ON THE NOTIONS OF CUT, DIMENSION AND TRANSCENDENCE DEGREE FOR MODELS OF ZFC

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Abstract. We define notions of generic cut, generic dimension and generic transcendence degree between models of ZFC and prove some results about them.

1. introduction

Given models $V \subseteq W$ of ZFC, we define the notions of generic cut, generic dimension and generic transcendence degree of W over V, and prove some results about them. We usually assume that W is a generic extension of V by a set or a class forcing notion, but some of our results work for general cases.

2. generic dimension and generic transcendence degree

In this section we define the notions of generic dimension and generic transcendence degree between two models of ZFC, and prove some results about them. Let's start with some definitions.

Definition 2.1. *Suppose that $V \subseteq W$ are models of ZFC with the same ordinals.*

(1) *Let $X = \langle x_i : i \in I \rangle \in W$, where I, the set of indices, is in V. The elements of X are called mutually generic over V, if for any partition $I = I_0 \cup I_1$ of I in V, $\langle x_i : i \in I_0 \rangle$ is generic over $V[\langle x_i : i \in I_1 \rangle]$, for some forcing notion $\mathbb{P} \in V$.*
(2) *The $\kappa-$generic transcendence degree of W over V, $\kappa - g.tr.deg_V(W)$, is defined to be $\sup\{|A| : A \in W$ and for all $X \in [A]^{<\kappa}$, the elements of X are mutually generic over $V\}$.*

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(3) *Upward generic dimension of* W *over* V, $[W : V]^U$, *is defined to be* $\sup\{\alpha :$ *there exists a* $\subset -$*increasing chain* $\langle V_i : i < \alpha\rangle$ *of set generic extensions of* V, *where* $V_0 = V$, *and each* $V_i \subseteq W\}$.

(4) *Downward generic dimension of* W *over* V, $[W : V]_D$, *is defined to be* $\sup\{\alpha :$ *there exists a* $\subset -$*decreasing chain* $\langle W_i : i < \alpha\rangle$ *of grounds* [1] *of* W, *where* $W_0 = W$, *and each* $W_i \supseteq V\}$.

The next lemma is trivial.

Lemma 2.2. (1) $\kappa < \lambda \Rightarrow \kappa - g.tr.deg_V(W) \geq \lambda - g.tr.deg_V(W)$.

(2) $[W : V]^U \geq \kappa - g.tr.deg_V(W)$, *for* κ *such that* $\kappa - g.tr.deg_V(W) = \kappa^+ - g.tr.deg_V(W)$ *(such a* κ *exists by* (1)*).*

Lemma 2.3. [2] *Let* $V[G]$ *be a generic extension of* V *by some forcing notion* $\mathbb{P} \in V$, *and let* W *be a model of ZFC such that* $V \subseteq W \subseteq V[G]$. *Then there are* $\mathbb{Q} \in V$ *and* $\pi : \mathbb{P} \to \mathbb{Q}$ *such that:*

(1) $|\mathbb{Q}| \leq |\mathbb{P}|$,

(2) $\pi[G]$ *generate a filter* H *which is* $\mathbb{Q}-$*generic over* V,

(3) $W = V[H]$.

Proof. Let $\mathbb{B} = r.o(\mathbb{P})$, and let $e : \mathbb{P} \to \mathbb{B}$ be the induced embedding. Let $\bar{G}$ be the filter generated by $e[G]$, so that $V[G] = V[\bar{G}]$. Then for some complete subalgebra $\mathbb{C}$ of $\mathbb{B}$, we have $W = V[\bar{G} \cap \mathbb{C}]$. Let $\pi : \mathbb{B} \to \mathbb{C}$ be the standard projection map, given by $\pi(b) = \min\{c \in \mathbb{C} : c \geq b\}$. Let $\mathbb{Q} = \pi[\mathbb{P}]$, and consider $\pi \restriction \mathbb{P} : \mathbb{P} \to \mathbb{Q}$. $\mathbb{Q}$ is a dense subset of $\mathbb{C}$, and so it is easily seen that $\pi \restriction \mathbb{P}$ and $\mathbb{Q}$ are as required. □

Corollary 2.4. *Let* $V[G]$ *be a generic extension of* V *by some forcing notion* $\mathbb{P} \in V$. *Then* $[V[G] : V]^U, [V[G] : V]_D \leq (2^{|\mathbb{P}|})^+$.

Question 2.5. *In the above Corollary, can we replace* $2^{|\mathbb{P}|}$ *by* $|\mathbb{P}|^{<\kappa}$, *where* κ *is such that* $\mathbb{P}$ *satisfies the* $\kappa - c.c.$?

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[1] Recall that V is a ground of W, if W is a set generic extension of V by some forcing notion in V.

[2] We thank Monroe Eskew for bringing this lemma to our attention.

Theorem 2.6. *Let $\mathbb{P} = \mathrm{Add}(\omega, 1)$ be the Cohen forcing for adding a new Cohen real and let G be $\mathbb{P}-$generic over V. Then:*

(1) $\omega - g.tr.deg_V(V[G]) = 2^{\aleph_0}$.

(2) *For $\kappa > \omega$ we have* $\kappa - g.tr.deg_V(V[G]) = \aleph_0$.

(3) $[V[G] : V]^U = \aleph_1$.

(4) $[V[G] : V]_D = \aleph_1$.

In the proof of the above theorem, we will use the following.

Lemma 2.7. (1) *Forcing with* $\mathrm{Add}(\omega, 1)$ *over V, can not add a generic sequence for* $\mathrm{Add}(\omega, \omega_1)$ *over V.*

(2) *A sequence $\langle x_\alpha : \alpha < \aleph_1\rangle$ of reals is $\mathrm{Add}(\omega, \omega_1)$-generic over V, iff for any countable set $I \in V, I \subseteq \omega_1$, the sequence $\langle x_\alpha : \alpha \in I\rangle$ is $\mathrm{Add}(\omega, I)$-generic over V, where $\mathrm{Add}(\omega, I)$ is the Cohen frcing for adding I-many Cohen reals indexed by I.*

(3) *If $\mathbb{P}$ is a non-trivial countable forcing notion, then $\mathbb{P} \simeq \mathrm{Add}(\omega, 1)$.*

A generalized version of (1) is proved in [3], (2) follows easily using the fact that the forcing notion $\mathrm{Add}(\omega, \omega_1)$ satisfies the countable chain condition, and (3) is well-known.

Proof. (1) : In V, fix a canonical enumeration $F : 2^{<\omega} \to \omega$ such that if $|s| < |t|$, then $F(s) < F(t)$. For any $t \in (2^\omega)^V$, define g_t by $g_t(n) = g(F(t \restriction n))$. Then $\langle g_t : t \in (2^\omega)^V)\rangle$ witnesses $\omega - g.tr.deg_V(V[G]) = 2^{\aleph_0}$.

(2) : Let $\kappa > \aleph_0$. As $\mathbb{P} \simeq \mathrm{Add}(\omega, \omega)$, it is clear that $\kappa - g.tr.deg_V(V[G]) \geq \aleph_0$. On the other hand, if $\kappa - g.tr.deg_V(V[G]) > \aleph_0$, then let $\langle x_\alpha : \alpha < \aleph_1\rangle \in V[G]$ be such that for any countable set $I \in V, I \subseteq \omega_1$, the sequence $\langle x_\alpha : \alpha \in I\rangle$ is a set of mutually generics over V. By Lemmas 2.3 and 2.7(3), each x_α can be viewed as a Cohen real, so it follows from Lemma 2.7(2) that the sequence $\langle x_\alpha : \alpha < \aleph_1\rangle$ is $\mathrm{Add}(\omega, \omega_1)$-generic over V, which contradicts Lemma 2.7(1).

(3) : Given any $\alpha < \aleph_1$, we have $\mathbb{P} \simeq \mathrm{Add}(\omega, \alpha)$, so let $\langle x_\beta : \beta < \alpha\rangle \in V[G]$ be $\mathrm{Add}(\omega, \alpha)$-generic over V, and define $V_\beta = V[\langle x_i : i < \beta\rangle]$, for $\beta \leq \alpha$ Then $V = V_0 \subset V_1 \subset \cdots \subset V_\alpha$ is an increasing chain of length α. So $[V[G] : V]^U \geq \alpha$.

Now assume on the contrary that, we have an increasing chain $V = V_0 \subset V_1 \subset \cdots \subset V_\alpha \cdots \subseteq V[G], \alpha < \aleph_1$, of generic extensions of V. For each $\alpha < \aleph_1$, set $V_\alpha = V[G_\alpha]$, where

$G_\alpha \in V[G]$ is generic over some forcing notion in V. Again using Lemmas 2.3 and 2.7(3), we can assume that each $G_{\alpha+1}$ is some Cohen real $x_{\alpha+1}$ over V_α. Thus we have a sequence $\langle x_\alpha : \alpha < \aleph_1 \rangle$ of reals, each x_α is Cohen generic over $V[\langle x_\beta : \beta < \alpha \rangle]$. By Lemma 2.7(2), the sequence $\langle x_\alpha : \alpha < \aleph_1 \rangle$ is $\mathrm{Add}(\omega, \omega_1)$-generic over V, which contradicts Lemma 2.7(1).

(4) can be proved similarly. □

Using forcing notions producing minimal generic extensions, we can prove the following.

Theorem 2.8. *For any $0 < n < \omega$, there is a generic extension $V[G]$ of the V in which $[V[G] : V]^U = [V[G] : V]_D = n$.*

V. Kanovei noticed the following:

Theorem 2.9. *There is a generic extension $L[G]$ of L in which $[L[G] : L]^U = \omega + 1$.*

It follows from the results in [5] that:

Theorem 2.10. *Given any $\alpha \leq \omega_1$, there exists a cofinality preserving generic extension $L[G]$ of L in which $[L[G] : L]_D = \alpha + 1$.*

Question 2.11. *Given cardinals $\kappa_0 \geq \kappa_1 \geq ... \geq \kappa_n$, is there a forcing extension $V[G]$ of V, in which $\aleph_i - g.tr.deg_V(V[G]) = \kappa_i, i = 0, ..., n$ and $\lambda - g.tr.deg_V(V[G]) = \kappa_n$, for all $\lambda \geq \aleph_n$?*

Remark 2.12. *Force with $\mathbb{P} = \prod_{i=0}^{n} \mathrm{Add}(\aleph_i, \kappa_i)$, and let G be $\mathbb{P}$-generic over V. Then for all $0 \leq i \leq n, \aleph_i - g.tr.deg_V(V[G]) \geq \kappa_i$.*

Question 2.13. *Let $\alpha_0, \alpha_1 \leq \lambda$, where α_0, α_1 are ordinals and λ is a cardinal. Is there a generic extension $V[G]$ of V (possibly by a class forcing notion) in which $[V[G] : V]^U = \alpha_0, [V[G] : V]_D = \alpha_1$ and the number of intermediate submodels of V and $V[G]$ is λ?*

Using results of [1] and the fact that $0^\sharp$ can not be produced by set forcing, we have:

Theorem 2.14. *For all $\kappa, \kappa - g.tr.deg_L(L[0^\sharp]) = \infty$. Also we have $[L[0^\sharp] : L]^U = \infty$ and $[L[0^\sharp] : L]_D = 0$.*

3. Some non-absoluteness results

In this section we present some results which say that the notions of generic dimension and generic transcendence degree are not absolute between different models of ZFC.

Theorem 3.1. *There exists a generic extension V of L, such that if R is $\mathrm{Add}(\omega, 1)$-generic over V, then:*

(1) $\aleph_1 - g.tr.deg_V(V[R]) = \aleph_0$,

(2) $\aleph_1 - g.tr.deg_L(V[R]) \geq \aleph_1$.

Proof. By [2], there is a cofinality preserving generic extension V of L, such that adding a Cohen real R over V, adds a generic filter for $\mathrm{Add}(\omega, \omega_1)$ over L.

Now (1) follows from Theorem 2.6(2), and (2) follows from the fact that there exists a sequence $\langle x_\alpha : \alpha < \aleph_1\rangle$ of reals, which is $\mathrm{Add}(\omega, \omega_1)$-generic over L. □

Theorem 3.2. *Assume $0^\sharp$ exists, κ is a regular cardinal in L, and let $\mathbb{P} = Sacks(\kappa, 1)_L$, the forcing for producing a minimal extension of L by adding a new subset of κ. Let G be $\mathbb{P}$-generic over V. Then:*

(1) $[L[G] : L]^U = [L[G] : L]_D = 1$,

(2) $[V[G] : V]^U, [V[G] : V]_D \geq \kappa$.

Proof. (1) is trivial, as forcing with $\mathbb{P}$ over L produces a minimal generic extension of L. (2) Follows from a result of Stanley [7], which says that forcing with $\mathbb{P}$ over V collapses κ into ω. □

4. generic cut for pairs of models of ZFC

In this section, we define the notion of generic cut between two models of ZFC, and prove some results about it. Let's start with the main definition.

Definition 4.1. *Suppose that $V \subseteq W$ are models of ZFC with the same ordinals and α, β are ordinals. An $(\alpha, \beta)-$generic cut of (V, W), is a pair $\langle \vec{V}, \vec{W}\rangle$, where*

(1) $\vec{V} = \langle V_i : i < \alpha\rangle$ *is a $\subset$-increasing chain of generic extensions of V, with $V_0 = V$,*

(2) $\vec{W} = \langle W_j : j < \beta\rangle$ *is a $\subset$-decreasing chain of grounds of W, with $W_0 = W$,*

(3) $\forall i < \alpha, j < \beta, V_i \subset W_j$,

(4) *There is no inner model* $V \subseteq M \subseteq W$ *of ZFC such that* $\forall i < \alpha, j < \beta, V_i \subset M \subset W_j$.

Note that if W is a set generic extension of V, then any V_i is a ground of W and each W_j is a generic extension of V.

Lemma 4.2. *Suppose that* $V \subseteq W$ *are models of ZFC and there exists an* (α, β)*-generic cut of* (V, W)*. Then* $\alpha \leq [W : V]^U$ *and* $\beta \leq [W : V]_D$.

The next theorem shows that there are no generic cuts in the extension by Cohen forcing.

Theorem 4.3. *Let* $\mathbb{P} = \mathrm{Add}(\omega, 1)$ *be the Cohen forcing for adding a new Cohen real and let* G *be* $\mathbb{P}-$*generic over* V*.Then there exists no* (α, β)*-generic cut of* $(V, V[G])$.

Proof. Assume towards a contradiction that $(\vec{V}, \vec{W})$ witnesses an (α, β)-generic cut of $(V, V[G])$. We consider several cases:

(1) At least one of α or β is uncountable. Suppose for example that $\alpha \geq \aleph_1$. It follows that $\langle V_i : i < \aleph_1 \rangle$ is a $\subset$ $-$increasing chain of generic extensions of V, which are included in $V[G]$, which contradicts Theorem 2.6(3). So from now on we assume that both α, β are countable.

(2) Both of $\alpha = \alpha^- + 1$ and $\beta = \beta^- + 1$ are successor ordinals. Then we have $V \subseteq V_{\alpha^-} \subset W_{\beta^-} \subseteq V[G]$, and there are no inner models M of $V[G]$ with $V_{\alpha^-} \subset M \subset W_{\beta^-}$, which is clearly impossible (in fact there should be $2^{\aleph_0}$ such M's).

(3) Both of $\alpha, \beta < \aleph_1$ are limit ordinals. We can imagine each $V_i, i < \alpha$, is of the form $V_i = V[a_i]$, for some Cohen real a_i and similarly each $W_j, j < \beta$, is of the form $W_j = V[b_j]$, for some Cohen real b_j. Let $a \in V[G]$ be a Cohen real over V, coding all of a_i's, $i < \alpha$. Then for all $i < \alpha, j < \beta, V_i \subset V[a] \subset W_j$, a contradiction.

(4) One of α or β is a limit ordinal > 0 and the other one is a successor ordinal. Let's assume that α is a limit ordinal and $\beta = \beta^- + 1$ is a successor ordinal. As $cf(\alpha) = \omega$, we can just consider the case where $\alpha = \omega$. Then for all $i < \omega$ we have $V_i \subset W_{\beta^-}$. We can assume that each V_i is of the form $V_i = V[a_i]$, for some Cohen real a_i, and that $W_{\beta^-} = V[b]$, for some Cohen real b. Using a fix bijection $f : \omega \leftrightarrow \omega \times \omega, f \in V$, we can imagine b as an ω-sequence $\langle b_i : i < \omega \rangle$ of reals which is $\mathrm{Add}(\omega, \omega)$-generic over

V, so that $W_{\beta^-} = V[\langle b_i : i < \omega\rangle]$. We can further suppose that each b_i codes a_i (i.e., $a_i \in V[b_i]$) Let us now define a new sequence $\langle c_i : i < \omega\rangle$ of reals in $V[\langle b_i : i < \omega\rangle]$, so that $c_i(0) = 0$, and $c_i \upharpoonright [1, \omega) = b_i \upharpoonright [1, \omega)$. Finally let $M = V[\langle c_i : i < \omega\rangle]$. It is clear that each $V_i \subset M$. But also $M \subset W_{\beta^-}$, as the real $t \in W_{\beta^-}$ defined by $t(i) = b_i(0)$ is not in M (by a genericity argument). We get a contradiction.

(5) One of α or β is 0 and the other one is a limit or a successor ordinal. Then as above we can get a contradiction.

□

Theorem 4.4. *Assume α, β are ordinals. Then there exists a generic extension $V[G]$ of V, such that there is an $(\alpha + 1, \beta + 1)-$generic cut of $(V, V[G])$,*

Proof. Let $\mathbb{P}_1 = \text{Add}(\omega, \alpha)$, and let $G_1 = \langle a_i : i < \alpha\rangle$ be a generic filter over V. Force over $V[G_1]$ by any forcing notion which produces a minimal extension $V[G_1][G_2]$ of $V[G_1]$. Finally force over $V[G_1][G_2]$ by $\mathbb{P}_3 = \text{Add}(\omega, \beta)$, and let $G_3 = \langle b_j : j < \beta\rangle$ be a generic filter over $V[G_1][G_2]$. Let

- $\vec{V} = \langle V_i : i \leq \alpha\rangle$, where $V_i = V[\langle a_\xi : \xi < i\rangle]$ for $i < \alpha$, and $V_\alpha = V[G_1]$,
- $\vec{W} = \langle W_j : j \leq \beta\rangle$, where $W_j = V[G_1][G_2][\langle b_\xi : j \leq \xi < \beta\rangle]$ for $j < \beta$, and $W_\beta = V[G_1][G_2]$.

Then $(\vec{V}, \vec{W})$ witnesses an $(\alpha + 1, \beta + 1)-$generic cut of $(V, V[G])$. □

Remark 4.5. *If V satisfies GCH, then we can find $V[G]$, so that it also satisfies the GCH; it suffices to work with $\text{Add}(|\alpha|^+, \alpha)$ and $\text{Add}(|\beta|^+, \beta)$ instead of $\text{Add}(\omega, \alpha)$ and $\text{Add}(\omega, \beta)$ respectively.*

Theorem 4.6. *Assume λ_1, λ_2 are infinite regular cardinals and κ is a measurable cardinal above them. Then in a generic extension $V[G]$ of V, there exists a $(\lambda_1, \lambda_2)-$generic cut of $(V, V[G])$.*

Proof. By [6], we can find a generic extension $V[G_1]$ of V, by a forcing of size $< \kappa$, such that in $V[G_1]$, there exists a (λ_1, λ_2)-gap of $P(\omega)/fin$. κ remains measurable in $V[G_1]$, so let U be a normal measure on κ in $V[G_1]$, and force with the corresponding Prikry forcing $\mathbb{P}_U$, and let G_2 be $\mathbb{P}_U$-generic over $V[G_1]$. Let $V[G] = V[G_1][G_2]$. By [4],

$$(\{M : M \text{ is a model of } ZFC, V[G_1] \subseteq M \subseteq V[G]\}, \subseteq) \cong (P(\omega)/fin, \subseteq^*).$$

Now the result should be clear, as a (λ_1, λ_2)-gap in $P(\omega)/fin$, produces the corresponding $(\lambda_1, \lambda_2)-$generic cut $(\vec{V}, \vec{W})$ of $(V[G_1], V[G])$, which in turn produces the same $(\lambda_1, \lambda_2)-$generic cut of $(V, V[G])$ (by adding V at the beginning of $\vec{V}$). □

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