(WEAK) DIAMOND CAN FAIL AT THE LEAST INACCESSIBLE CARDINAL

MOHAMMAD GOLSHANI

Abstract. Starting from suitable large cardinals, we force the failure of (weak) diamond at the least inaccessible cardinal. The result improves an unpublished theorem of Woodin and a recent result of Ben-Neria, Garti and Hayut.

1. Introduction

We study the combinatorial principles diamond, introduced by Jensen [8], and weak diamond, introduced by Devlin-Shelah [5], and prove the consistency of their failure at the least inaccessible cardinal.

Suppose κ is an uncountable regular cardinal. Recall that diamond at κ, denoted $\diamondsuit_\kappa$, asserts the existence of a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ such that for each $\alpha < \kappa, S_\alpha \subseteq \alpha$, and if $X \subseteq \kappa$, then $\{\alpha < \kappa \mid X \cap \alpha = S_\alpha\}$ is stationary in κ.

Also, the weak diamond at κ, denoted Φ_κ, is the assertion "For every $c : 2^{<\kappa} \to 2$, there exists $g : \kappa \to 2$ such that for all $f : \kappa \to 2$, the set $\{\alpha < \kappa \mid c(f \restriction \alpha) = g(\alpha)\}$ is stationary in κ".

It is easily seen that $\diamondsuit_{\kappa^+}$ implies $2^\kappa = \kappa^+$, and in fact by a celebrated theorem of Shelah [10], for all uncountable cardinals $\kappa, \diamondsuit_{\kappa^+}$ is equivalent to $2^\kappa = \kappa^+$. It follows that it is easy to force the failure of diamond at successor cardinals.

Also, by [5], Φ_{κ^+} is equivalent to $2^\kappa < 2^{\kappa^+}$, hence, $\diamondsuit_{\kappa^+}$ implies Φ_{κ^+} and again it is easy to force its failure.

Unlike the case of successor cardinals, it is difficult to force the failure of diamond or weak diamond at an inaccessible cardinal. By an old unpublished result of Woodin [3], if

This is the preliminary version of the paper.

The author's research has been supported by a grant from IPM (No. 91030417). He thanks Hugh Woodin [11] for allowing him to use his unpublished ideas. He also thanks James Cummings [4] for explaining him Woodin's proof.

$2^\kappa > \kappa^+$ and if u is a measure sequence on κ of length κ^+, then in the generic extension by Radin forcing using u, diamond fails at κ. Recent result of Ben-Neria, Garti and Hayut shows that in fact the weak diamond at κ fails in the resulting model.

However in the model above, κ is not small, in the sense that it is a limit of some very large cardinals. In this paper we extend the above results of Woodin and Ben-Neria, Garti and Hayut to obtain the failure of (weak) diamond at the least inaccessible cardinal.

Theorem 1.1. *Assume κ is a $(\kappa+3)$-strong cardinal. Then there is a generic extension of the universe in which κ is the least inaccessible cardinal and Φ_κ (and hence also $\diamondsuit_\kappa$) fails.*

Remark 1.2. *By* [12]*, some large cardinal assumptions are needed to get the result.*

In Section 2 we define the generic extension we are looking for and prove some of its basic properties. Then in Section 3 we extend Woodin's theorem by showing that $\diamondsuit_\kappa$ fails in the resulting model. Finally in Section 4, we extend the Ben-Neria, Garti and Hayut result and show that Φ_κ also fails in the model. We may mention that, though a model for the failure of Φ_κ is necessarily a model in which $\diamondsuit_\kappa$ fails, but we have decided to include the proofs of the failure of both $\diamondsuit_\kappa$ and Φ_κ, as we found both of the proofs interesting. Also the proof of the failure of $\diamondsuit_\kappa$ is based on the unpublished work of Woodin [3] and so it will make public the basic ideas first developed by Woodin in connection to the failure of diamond at inaccessible cardinals.

2. Radin forcing with interleaved collapses

In this section we define the main forcing notion used in the proof of 1.1 which is based on ideas from [2] and [7]. We assume that the ground model satisfies the following:

- κ is $(\kappa+2)$-strong, as witnessed by $j : V \to M \supseteq V_{\kappa+2}$ with $\mathrm{crit}(j) = \kappa$.
- For each inaccessible cardinal $\alpha \leq \kappa, 2^\alpha = \alpha^{++}$, further, if $0 < n < \omega$, then $2^{\alpha^{+n}} = \alpha^{+n+1}$.
- $M \models$"$2^\kappa = \kappa^{++}$ and for all $0 < n < \omega, 2^{\kappa^{+n}} = \kappa^{+n+1}$".
- There is $F \in V$ which is $\mathbb{P} = \mathrm{Col}(\kappa^{+4}, < i(\kappa))_N$-generic over N, where $i : V \to N$ is the ultrapower of V by $U_j = \{X \subseteq \kappa : \kappa \in j(X)\}$.
- j is generated by a (κ, κ^{+3})-extender.

Such a model can be constructed starting from GCH and a $(\kappa+3)$-strong cardinal κ [2]. Note that $F \in M$, as it can be coded by an element of $V_{\kappa+2}$, also if $k : N \to M$ is the induced elementary embedding, then $\mathrm{crit}(k) = \kappa_N^{+3} < \kappa_M^{+3} = \kappa^{+3}$ and F can be transferred along k.

Set

$$P^* = \{f : \kappa \to V_\kappa \mid \mathrm{dom}(f) \in U_j \text{ and } \forall\alpha, f(\alpha) \in \mathrm{Col}(\alpha^{+4}, <\kappa)\}.$$

$$F^* = \{f \in P^* \mid i(f)(\kappa) \in F\}.$$

Then U_j can be read off F^* as $U_j = \{X \subseteq \kappa \mid \exists f \in F^*, X = \mathrm{dom}(f)\}$.

2.1. **Measure sequences.** The following definitions are based on [2] with modifications required for our purposes.

Definition 2.1. *A constructing pair, is a pair (j, F) where*

- *$j : V \to M$ is a non-trivial elementary embedding into a transitive inner model, and if $\kappa = \mathrm{crit}(j)$, then $M^\kappa \subseteq M$.*
- *F is $\mathrm{Col}(\kappa^{+4}, < i(\kappa))_N$-generic over N, where $i : V \to N \simeq \mathrm{Ult}(V, U_j)$ is the ultrapower embedding approximating j. Also factor j through i, say $j = k \circ i$.*
- *$F \in M$.*
- *F can be transferred along k to give a $\mathrm{Col}(\kappa^{+4}, < j(\kappa))_M$-generic over M.*

In particular note that the pair (j, F) constructed above is a constructing pair.

Definition 2.2. *If (j, F) is a constructing pair as above, then $F^* = \{f \in P^* \mid i(f)(\kappa) \in F\}$.*

Definition 2.3. *Suppose (j, F) is a constructing pair as above. A sequence w is constructed by (j, F) iff*

- *$w \in M$.*
- *$w(0) = \kappa = \mathrm{crit}(j)$.*
- *$w(1) = F^*$.*
- *For $1 < \beta < \mathrm{lh}(w), w(\beta) = \{X \subseteq V_\kappa \mid w \upharpoonright \beta \in j(X)\}$.*
- *$M \models |\mathrm{lh}(w)| \leq w(0)^{++}$.*

If w is constructed by (j, F), then we set $\kappa_w = w(0)$, and if $\mathrm{lh}(w) \geq 2$, then we define

$F^*_w = w(1)$.

$\mu_w = \{X \subseteq \kappa_w \mid \exists f \in F^*_w, X = \mathrm{dom}(f)\}$.

$\bar{\mu}_w = \{X \subseteq V_{\kappa_w} \mid \{\alpha \mid \langle\alpha\rangle \in \mu_w\} \in \mu_w\}$.

$F_w = \{[f]_{\mu_w} \mid f \in F^*_w\}$.

$\mathcal{F}_w = \bar{\mu}_w \cap \bigcap\{w(\alpha) \mid 1 < \alpha < \mathrm{lh}(w)\}$.

Definition 2.4. *Define inductively*

$\mathcal{U}_0 = \{w \mid \exists(j, F)$ *such that* (j, F) *constructs* $w\}$.

$\mathcal{U}_{n+1} = \{w \in \mathcal{U}_n \mid \mathcal{U}_n \cap V_{\kappa_w} \in \mathcal{F}_w\}$.

$\mathcal{U}_\infty = \bigcap_{n\in\omega} \mathcal{U}_n$.

The elements of $\mathcal{U}_\infty$ *are called measure sequences.*

Now let u be the measure sequences constructed using (j, F) above. It is easily seen that for each $\alpha < \kappa^{+3}, u \restriction \alpha$ exists and is in $\mathcal{U}_\infty$.

In the next subsection, we assign to each $w \in \mathcal{U}_\infty$ a forcing notion $\mathbb{R}_w$, which is Radin forcing with interleaved collapses. The forcing notion used for the proof of 1.1 is then $\mathbb{R}_{u\restriction\kappa^+}$.

2.2. **Definition of forcing.** We are now ready to define our main forcing notion. The forcing is very similar to the one defined by Cummings [2], but we follow the presentation developed in [7].

Definition 2.5. *Assume* w *is a measure sequence.* $\mathbb{P}_w$ *consists of all tuples* $s = (w, \lambda, A, H, h)$, *where*

(1) w *is a measure sequence.*

(2) $\lambda < \kappa_w$.

(3) $A \in \mathcal{F}_w$.

(4) $H \in F^*_w$ *with* $\mathrm{dom}(H) = \{\kappa_v > \lambda \mid v \in A\}$.

(5) $h \in \mathrm{Col}(\lambda^{+4}, < \kappa_w)$.

Note that if $\mathrm{lh}(w) = 1$, *then the above tuple is of the form* $s = (w, \lambda, \emptyset, \emptyset, h)$ *(where* $\lambda < \kappa_w$ *and* $h \in \mathrm{Col}(\lambda^{+4}, < \kappa_w)$*).*

We define the order $\leq^*$ on $\mathbb{P}_w$ as follows.

Definition 2.6. *Assume* $s = (w, \lambda, A, H, h)$ *and* $s' = (w', \lambda', A', H', h')$ *are in* $\mathbb{P}_w$. *Then* $s \leq^* s'$ *iff*

(1) $w = w'$ *and* $\lambda = \lambda'$.

(2) $A \subseteq A'$.

(3) *For all* $u \in A, H(\kappa_u) \supseteq H'(\kappa_u)$.

(4) $h \supseteq h'$.

We now define our main forcing notion.

Definition 2.7. *If* w *is a measure sequence, then* $\mathbb{R}_w$ *is the set of finite sequences* $p = \langle p_k \mid k \leq n\rangle$, *where*

(1) $p_k = (w_k, \lambda_k, A_k, H_k, h_k) \in \mathbb{P}_{w_k}$, *for each* $k \leq n$.

(2) $w_n = w$.

(3) *If* $k < n$, *then* $\lambda_{k+1} = \kappa_{w_k}$.

Definition 2.8. *Assume* $p = \langle p_k \mid k \leq n\rangle$ *and* $p' = \langle p'_k \mid k \leq n'\rangle$ *are in* $\mathbb{R}_w$. *Then* $p \leq^* p'$ *(*p *is a direct extension or a Prikry extension of* p'*) iff* $n = n'$ *and for all* $k \leq n, p_k \leq^* p'_k$.

We now define one point extension of a condition, which allows us to define the extension relation on $\mathbb{R}_w$.

Definition 2.9. *Assume* $p = (w, \lambda, A, H, h) \in \mathbb{P}_w$ *and* $w' \in A$. *Then* $\mathrm{Add}(p, w')$ *is the condition* $\langle p_0, p_1\rangle \in \mathbb{R}_w$ *defined by*

(1) $p_0 = (w', \lambda, A \cap V_{\kappa_{w'}}, H \restriction V_{\kappa_{w'}}, h)$.

(2) $p_1 = (w, \kappa_{w'}, A \setminus V_\eta, H \restriction \mathrm{dom}(H) \setminus V_\eta, H(\kappa_{w'}))$, *where* $\eta = \sup \mathrm{range}(H(\kappa_{w'}))$.

In the case that this does not yield a member of $\mathbb{R}_w$, *then* $\mathrm{Add}(s, w')$ *is undefined.*

If $p = \langle p_0, \ldots, p_n\rangle \in \mathbb{R}_w$ *and* $u \in A_k$ *for some* $k \leq n$ *then* $\mathrm{Add}(p, u)$ *is the member of* $\mathbb{R}_w$ *obtained by replacing* p_k *with the two members of* $\mathrm{Add}(p_k, u)$.

The forcing order $\leq$ *on* $\mathbb{R}_w$ *is the smallest transitive relation containing the direct order* $\leq^*$ *and all pairs of the form* $(p, \mathrm{Add}(p, u))$.

2.3. **Basic properties of the forcing notion** $\mathbb{R}_w$**.** We now state the main properties of the forcing notion $\mathbb{R}_w$.

Lemma 2.10. $(\mathbb{R}_w, \leq)$ *satisfies the* κ_w^+*-c.c.*

Proof. Assume on the contrary that $A \subseteq \mathbb{R}_w$ is an antichain of size κ_w^+. We can assume that all $p \in A$ have the same length n. Write each $p \in A$ as $p = d_p^\frown p_n$, where $d_p \in V_{\kappa_w}$ and $p_n = (w, \lambda^p, A^p, H^p, h^p)$. By shrinking A, if necessary, we can assume that there are fixed $d \in V_{\kappa_w}$ and $\lambda < \kappa_w$ such that for all $p \in A$, $d_p = d$ and $\lambda_p = \lambda$.

Note that for $p \neq q$ in A, as p and q are incompatible, we must have h^p is incompatible with h^q. But $\mathrm{Col}(\lambda^{+4}, < \kappa_w)$ satisfies the κ_w-c.c., and we get a contradiction. □

Lemma 2.11. *(The factorization lemma) Assume* $p = \langle p_0, \ldots, p_n \rangle \in \mathbb{R}_w$ *with* $p_i = (w_i, \lambda_i, A_i, H_i, h_i)$ *and* $m < n$. *Set* $p^{\leq m} = \langle p_0, \ldots, p_m \rangle$ *and* $p^{>m} = \langle p_{m+1}, \ldots, p_n \rangle$.

(*a*) $p^{\leq m} \in \mathbb{R}_{w_m}$, $p^{>m} \in \mathbb{R}_w$ *and there exists* $i : \mathbb{R}_w/p \to \mathbb{R}_{w_m}/p^{\leq m} \times \mathbb{R}_w/p^{>m}$ *which is an isomorphism with respect to both* $\leq^*$ *and* $\leq$.

(*b*) *If* $m+1 < n$, *then there exists* $i : \mathbb{R}_w/p \to \mathbb{R}_{w_m}/p^{\leq m} \times \mathrm{Col}(\kappa_{w_m}^{+4}, < \kappa_{w_{m+1}}) \times \mathbb{R}_w/p^{>m+1}$ *which is an isomorphism with respect to both* $\leq^*$ *and* $\leq$. □

The next lemma can be proved as in [2] (in fact in [2] the lemma is proved for a more complicated forcing notion).

Lemma 2.12. $(\mathbb{R}_w, \leq, \leq^*)$ *satisfies the Prikry property.*

Now suppose that $w = u \restriction \kappa^+$, where u is the measure sequence constructed by (j, F) and let $G \subseteq \mathbb{R}_w$ be generic over V. Set

$$C = \{\kappa_u \mid \exists p \in G, \exists i < \mathrm{lh}(p), p_i = (u, \lambda, A, H, h)\}.$$

By standard arguments, C is a club of κ. Let $\langle \kappa_i : i < \kappa \rangle$ be the increasing enumeration of the club C and let $\vec{u} = \langle u_i \mid i < \kappa \rangle$ be the enumeration of $\{u \mid \exists p \in G, \exists i < \mathrm{lh}(p), p_i = (u, \lambda, A, H, h)\}$ such that for $i < j < \kappa, \kappa_{u_i} = \kappa_i < \kappa_j = \kappa_{u_j}$. Also let $\vec{F} = \langle F_i \mid i < \kappa \rangle$ be such that each F_i is $\mathrm{Col}(\kappa_i^{+4}, < \kappa_{i+1})$-generic over V produced by G.

Lemma 2.13. (a) $V[G] = V[\vec{u}, \vec{F}]$.

(b) *For every limit ordinal* $j < \kappa, \langle \vec{u} \restriction j, \vec{F} \restriction j \rangle$ *is* $\mathbb{R}_{u_j}$*-generic over* V, *and* $\langle \vec{u} \restriction [j, \kappa), \vec{F} \restriction [j, \kappa) \rangle$ *is* $\mathbb{R}_w$*-generic over* $V[\vec{u} \restriction j, \vec{F} \restriction j]$.

(c) *For every* $\gamma < \kappa$ *and every* $A \subseteq \gamma$ *with* $A \in V[\vec{u}, \vec{F}]$, *we have* $A \in V[\vec{u} \restriction j, \vec{F} \restriction j]$, *where* j *is the least ordinal such that* $\gamma < \kappa_j$.

As $\mathrm{lh}(w) = \kappa^+$, it follows from Mitchell [9] (see also [6]) that

Lemma 2.14. *κ remains strongly inaccessible in $V[G]$.*

It follows that

$$CARD^{V[G]} \cap [\kappa_0, \kappa) = \lim(C) \cup \{\alpha, \alpha^+, \alpha^{++}, \alpha^{+3}, \alpha^{+4} \mid \alpha \in C\}.$$

As every limit point of C is singular in $V[G]$, it follows that κ is the least inaccessible cardinal above κ_0.

3. Diamond at κ in $V[G]$

Lemma 3.1. *$\diamondsuit_\kappa$ fails in $V[G]$*

Proof. As in Woodin [3], we show that there is no diamond sequence in the extension which guesses every subset of κ from the ground model V. Assume not and let $\langle \dot{S}_\alpha \mid \alpha < \kappa \rangle$ be an $\mathbb{R}_w$-name for a diamond sequence at κ.

Let $p \in \mathbb{R}_w$ and write it as $p = \vec{d}^\frown(w, \lambda, A, H, h)$, where $\vec{d} \in V_\kappa$. We may further assume that for each $u \in A$, $\mathrm{Add}(p, u)$ is defined. Thus for each $u \in A$, we can form the condition $\mathrm{Add}(p, u) = \vec{d}^\frown\langle p_0^u, p_1^u \rangle \in \mathbb{R}_w$, where

$$p_0^u = (u, \lambda, A \cap V_{\kappa_u}, H \restriction V_{\kappa_u}, h)$$

and

$$p_1^u = (w, \kappa_u, A \setminus V_{\eta_u}, H \restriction \mathrm{dom}(H) \setminus V_{\eta_u}, H(\kappa_u)),$$

where $\eta_u = \sup \mathrm{range}(H(\kappa_u))$.

By the factorization lemma 2.11,

$$\mathbb{R}_w / \mathrm{Add}(p, u) \simeq (\mathbb{R}_u / \vec{d}^\frown p_0^u) \times (\mathbb{R}_w / p_1^u).$$

As forcing with $\mathbb{R}_w / p_1^u$ does not add new subsets to κ_u,, we can look at $\dot{S}_{\kappa_u}$ as an $\mathbb{R}_w$-name for an $\mathbb{R}_u$-name for a subset of κ_u. So by the Prikry property 2.12, we can find $q_1^u = (w, \lambda, A_u, H_u, h_u) \leq^* p_1^u$ and an $\mathbb{R}_u$-name τ_u such that $\vec{d}^\frown\langle p_0^u, q_1^u \rangle$ forces $\dot{S}_{\kappa_u}$ to be the realization of τ_u by the generic up to the point u, i.e.,

$$\vec{d}^\frown\langle p_0^u, q_1^u \rangle \Vdash \text{“}\dot{S}_{\kappa_u} = \tau_u\text{”}.$$

Let $A_1 = \triangle_{u\in A} A_u = \{w' \in \mathcal{U}_\infty \cap V_\kappa \mid \forall u \in A,\ \kappa_u < \kappa_{w'} \Rightarrow w' \in A_u\} \in \mathcal{F}_w$ be the diagonal intersection of A_u's, $u \in A$.

Now consider the sequence $\langle [H_u]_{\mu_w} : u \in A_1 \rangle \subseteq F_w$. Let $H^* \in F^*_w$ be such that $[H^*]_{\mu_w} \leq [H_u]_{\mu_w}$ for all $u \in A_1$. Thus for each $u \in A_1$, we can find some a measure one set $B_u \in \mathcal{F}_w$ on which H^* extends H_u. Set $A_2 = A_1 \cap \triangle_{u\in A_1} B_u$.

Now let $A^* \in \mathcal{F}_w, A^* \subseteq A_2$, be such that for all $w', w'' \in A^*, h_{w'}$ and $h_{w''}$ are compatible. Set

$$p^* = \vec{d}^\frown (w, \lambda, A^*, H^*, h) \leq^* p.$$

Then $p^* \leq^* p$ and for every $u \in A^*$, we have $\mathrm{Add}(p^*, u)$ and

$$\mathrm{Add}(p^*, u) \Vdash \text{“}\dot{S}_{\kappa_u} = \tau_u\text{”}.$$

Let τ be defined on A^* so that $\tau : u \mapsto \tau_u$. In M, consider the map $j(\tau)$. Then for each $\alpha < \kappa^+$, we will have $j(\tau)_{u\restriction\alpha}$ which is an $\mathbb{R}_{u\restriction\alpha}$-name for a member of $P(\kappa) \cap M = P(\kappa) \cap V$.

By the chain condition property 2.10, there are only κ possibilities for the value of this name, and since $2^\kappa > \kappa^+$, we can find $S \subseteq \kappa, S \in V$, such that for all $\alpha < \kappa^+, \Vdash_{\mathbb{R}_{u\restriction\alpha}}$“$\check{S} \neq j(\tau)_{u\restriction\alpha}$”.

Consider $j(p^*) = \vec{d}^\frown (j(w), \lambda, j(A^*), j(H^*), j(h))$. For each $\alpha < \kappa^+$, $\mathrm{Add}(j(p^*), w \restriction \alpha) \in \mathbb{R}^M_{j(w)}$ and it forces that $j(\dot{S})_\kappa$ is the realization of $j(\tau)_{u\restriction\alpha}$. So this condition forces that $j(\dot{S})_\kappa \neq \check{S}$, and since $S = j(S) \cap \kappa$, it follows from Łoś's theorem that for each $0 < \alpha < \kappa^+$,

$$A^{**} = \{u \in A^* \mid \mathrm{Add}(p^*, u) \Vdash \text{“ } \dot{S}_{\kappa_u} \neq \check{S} \cap \kappa_u\} \in w(\alpha).$$

Let $p^{**} = \vec{d}^\frown (w, \lambda, A^{**}, H^* \restriction A^{**}, h)$, where $H^* \restriction A^{**} = H^* \restriction \{\kappa_u > \lambda \mid \kappa_u \in A^{**}\}$.

Then

$$p^{**} \Vdash \text{“}\dot{C} \cap \{\alpha < \kappa : \dot{S}_\alpha = \check{S} \cap \alpha\} \text{ is bounded in } \kappa\text{”}.$$

We get a contradiction and the lemma follows. □

Force over $V[G]$ with $\mathrm{Col}(\omega, \kappa_0)$ an let H be $\mathrm{Col}(\omega, \kappa_0)$-generic over $V[G]$. As the forcing is small, we can easily show that

$$V[G][H] \models \text{“}\kappa \text{ is the least inaccessible cardinal and } \diamondsuit_\kappa \text{ fails”}.$$

So in $V[G][H]$, $\diamondsuit_\kappa$ fails.

4. Weak diamond at κ in $V[G]$

In this section, we improve the conclusion of the last section, by showing that in fact Φ_κ fails in the model constructed above. It suffices to prove the following.

Lemma 4.1. *Φ_κ fails in $V[G]$.*

Proof. The proof follows ideas developed in [1]. Note that in M, we have $2^\kappa = 2^{\kappa^+} = \kappa^{++}$, so in V, there exists a partial function $H : \kappa \to V_\kappa$ such that $\mathrm{dom}(H) = \{\alpha < \kappa \mid 2^\alpha = 2^{\alpha^+} = \alpha^{++}\}$ [1] and for all $\alpha \in \mathrm{dom}(H)$,

$$H(\alpha) : 2^\alpha \leftrightarrow (V_\alpha^{<\omega} \times \alpha \times P(V_\alpha) \times P(V_\alpha \times V_\alpha) \times P(V_\alpha))^{\alpha \times \alpha^+}$$

is a bijection.

As in [1], let us identify a condition $p = \vec{d}^\frown(w, \lambda, A, H, h) \in \mathbb{R}_w$ with

$$\langle \vec{d}, \lambda, A, H, h \rangle \in V_\kappa^{<\omega} \times \kappa \times P(V_\kappa) \times P(V_\kappa \times V_\kappa) \times P(V_\kappa)$$

and call it a simple representation of p. As $\mathbb{R}_w$ satisfies the κ^+-c.c., every antichain in $\mathbb{R}$ can be represented as an element of $(V_\kappa^{<\omega} \times \kappa \times P(V_\kappa) \times P(V_\kappa \times V_\kappa) \times P(V_\kappa))^\kappa$.

Define $F : 2^{<\kappa} \to V_\kappa, F \in V$ as follows: for $t \in 2^{<\kappa}$ with $\mathrm{dom}(t) = \alpha, F(t) \in V_\kappa$ is a function such that

- $\mathrm{dom}(F(t)) = \{u \in \mathcal{U}_\infty \cap V_\kappa \mid \kappa_u = \alpha\}$.
- $F(t)(u) = \{q \in \mathbb{R}_u \mid q$ is simply represented by an element of $H(\alpha)(t)(\mathrm{lh}(u))\}$.

Note that $H(\alpha)(t) \in (V_\alpha^{<\omega} \times \alpha \times P(V_\alpha) \times P(V_\alpha \times V_\alpha) \times P(V_\alpha))^{\alpha \times \alpha^+}$, and so $F(t)(u)$ is well-defined. Now define the coloring $c : 2^{<\kappa} \to 2, c \in V[G]$, as follows:

- If $t \in 2^{\kappa_i}$, for some $i < \kappa$, then
$$c(t) = \begin{cases} 1 & \text{if } F(t)(u_i) \cap (G_{u_i}) \neq \emptyset, \\ 0 & \text{if } Otherwise. \end{cases}$$
where G_{u_i} is the $\mathbb{R}_{u_i}$-generic filter generated by G.
- $c(t) = 0$ for every other $t \in 2^{<\kappa}$.

[1]In fact we have $\mathrm{dom}(H) = \{\alpha < \kappa \mid \alpha$ is an inaccessible cardinal$\}$.

We show that c exemplify the failure of Φ_κ in $V[G]$. Thus suppose that $g : \kappa \to 2, g \in V[G]$. We find some $f : \kappa \to 2, f \in V[G]$ such that the set $\{\alpha < \kappa \mid c(f \restriction \alpha) \neq g(\alpha)\}$ contains a club of κ.

Let $\dot{g}$ be an $\mathbb{R}_w$-name for g, and suppose $p = \vec{d}^\frown(w, \lambda, A, H, h) \in \mathbb{R}_w$. As before, we can assume that for each $u \in A$, $\mathrm{Add}(p, u)$ is defined. Thus for each $u \in A$, we can form the condition $\mathrm{Add}(p, u) = \vec{d}^\frown\langle p_0^u, p_1^u\rangle \in \mathbb{R}_w$, where p_0^u and p_1^u are defined as in the proof of Lemma 3.1. By the factorization lemma 2.11,

$$\mathbb{R}_w / \mathrm{Add}(p, u) \simeq (\mathbb{R}_u / \vec{d}^\frown p_0^u) \times (\mathbb{R}_w / p_1^u).$$

By the Prikry property 2.12, we can find $q_1^u = (w, \lambda, A_u, H_u, h_u) \leq^* p_1^u$ and an $\mathbb{R}_u$-name σ_u for an ordinal in $\{0, 1\}$ such that

$$\vec{d}^\frown\langle p_0^u, q_1^u\rangle \Vdash \text{"}\dot{g}(\kappa_u) = \sigma_u\text{"}.$$

Now define $p^* \leq^* p$ as before.

Consider the map $\sigma : u \mapsto \sigma_u$ which is define on A^*. In M, define the function $h : \kappa^+ \to (V_\kappa^{<\omega} \times \kappa \times P(V_\kappa) \times P(V_\kappa \times V_\kappa) \times P(V_\kappa))^\kappa$ as follows: for every $\tau < \kappa^+$ let $h(\tau) \in (V_\kappa^{<\omega} \times \kappa \times P(V_\kappa) \times P(V_\kappa \times V_\kappa) \times P(V_\kappa))^\kappa$ be a simple representation of a maximal antichain $A_\tau \subseteq \mathbb{R}_{w \restriction \tau}$ of conditions $q \in \mathbb{R}_{w \restriction \tau}$ which force $j(\sigma)_{w \restriction \tau} = 0$. h can be identified with an element of $(V_\kappa^{<\omega} \times \kappa \times P(V_\kappa) \times P(V_\kappa \times V_\kappa) \times P(V_\kappa))^{\kappa \times \kappa^+}$, and so we can consider the function $f = H(\kappa)^{-1}(h) : \kappa \to 2$, where $H(\kappa) = j(H)(\kappa)$.

Set $A^{**} = \{u \in A^* \mid c(f \restriction \kappa_u)(u)$ is a maximal antichain of $\mathbb{R}_u$ of conditions $q \Vdash$"$\sigma_u = 0$"$\}$. Then it is easily seen by our choice of f that $A^{**} \in \mathcal{F}_w$. Set $p^{**} = \vec{d}^\frown(w, \lambda, A^{**}, H^* \restriction A^{**}, h)$. Then

$$p^{**} \Vdash \text{"}\dot{C} \cap \{\alpha < \kappa \mid c(f \restriction \alpha) = \dot{g}(\alpha)\} \text{ is bounded in } \kappa\text{"}.$$

The result follows immediately. □

In the model of 1.1, GCH fails cofinally often below κ, and we do not know the answer to the following.

Question 4.2. *Is it consistent with* GCH *that* $\diamondsuit_\kappa$ *(or* Φ_κ*) fails for the least inaccessible cardinal.*

Also it is possible to extend our result to make κ the least Mahlo cardinal (by taking $\mathrm{lh}(w) = \kappa^+ \cdot \kappa^+$), the least greatly Mahlo cardinal (by taking $\mathrm{lh}(w) = (\kappa^+)^{\kappa^+}$) and so on. On the other hand $\diamondsuit_\kappa$ (and hence Φ_κ) holds if κ is large enough, say a measurable cardinal. However the answer to the following is unknown.

Question 4.3. *Is it consistent that $\diamondsuit_\kappa$ (or Φ_κ) fails for κ a weakly compact cardinal.*

References

[1] Ben-Neria, Omer; Garti, Shimon, Hayut, Yair, Weak prediction principles, preprint.

[2] Cummings, James; A model in which GCH holds at successors but fails at limits. Trans. Amer. Math. Soc. 329 (1992), no. 1, 1–39.

[3] Cummings, James; Woodin's theorem on killing diamond via Radin forcing, unpublished note.

[4] Cummings, James; Personal communication, 2010.

[5] Devlin, K.J., Shelah, S., A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. Israel J. Math. 29, 239-247 (1978).

[6] Gitik, Moti; Prikry-type forcings. Handbook of set theory. Vols. 1, 2, 3, 1351–1447, Springer, Dordrecht, 2010.

[7] Golshani, Mohammad; Mitchell, William; On a question of Hamkins and Lowe on the modal logic of collapse forcing. In preparation.

[8] R. B. Jensen, The fine structure of the constructible hierarchy, Ann. Math. Logic 4 (1972) 229-308; erratum, ibid. 4 (1972) 443. With a section by Jack Silver.

[9] Mitchell, William How weak is a closed unbounded ultrafilter? Logic Colloquium '80 (Prague, 1980), pp. 209–230, Stud. Logic Foundations Math., 108, North-Holland, Amsterdam-New York, 1982.

[10] Shelah, Saharon Diamonds. Proc. Amer. Math. Soc. 138 (2010), no. 6, 2151-2161.

[11] Woodin, Hugh; Personal communication.

[12] Zeman, Martin, $\diamondsuit$ at Mahlo cardinals. J. Symbolic Logic 65 (2000), no. 4, 1813-1822.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran-Iran.

E-mail address: golshani.m@gmail.com