Diophantine approximation of Cohen reals

Mohammad Golshani (IPM)

January 8, 2020

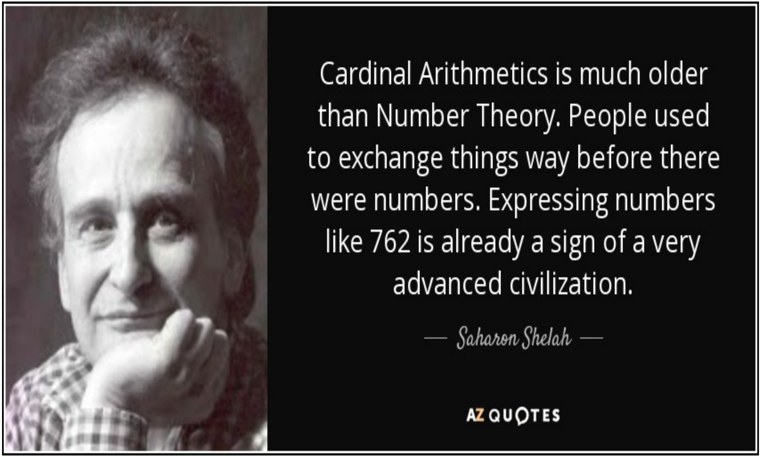
Cardinal Arithmetics is much older than Number Theory. People used to exchange things way before there were numbers. Expressing numbers like 762 is already a sign of a very advanced civilization.
— Saharon Shelah —
AZ QUOTES

Table of Contents

Mahler's classification of reals

- The set of real numbers splits into algebraic and transcendental numbers.

Mahler's classification of reals

- The set of real numbers splits into algebraic and transcendental numbers.
- Kurt Mahler in 1932 partitioned the real numbers into four classes, called $A, S, T,$ and U.

Mahler's classification of reals

- The set of real numbers splits into algebraic and transcendental numbers.
- Kurt Mahler in 1932 partitioned the real numbers into four classes, called $A, S, T,$ and U.
- Class A consists of algebraic numbers, while classes S, T and U contain transcendental numbers.

Mahler's classification of reals

- The set of real numbers splits into algebraic and transcendental numbers.
- Kurt Mahler in 1932 partitioned the real numbers into four classes, called $A, S, T,$ and U.
- Class A consists of algebraic numbers, while classes S, T and U contain transcendental numbers.
- There is another classification of reals known as Koksma's classification (introduced in 1939), where instead of looking at the approximation of 0 by integer polynomials evaluated at the real number ξ, the approximation of ξ by algebraic numbers is considered.

Mahler's classification of reals

- The set of real numbers splits into algebraic and transcendental numbers.
- Kurt Mahler in 1932 partitioned the real numbers into four classes, called A, S, T, and U.
- Class A consists of algebraic numbers, while classes S, T and U contain transcendental numbers.
- There is another classification of reals known as Koksma's classification (introduced in 1939), where instead of looking at the approximation of 0 by integer polynomials evaluated at the real number ξ, the approximation of ξ by algebraic numbers is considered.
- Koksma's classes are denoted by A^*, S^*, T^* and U^*.

Mahler's classification of reals

- The set of real numbers splits into algebraic and transcendental numbers.
- Kurt Mahler in 1932 partitioned the real numbers into four classes, called A, S, T, and U.
- Class A consists of algebraic numbers, while classes S, T and U contain transcendental numbers.
- There is another classification of reals known as Koksma's classification (introduced in 1939), where instead of looking at the approximation of 0 by integer polynomials evaluated at the real number ξ, the approximation of ξ by algebraic numbers is considered.
- Koksma's classes are denoted by A^*, S^*, T^* and U^*.
- It is well-know that the classifications of Mahler and of Koksma coincide, in the sense that for any real number ξ, ξ is an A (resp. S, T or U) number $\iff$ ξ is an A^* (resp. S^*, T^* or U^*) number.

Mahler's classification of reals

- We work with Koksma's classification.

Mahler's classification of reals

- We work with Koksma's classification.
- For an algebraic real number α let $\deg(\alpha) = \deg(P)$ and $H(\alpha) = H(P)$, the height of P, where $P(X) \in \mathbb{Z}[X]$ is the minimal polynomial of α.

Mahler's classification of reals

- We work with Koksma's classification.
- For an algebraic real number α let $\deg(\alpha) = \deg(P)$ and $H(\alpha) = H(P)$, the height of P, where $P(X) \in \mathbb{Z}[X]$ is the minimal polynomial of α.
- For a positive integer n and real numbers ξ and $H \geq 1$, define $w_n^*(\xi, H) = \min\{|\xi - \alpha| :$ α real algebraic $\deg(\alpha) \leq n,\ H(\alpha) \leq H,\ \alpha \neq \xi\}$.

Mahler's classification of reals

- We work with Koksma's classification.
- For an algebraic real number α let $\deg(\alpha) = \deg(P)$ and $H(\alpha) = H(P)$, the height of P, where $P(X) \in \mathbb{Z}[X]$ is the minimal polynomial of α.
- For a positive integer n and real numbers ξ and $H \geq 1$, define $w_n^*(\xi, H) = \min\{|\xi - \alpha| :$ α real algebraic $\deg(\alpha) \leq n,\ H(\alpha) \leq H,\ \alpha \neq \xi\}$.
- Set

$$w_n^*(\xi) = \limsup_{H \to \infty} \frac{-\log(H w_n^*(\xi, H))}{\log H}$$

and

$$w^*(\xi) = \limsup_{n \to \infty} \frac{w_n^*(\xi)}{n}.$$

Mahler's classification of reals

- In other words, $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w-1}.$$

Mahler's classification of reals

- In other words, $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w-1}.$$

- Now Koksma's classes A^*, S^*, T^* and U^* are defined as follows.

Mahler's classification of reals

- In other words, $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w-1}.$$

- Now Koksma's classes A^*, S^*, T^* and U^* are defined as follows.
- Let ξ be a real number.

Mahler's classification of reals

- In other words, $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w-1}.$$

- Now Koksma's classes A^*, S^*, T^* and U^* are defined as follows.
- Let ξ be a real number.
- ξ is an A^*-number if $w^*(\xi) = 0$.

Mahler's classification of reals

- In other words, $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w-1}.$$

- Now Koksma's classes A^*, S^*, T^* and U^* are defined as follows.
- Let ξ be a real number.
- ξ is an A^*-number if $w^*(\xi) = 0$.
- ξ is an S^*-number if $0 < w^*(\xi) < \infty$.

Mahler's classification of reals

- In other words, $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w-1}.$$

- Now Koksma's classes A^*, S^*, T^* and U^* are defined as follows.
- Let ξ be a real number.
- ξ is an A^*-number if $w^*(\xi) = 0$.
- ξ is an S^*-number if $0 < w^*(\xi) < \infty$.
- ξ is a T^*-number if $w^*(\xi) = \infty$ and $w_n^*(\xi) < \infty$ for any $n \geq 1$.

Mahler's classification of reals

- In other words, $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w-1}.$$

- Now Koksma's classes A^*, S^*, T^* and U^* are defined as follows.
- Let ξ be a real number.
- ξ is an A^*-number if $w^*(\xi) = 0$.
- ξ is an S^*-number if $0 < w^*(\xi) < \infty$.
- ξ is a T^*-number if $w^*(\xi) = \infty$ and $w_n^*(\xi) < \infty$ for any $n \geq 1$.
- ξ is a U^*-number if $w^*(\xi) = \infty$ and $w_n^*(\xi) = \infty$ for some n onwards.

Mahler's classification of reals

- **Lemma**
  (Sprindzuk, 1965) There exists a set A of measure zero which contains all transcendental numbers ξ with $w^*(\xi) > 1$.

Hausdorff dimension

- For a given $S \subseteq \mathbb{R}$ let $diam(S)$ denote the diameter of S; $diam(S) = \sup\{|x - y| : x, y \in S\}$.

Hausdorff dimension

- For a given $S \subseteq \mathbb{R}$ let $diam(S)$ denote the diameter of S; $diam(S) = \sup\{|x - y| : x, y \in S\}$.
- Suppose $S \subseteq \mathbb{R}$ has $diam(S) < \infty$ and $\alpha > 0$.

Hausdorff dimension

- For a given $S \subseteq \mathbb{R}$ let $diam(S)$ denote the diameter of S; $diam(S) = \sup\{|x - y| : x, y \in S\}$.
- Suppose $S \subseteq \mathbb{R}$ has $diam(S) < \infty$ and $\alpha > 0$.
- The α-Hausdorff content of S is defined as $H_\alpha(S) = \inf\{\sum_{n\in\omega} diam(C_n)^\alpha : \langle C_n : n \in \omega\rangle$ is an open cover of $S\}$.

Hausdorff dimension

- For a given $S \subseteq \mathbb{R}$ let $diam(S)$ denote the diameter of S; $diam(S) = \sup\{|x - y| : x, y \in S\}$.
- Suppose $S \subseteq \mathbb{R}$ has $diam(S) < \infty$ and $\alpha > 0$.
- The α-Hausdorff content of S is defined as $H_\alpha(S) = \inf\{\sum_{n\in\omega} diam(C_n)^\alpha : \langle C_n : n \in \omega\rangle$ is an open cover of $S\}$.
- The Hausdorff dimension of S is defined by $\dim_H(S) = \inf\{\alpha > 0 : H_\alpha(S) = 0\} = \sup\{\alpha \geq 0 : H_\alpha(S) > 0\}$.

Hausdorff dimension

- For a given $S \subseteq \mathbb{R}$ let $diam(S)$ denote the diameter of S; $diam(S) = \sup\{|x - y| : x, y \in S\}$.
- Suppose $S \subseteq \mathbb{R}$ has $diam(S) < \infty$ and $\alpha > 0$.
- The α-Hausdorff content of S is defined as $H_\alpha(S) = \inf\{\sum_{n\in\omega} diam(C_n)^\alpha : \langle C_n : n \in \omega\rangle$ is an open cover of $S\}$.
- The Hausdorff dimension of S is defined by $\dim_H(S) = \inf\{\alpha > 0 : H_\alpha(S) = 0\} = \sup\{\alpha \geq 0 : H_\alpha(S) > 0\}$.

- **Lemma**

  (Kasch, Volkmann, 1958) The set of T-numbers and the set of U-numbers have Hausdorff dimension zero.

Cohen forcing

- Let $\mathbb{P}$ be the set of finite partial function from ω to 2, ordered by reverse inclusion.

Cohen forcing

- Let $\mathbb{P}$ be the set of finite partial function from ω to 2, ordered by reverse inclusion.
- $\mathbb{P}$ is called Cohen forcing.

Cohen forcing

- Let $\mathbb{P}$ be the set of finite partial function from ω to 2, ordered by reverse inclusion.
- $\mathbb{P}$ is called Cohen forcing.
- Let G be $\mathbb{P}$-generic over V and let $r_G = \bigcup_{p \in G} p$.

Cohen forcing

- Let $\mathbb{P}$ be the set of finite partial function from ω to 2, ordered by reverse inclusion.
- $\mathbb{P}$ is called Cohen forcing.
- Let G be $\mathbb{P}$-generic over V and let $r_G = \bigcup_{p \in G} p$.
- r_G is a real, called Cohen real.

Some corollaries

- **Lemma**
  (joint with Will Brian, proved independently by Glenn David Dean) Suppose r is a Cohen real. Then r is in class U (and indeed it is a Liouville number, i.e., $w_1^*(r) = \infty$).

Some corollaries

- Lemma
  (joint with Will Brian, proved independently by Glenn David Dean) Suppose r is a Cohen real. Then r is in class U (and indeed it is a Liouville number, i.e., $w_1^*(r) = \infty$).
  - Corollary 1 (joint with Will Brian) Suppose W is a generic extension of V. Then the set

    $$\{r \in W : r \text{ is Cohen generic over } V\}$$

    has Hausdorff measure zero.

Some corollaries

- Lemma
  (joint with Will Brian, proved independently by Glenn David Dean) Suppose r is a Cohen real. Then r is in class U (and indeed it is a Liouville number, i.e., $w_1^*(r) = \infty$).
  - Corollary 1 (joint with Will Brian) Suppose W is a generic extension of V. Then the set
  $$\{r \in W : r \text{ is Cohen generic over } V\}$$
  has Hausdorff measure zero.
  - Corollary 2 (joint with Will Brian) There exists a perfect set of Liouville (and hence U) numbers.

Thank You