming@andrew.cmu.edu

Kurt Gödel Research Center for Mathematical Logic, Währinger Strasse 25, 1090 Vienna, Austria.

E-mail address: sdf@logic.univie.ac.at

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran-Iran.

E-mail address: golshani.m@gmail.com